A New Relativistic Hydrodynamic Code

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ABSTRACT

Relativistic temperature of gas raises the issue of equation of state (EoS) in relativistic hydrodynamics. We present a code for relativistic hydrodynamics with an EoS that is simple but approximates very closely the EoS of single-component perfect gas in the relativistic regime. Tests with a code based on the TVD scheme are presented to highlight differences induced by different EoS.
1. Introduction

Many high-energy astrophysical phenomena involve relativistic flows that are highly nonlinear and intrinsically complex. Understanding such relativistic flows is important for correctly interpreting the phenomena, but often studying them is possible only through numerical simulations.

Gas in relativistic hydrodynamics (RHDs) is characterized by relativistic fluid speed \( v \sim c \) and/or relativistic temperature (internal energy much greater than rest energy), and the latter brings us to the issue of equation of state (hereafter EoS) of the gas. The EoS most commonly used in numerical RHDs, which is originally designed for the non-relativistic gas with constant ratio of specific heats, however, is essentially valid only for the gas of either subrelativistic or ultrarelativistic temperature. In other words, that is not derived from relativistic kinetic theory. On the other hand, the EoS of single-component perfect gas in relativistic regime was presented (see Synge 1957). But it’s form is too complicated to be implemented in numerical schemes.

In this poster, we present a new code for RHDs. For it, we propose a new EoS which is an algebraic function of temperature. Our new EoS is simple to be implemented to numerical codes with minimum efforts and minimum computational cost, but at the same time approximates very closely the EoS of single-component perfect gas in relativistic regime. We also present the Lorentz transformation (hereafter LT) from the conserved quantities to the local quantities for the EoS. Then we present the entire eigenstructure of RHDs for general EoS. Such that one has to define the chosen equation of state in the code, the code does the rest. Finally we present shock tube tests and compare our EoS to those previous used in numerical codes.
2. Relativistic Hydrodynamics

2.1. Basic Equations

The special RHD equations for an ideal fluid can be written in the laboratory frame of reference as a hyperbolic system of conservation equations

\[ \frac{\partial D}{\partial t} + \frac{\partial}{\partial x_j} (Dv_j) = 0, \quad (1a) \]

\[ \frac{\partial M_i}{\partial t} + \frac{\partial}{\partial x_j} (M_i v_j + p \delta_{ij}) = 0, \quad (1b) \]

\[ \frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j} [(E + p) v_j] = 0, \quad (1c) \]

where \( D, M_i, \) and \( E \) are the mass density, momentum density, and total energy density in the reference frame, respectively (see, e.g., Landau & Lifshitz 1959; Wilson & Mathews 2003). The quantities in the reference frame are related to those in the local frame via LT

\[ D = \Gamma \rho, \quad (2a) \]

\[ M_i = \Gamma^2 \rho h v_i, \quad (2b) \]

\[ E = \Gamma^2 \rho h - p, \quad (2c) \]

where \( \rho, v_i, p, \) and \( h \) are the proper mass density, fluid three-velocity, isotropic gas pressure and specific enthalpy, respectively, and the Lorentz factor is given by

\[ \Gamma = \frac{1}{\sqrt{1 - v^2}} \quad \text{with} \quad v^2 = v_x^2 + v_y^2 + v_z^2. \quad (3) \]

In above, the Latin indices (e.g., \( i \)) represents spatial coordinates and conventional Einstein summation is used. The speed of light is set to unity \( (c \equiv 1) \) throughout this poster.
2.2. Equation of State

The above system of equations is closed with an EoS. Here we first present the EoS that have been used previously, and then propose a new EoS.

Without loss of generality the EoS is given as

\[ h \equiv h(p, \rho). \] (4)

Then the general form of polytropic index, \( n \), and the general form of sound speed, \( c_s \), respectively can be written as

\[ n = \rho \frac{\partial h}{\partial p} - 1, \quad c_s^2 = -\frac{\rho}{n h} \frac{\partial h}{\partial \rho}. \] (5)

In addition we introduce a variable \( \gamma_h \), with which the EoS property will be conveniently presented,

\[ \gamma_h = \frac{h(\Theta) - 1}{\Theta}. \] (6)

The most commonly used EoS, which is called the ideal EoS (hereafter ID), is given as

\[ h = 1 + \frac{\gamma \Theta}{\gamma - 1} \] (7)

with a constant \( \gamma \). Here \( \Theta = p/\rho \) is effectively temperature and \( \gamma = c_p/c_v \) is the ratio of specific heats. For it, \( \gamma_h = \gamma/\gamma - 1 \) does not depend on \( \Theta \). ID may be correctly applied to the gas of either subrelativistic temperature with \( \gamma = 5/3 \) or ultrarelativistic temperature with \( \gamma = 4/3 \). But ID is rented from non-relativistic hydrodynamics, and hence is not consistent with relativistic kinetic theory. For example, we have

\[ n = \frac{1}{\gamma - 1}, \quad c_s^2 = \frac{\gamma \Theta (\gamma - 1)}{\gamma \Theta + \gamma - 1}. \] (8)

In the high temperature limit, i.e., \( \Theta \to \infty \), and for \( \gamma > 2 \), \( c_s > 1 \) i.e., superluminal sound speed (see Taub 1948). More importantly, Taub (1948) showed in his work that the choice of EoS is not arbitrary and has to satisfy the inequality,

\[ (h - \Theta)(h - 4\Theta) \geq 1. \] (9)

This rules out ID for \( \gamma > 4/3 \).

The correct EoS for the single-component perfect gas in relativistic regime (hereafter RP) was given by Synge (1957),

\[ h = \frac{K_3(1/\Theta)}{K_2(1/\Theta)}, \] (10)
where $K_2$ and $K_3$ are the modified Bessel functions of the second kind of order two and three, respectively. In the extreme non-relativistic limit ($\Theta \to 0$), $\gamma_h \to 5/2$, and in the extreme ultrarelativistic limit ($\Theta \to \infty$), $\gamma_h \to 4$. However, using the above EoS comes with a price of extra computational cost, since the thermodynamics of the fluid is expressed in terms of the modified Bessel functions and no analytic expression can be written for LT.

In a recent paper, Mignone et al. (2005) proposed an EoS which fits RP well. The EoS, which is abbreviated as TM following Mignone et al. (2005), is given by

$$h = \frac{5}{2} \Theta + \frac{3}{2} \sqrt{\Theta^2 + \frac{4}{9}}. \quad (11)$$

With TM the expressions of $n$ and $c_s$ become

$$n = \frac{3}{2} + \frac{3}{2} \frac{\Theta}{\sqrt{\Theta^2 + 4/9}}, \quad c_s^2 = \frac{5\Theta \sqrt{\Theta^2 + 4/9} + 3\Theta^2}{12\Theta \sqrt{\Theta^2 + 4/9} + 12\Theta^2 + 2}. \quad (12)$$

TM was derived from the lower bound of the Taub’s inequality, $(h - \Theta)(h - 4\Theta) = 1$. It produces right asymptotic values for $\gamma_h$.

In this poster we propose a new EoS, which is a simpler algebraic function of $\Theta$ and is also a better fit of RP compared to TM. We abbreviate our proposed EoS as RC and give it by

$$h = \frac{26\Theta^2 + 4\Theta + 1}{3\Theta + 2}. \quad (13)$$

With RC the expressions of $n$ and $c_s$ become

$$n = \frac{3}{2} \frac{9\Theta^2 + 12\Theta + 2}{(3\Theta + 2)^2}, \quad c_s^2 = \frac{\Theta(3\Theta + 2)(18\Theta^2 + 24\Theta + 5)}{3(6\Theta^2 + 4\Theta + 1)(9\Theta^2 + 12\Theta + 2)}. \quad (14)$$

RC satisfies the Taub’s inequality, $(h - \Theta)(h - 4\Theta) \geq 1$, for all $\Theta$. It also produces right asymptotic values for $\gamma_h$. For both TM and RC, correctly $c_s^2 \to 5\Theta/3$ in the extreme non-relativistic limit, and $c_s^2 \to 1/3$ in the extreme ultrarelativistic limit, respectively.

In Figure 1, $\gamma_h$, $n$, and $c_s$ are plotted with $\Theta$ to compare TM and RC to RP as well as ID. One can see the RC is a much better fit of RP than TM with

$$\frac{|h_{TM} - h_{RP}|}{h_{RP}} \lesssim 2\%, \quad \frac{|h_{RC} - h_{RP}|}{h_{RP}} \lesssim 0.8\%. \quad (15)$$

It is to be remembered that both $\gamma_h$ and $n$ are independent of $\Theta$, if ID is used.
3. Lorentz Transformation for RC

The RHD equations evolve the conserved quantities, $D$, $M_i$, and $E$, but we need to know the local quantities, $\rho$, $v_i$, $p$, to calculate the equations numerically. So the LT equations (2a–2c) need to be solved.

Combining the LT equations with the EoS of RC in (12), we get

$$M\sqrt{\Gamma^2 - 1} \left[ 3E\Gamma(8\Gamma^2 - 1) + 2D(1 - 4\Gamma^2) \right]$$
$$= 3\Gamma^2 \left[ 4(M^2 + E^2)\Gamma^2 - (M^2 + 4E^2) \right] - 2D(4E\Gamma - D)(\Gamma^2 - 1).$$  \hspace{1cm} (16)

Further simplification reduces it into an equation of 8th power in $\Gamma$.

Although the equation has to be solved numerically, it behaves very well. The physically meaningful solution should be between the upper limit, $\Gamma_u$, and the lower limit, $\Gamma_l$, that is derived inserting $D = 0$ into equation (16):

$$16(M^2 - E^2)^2\Gamma_l^6 - 8(M^2 - E^2)(M^2 - 4E^2)\Gamma_l^4 + (M^4 - 9M^2E^2 + 16E^4)\Gamma_l^2 + M^2E^2 = 0$$  \hspace{1cm} (18)

(a cubic equation of $\Gamma_l^2$). Out of the eight roots of equation (16), four are complex and four are real. Out of the four real roots, two are negative and two are positive. And out of the two real and positive roots, one is always larger than $\Gamma_u$, and the other is between $\Gamma_l$ and $\Gamma_u$ and so is the physical solution.

In codes equation (16) can be easily solved by the Newton-Raphson method. With an initial guess $\Gamma = \Gamma_l$ or any value smaller than it including 1, iteration can be proceeded upwards. Since the equation is extremely well-behaved, the iteration converges within a few steps. Once $\Gamma$ is known, the fluid speed is computed by

$$v = \frac{\sqrt{\Gamma^2 - 1}}{\Gamma},$$  \hspace{1cm} (19)

and the quantities $\rho$, $v_i$, $p$, are computed by

$$\rho = \frac{D}{\Gamma},$$  \hspace{1cm} (20a)

$$v_x = \frac{M_x}{M}v, \quad v_y = \frac{M_y}{M}v, \quad v_z = \frac{M_z}{M}v$$  \hspace{1cm} (20b)

$$p = \frac{(E - M_i v_i) - 2\rho + [(E - M_i v_i)^2 + 4\rho(E - M_i v_i) - 4\rho^2]^{\frac{1}{2}}}{6},$$  \hspace{1cm} (20c)

where

$$M_i v_i = M_x v_x + M_y v_y + M_z v_z.$$  \hspace{1cm} (21)
4. Eigenvalues and Eigenvectors

In building an upwind code to solve a hyperbolic system of conservation equations, eigenstructure (eigenvalues and eigenvectors of the Jacobian matrix) is required. Here we present our complete set of eigenvalues and eigenvectors without assuming any particular form of EoS.

Equations (1a)–(1c) can be written as

$$\frac{\partial \tilde{q}}{\partial t} + \frac{\partial \tilde{F}_j}{\partial x_j} = 0$$

(22)

with the state and flux vectors

$$\tilde{q} = \begin{bmatrix} D \\ M_i \\ E \end{bmatrix}, \quad \tilde{F}_j = \begin{bmatrix} D v_j \\ M_i v_j + p \delta_{ij} \\ (E + p) v_j \end{bmatrix},$$

(23)

or as

$$\frac{\partial \tilde{q}}{\partial t} + A_j \frac{\partial \tilde{q}}{\partial x_j} = 0, \quad A_j = \frac{\partial \tilde{F}_j}{\partial \tilde{q}}.$$  \hspace{1cm} (24)

Here $A_j$ is the $5 \times 5$ Jacobian matrix composed with the state and flux vectors.

The eigenvalues of $A_x$, the $x$-component of the Jacobian matrix, are

$$a_1 = \frac{(1 - c_s^2) v_x - c_s \Gamma \sqrt{Q}}{1 - c_s^2 v^2},$$  \hspace{1cm} (25a)

$$a_2 = a_3 = a_4 = v_x,$$  \hspace{1cm} (25b)

$$a_5 = \frac{(1 - c_s^2) v_x + c_s \Gamma \sqrt{Q}}{1 - c_s^2 v^2},$$  \hspace{1cm} (25c)

where $Q = 1 - v_x^2 - c_s^2(v_y^2 + v_z^2)$. The eigenvalues represent the five characteristic speeds associated with two sound wave modes ($a_1$ and $a_5$) and three entropy modes ($a_2$, $a_3$, and $a_4$).

The complete set of the right eigenvectors ($A_x \tilde{R} = a \tilde{R}$) is given by

$$\tilde{R}_1 = \left[ \frac{1 - a_1 v_x}{\Gamma}, \quad a_1 h(1 - v_x^2), \quad h(1 - a_1 v_x) v_y, \quad h(1 - a_1 v_x) v_z, \quad h(1 - v_x^2) \right]^T,$$

(26a)

$$\tilde{R}_2 = \tilde{X} \left[ X_1, \quad X_2, \quad X_3, \quad X_4, \quad X_5 \right]^T,$$

(26b)

$$\tilde{R}_3 = \frac{1}{1 - v_x^2} \left[ \frac{v_y}{\Gamma h}, \quad 2 v_x v_y, \quad 1 - v_x^2 + v_y^2, \quad v_y v_z, \quad 2 v_y \right]^T,$$

(26c)
\[ \vec{R}_4 = \frac{1}{1 - v_x^2} \begin{bmatrix} v_z, & 2v_x v_z, & v_y v_z, & 1 - v_x^2 + v_z^2, & 2v_z \end{bmatrix}^T, \]

\[ \vec{R}_5 = \left[ \frac{1 - a_5 v_x}{\Gamma}, \ a_5 h(1 - v_x^2), \ h(1 - a_5 v_x) v_y, \ h(1 - a_5 v_x) v_z, \ h(1 - v_x^2) \right]^T, \]

where

\[ X_1 = \frac{nc_s^2(v_y^2 + v_z^2) + (1 - v_x^2)}{\Gamma h}, \]

\[ X_2 = \left[ 2nc_s^2(v_y^2 + v_z^2) + (1 - nc_s^2)(1 - v_x^2) \right] v_x, \]

\[ X_3 = \left[ nc_s^2(v_y^2 + v_z^2) + (1 - v_x^2) \right] v_y, \]

\[ X_4 = \left[ nc_s^2(v_y^2 + v_z^2) + (1 - v_x^2) \right] v_z, \]

\[ X_5 = 2nc_s^2(v_y^2 + v_z^2) + (1 - nc_s^2)(1 - v_x^2). \]

The complete set of the left eigenvectors (\( \vec{L} A_x = a \vec{L} \)), which are orthonormal to the right eigenvectors, is

\[ \vec{L}_1 = \frac{1}{Y_1} \begin{bmatrix} Y_{11}, \ Y_{12}, \ Y_{13}, \ Y_{13}, \ Y_{15} \end{bmatrix}, \]

\[ \vec{L}_2 = \begin{bmatrix} h, & v_x, & v_y, & v_z, & -1 \end{bmatrix}, \]

\[ \vec{L}_3 = [-\Gamma hv_y, \ 0, \ 1, \ 0, \ 0], \]

\[ \vec{L}_4 = [-\Gamma hv_z, \ 0, \ 0, \ 1, \ 0], \]

\[ \vec{L}_5 = \frac{1}{Y_5} \begin{bmatrix} Y_{51}, \ Y_{52}, \ Y_{53}, \ Y_{53}, \ Y_{55} \end{bmatrix}, \]

where

\[ Y_{i1} = \frac{h}{\Gamma}(1 - a_i v_x)(1 - nc_s^2), \]

\[ Y_{i2} = na_i(1 - c_s^2 v_x^2) + a_i(1 + nc_s^2)v_x^2 - (1 + n)v_x, \]

\[ Y_{i3} = -(1 + nc_s^2)(1 - a_i v_x)v_y, \]

\[ Y_{i4} = -(1 + nc_s^2)(1 - a_i v_x)v_z, \]

\[ Y_{i5} = (1 + nc_s^2v_x^2) + (1 - c_s^2)nv_x^2 - a_i(1 + n)v_x, \]

\[ \dot{Y}_i = hn \left[ (a_i - v_x)^2 Q + \frac{c_s^2}{\Gamma^2} \right], \]

and index \( i = 1, 5 \).

With three degenerate modes that have same eigenvalues, \( a_2 = a_3 = a_4 \), we have a freedom to write down the right and left eigenvectors in a variety of different forms. We chose to present the ones that produce the best results with the TVD code.
5. Numerical Tests

The differences induced by different EoS are illustrated through a series of shock tube tests, which were performed using the TVD code built with the EoS in §2 and the eigenvalues and eigenvectors in §3. Two sets are considered.

For the first set with parallel velocity component only, two tests are presented:
P1: $\rho_L = 10, \rho_R = 1, p_L = 13.3, p_R = 10^{-6}$, and $v_{p,L} = v_{p,R} = 0$ initially, and $t_{\text{end}} = 0.45$,
P2: $\rho_L = \rho_R = 1, p_L = 10^3, p_R = 10^{-2}$, and $v_{p,L} = v_{p,R} = 0$ initially, and $t_{\text{end}} = 0.4$.

For the second set with transverse velocity component, two tests, where different transverse velocities were added to the test P2, are presented:
T1: initially $u_{t,R} = 0.99$ to the right state, $t_{\text{end}} = 0.45$,
T2: initially $u_{t,L} = 0.9$ and $u_{t,R} = 0.99$ to the left and right states, $t_{\text{end}} = 0.75$.
The box covers the region of $0 \leq x \leq 1$ in all the tests.

Figures 2, 3, 4, and 5 show the numerical solutions for RC and TM, but the analytic solutions for ID with $\gamma = 5/3$ and $4/3$. For ID numerical solutions are almost indistinguishable from analytic solutions, once they are calculated. The ID solutions are clearly different from the RC and TM solutions. The ID solution with $\gamma = 4/3$ looks to match the RC and TM solutions in P2, especially in the left region of contact discontinuity (hereafter CD) where the flow is overall highly relativistic with $\Theta \gg 1$. But the difference is obvious in the region between CD and shock, because $\Theta \sim 1$ there. On the other hand, the solutions of RC and TM look very much alike. It reflects the similarity in the distributions of specific enthalpy in equations (11) and (13). But yet there is a noticeable difference, especially in the density in the region between CD and shock, and the difference reaches up to $\sim 5\%$.

The most commonly used, ideal EoS, ID, can be used for entirely non-relativistic gas ($\Theta \ll 1$) with $\gamma = 5/3$ or for entirely ultrarelativistic gas ($\Theta \gg 1$) with $\gamma = 4/3$. However, if the transition from non-relativistic to relativistic with $\Theta \sim 1$ is involved, ID produces incorrect results and using it should be avoided. The EoS proposed by Mignone et al. (2005), TM, produces reasonably correct results with error of a few percent at most. The newly suggested EoS, RC, which approximates the EoS of relativistic perfect gas, RP, most accurately, produces thermodynamically the most accurate results. At the same time it is simple enough to be implemented to numerical codes with minimum efforts and minimum computational cost. The correctness and simplicity make RC suitable for astrophysical applications.
REFERENCES


Taub, A. H. 1948, Phys. Rev., 74, 328


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Fig. 1.— Comparison between different EoS. $\Gamma_h$, $n$, and $c_s$, vs $\Theta$ for RC (red-long dashed), TM (blue-short dashed), ID (green and cyan-dotted), and RP (black-solid).
Fig. 2.— Relativistic shock tube with parallel component of velocity only (P1) for RC (red), TM (blue), and ID (green and cyan).
Fig. 3.— Relativistic shock tube with parallel component of velocity only (P2) for RC (red), TM (blue), and ID (green and cyan).
Fig. 4.— Relativistic shock tube with transverse component of velocity (T1) for RC (red), TM (blue), and ID (green and cyan).
Fig. 5.— Relativistic shock tube with transverse component of velocity (T2) for RC (red), TM (blue), and ID (green and cyan).