Black holes in Gauss-Bonnet gravity

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• Motivation
• Lovelock gravity
• Dilatonic black holes
• Dilaton Gauss-Bonnet gravity
• Summary and recent result
We consider an extremal limit of dilatonic black hole in Einstein-Maxwell-dilaton system. It’s known that such limit is bad defined: the event horizon become singular and shrink to single point, so entropy for such solution is also zero.

From other hand quantum calculation gives non-zero value of entropy. So it claims that corrections from more general theory may resolve this problem.

One of such correction is Gauss-Bonnet term:

\[ R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} \]

which arise in two theories:
• one loop correction in heterotic string theory
• Lovelock theory
Lovelock theory

D. Lovelock JoMP v.12, n.3, 498 (1971)

\[ G_{ij} \rightarrow A^{ij} = A^{ji},\ A^{ij} = A^{ji}(g_{ab}, g_{ab,c}, g_{ab,cd}),\ A^{ij}_{;j} = 0,\ A^{ij} \mid_{\text{invacuo}} = 0 \]

LL action

\[
I_G = \kappa \int \sum_{p=0}^{[d/2]} \sum_{\alpha_p} L^{(p)}
\]

\[
L^{(p)} = \varepsilon_{a_1 \ldots a_d} R^{a_1 a_2} \ldots R^{a_{2p-1} a_{2p}} e^{a_2 p+1} \ldots e^{a_d}.
\]

But equation of motion possess in general, several constant curvature solutions with different radii, making the value of the cosmological constant ambiguous. In fact, the cosmological constant could change in different regions of a spatial section, or it could jump arbitrarily as the system evolves in time.

Some of solutions are ``spurious'' in the sense that perturbations around them yield ghosts.
These problems are overcome demanding the theory to have a unique cosmological constant.

The simplest class of such theory describe by action

\[ I_G = \kappa \int \sum_{p=0}^{k} \alpha_p L^{(p)} \]

where

\[ \alpha_p := c_p^k = \begin{cases} \frac{l^2(p-k)}{(d-2p)(p)} & , p \leq k \\ 0 & , p > k \end{cases} \]

\[ \kappa = \frac{1}{2(d-2)!\Omega_{d-2}G_k}, \Lambda = \frac{(d-1)(d-2)}{2l^2} \]

For a given dimension d, the coefficients \( c_p^k \) give rise to a family of inequivalent theories, labeled by the integer k, which represents the highest power of curvature in the Lagrangian.
Simplest test for Lovelock theory is existing of exact solution

For system consists of

Einstein-Hilbert action

\[ I_{EH} = -\frac{1}{2(d-2)\Omega_{d-2} G} \int d^d x \sqrt{-g} (R - 2\Lambda) \]

Maxwell term

\[ I_M = -\frac{1}{4\epsilon \Omega_{d-2}} \int \sqrt{-g} F_{\mu\nu} F^{\mu\nu} d^d x \]

LL term

\[ I_k = \kappa \int \sum_{p=0}^{k} c^k_p L^{(p)} \]
Spherical symmetric black hole solution

Metric:
\[ ds^2 = -f^2(r)dt^2 + \frac{dr^2}{f^2(r)} + r^2d\Omega^2_{d-2}. \]

, where
\[ \sigma = (\pm 1)^{(k+1)} \]
\[ f^2(r) = 1 + \frac{r^2}{l^2} - \sigma g_k(r), \]
\[ g_k(r) = \left( \frac{2G_k M + \delta_{d-2k,1}}{r^{d-2k-1}} - \frac{\varepsilon G_k}{(d-3)} \frac{Q^2}{r^{2(d-k-2)}} \right)^{\frac{1}{k}} \]

electromagnetic field strength
\[ F_{0r} = -\partial_r A_0(r). \]
\[ A_0(r) = \phi_\infty + \frac{\varepsilon}{(d-3)} \frac{Q}{r^{d-3}} \]
Dilatonic black holes

System consists of gravity, form field and scalar field:

\[ S = \int d^4x \sqrt{-g} \left( R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \frac{1}{q!} e^{a\phi} F_{[2]}^2 \right) \]

For the spherical symmetric case metric is:

\[ ds^2 = -e^{2B} dt^2 + e^{2A} dr^2 + e^{2C} d\Omega_2^2 \]

S-duality

\[ g_{\mu\nu} \rightarrow g_{\mu\nu}, \quad F \rightarrow e^{-a\phi} * F, \quad \phi \rightarrow -\phi \]

where * denotes a d-dimensional Hodge dual,

so we will restrict ourselves to purely magnetic solutions (for the case one charge).
Black hole is supported by the form field

- case of one charge (magnetically charged brane)

\[ F_{[2]} = b \, \text{vol}(\Omega_2) \]

- dyonic case

\[ F_{[2]} = b_1 \text{vol}_2 + b_2 e^{-a\phi} \ast \text{vol}_2, \quad \text{vol}_2 = \text{vol}(\Omega_2) \]

(analytic dyon solution known only for the cases of dilaton coupling \( a^2 = 1, 3 \))
For the case of outer spherical symmetric space exist two type of BH solutions:

- *asymptotically flat*
- *linear dilaton background (LDB) solution*


Clement, Gal'tsov, Leygnac and DO, PRD73, 045018 (2006); hep-th/0512013
• asymptotically flat solution - dilatonic black hole

\[ ds^2 = -f_+ f_-^{-\frac{1-a^2}{1+a^2}} \, dt^2 + f_+^{-1} f_-^{-\frac{1-a^2}{1+a^2}} \, dr^2 + r^2 f_-^{\frac{2a^2}{1+a^2}} \, d\Omega_2^2 \]

where

\[ f_\pm(\xi) = 1 - \frac{r_\pm}{r} \]

\[ e^{a\phi} = e^{a\phi_0} f_-^{\frac{2a^2}{\lambda}} \]

\[ F_{[2]} = 2\sqrt{\frac{r_+ r_-}{1+a^2}} e^{-\frac{a}{2} \phi_0} \, \text{vol}(\Omega_2) \]

Scalar curvature:

\[ R = \frac{2a^2}{(1+a^2)^2} |r|^{-4} f_-^{-\frac{2a^2}{1+a^2}} \left[ r_-^2 \frac{f_+}{f_-} \right] \]

\[ S = 4\pi \Omega_2 |r_+|^{\frac{2}{1+a^2}} (r_+ - r_-) \frac{2a^2}{1+a^2} \quad \text{- entropy} \]
But if we want to get extremal solution \( r_- \to r_+ \)

We can’t do it with getting regular horizon, because the resulting point \( r = r_- = r_+ \), as we can see from scalar curvature

\[
R \Box f_- \frac{-2a^2}{1+a^2}
\]

will be singular point and radius of horizon for such solution will be zero, just as entropy.

By another hand quantum calculation give non-zero value of entropy.
- black hole on linear dilaton background (LDB)

\[
\begin{align*}
 ds^2 &= -\left(\frac{r}{r_0}\right)^{2\left(1+a^2\right)} \left(1 - \frac{c}{r}\right) dt^2 + \left(\frac{r_0}{r}\right)^{2\left(1+a^2\right)} \left(1 - \frac{c}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2 \\
 e^{a\phi} &= e^{a\phi_0} \left(\frac{r}{r_0}\right)^{2\left(1+a^2\right)} \\
 F_{[2]} &= \frac{2e^{-a\phi_0/2}}{\sqrt{1+a^2}} \frac{r_0}{\text{vol}(\Omega_2)}
\end{align*}
\]

(neutral solution on charged background)

Scalar curvature:

\[
R = \frac{2}{(1+a^2)^2 r^2} \left(\frac{r}{r_0}\right)^{2\left(1+a^2\right)} \left(a^2 - \frac{a^2 c}{r}\right)
\]
• asymptotically flat solution - dyonic black hole ( $a^2 = 1$ )

\[
d s^2 = -\frac{f_+f_-}{f_1}dt^2 + \left[ \frac{f_+f_-}{f_1} \right]^{-1}dr^2 + r^2 f_1 d\Omega_2^2
\]

\[
e^{a\phi} = \frac{1}{f_1}
\]

\[
F_{[2]} = \sqrt{2} \left[ \sqrt{(r_+ - r_1)(r_- - r_1)} \text{vol}(\Omega_2) - \sqrt{r_+r_-} \frac{dr}{r^2} \wedge dt \right]
\]

Another solution for a dilaton coupling $a^2 = 3$

is received by Toda lattice method.

Both solution have a regular extremal limit, but if we consider a expansion on horizon, then we see that extremal limit exist only for some particular value of a dilaton coupling $a^2 = 1, 3, 6, 11, \ldots, \frac{n(n+1)}{2}$
\textbf{• dyonic solution - black hole on LDB} \quad (a^2 = 1)

\begin{align*}
\text{charged solution on charged background)}
\end{align*}

\begin{align*}
\begin{align*}
ds^2 &= -f_+ f_- \frac{r}{r_0} dt^2 + \frac{r_0}{r} \left( \frac{dr^2}{f_+} + r^2 d\Omega_k^2 \right) \\
e^{a\phi} &= \frac{r}{r_0} \\
F_{[n]} &= \sqrt{2} \left[ r_0 \text{vol}(\Omega_2) - r_+ r_- \frac{dr}{r^2} \right]
\end{align*}
\end{align*}

\textbf{• dyonic solution - black hole on LDB} \quad (a = 0)

\begin{align*}
\text{(neutral solution on dyon background)}
\end{align*}

\begin{align*}
\begin{align*}
ds^2 &= -\left[ \frac{r}{r_0} \right]^2 f_+ dt^2 + \left[ \frac{r}{r_0} \right]^2 \left( \frac{dr^2}{f_+} + r^2 d\Omega_2^2 \right) \\
e^{a\phi} &= 1 \\
F_{[2]} &= \sqrt{2} \left[ r_0 \text{vol}(\Omega_2) - \frac{dr}{r_0} \wedge dt \right]
\end{align*}
\end{align*}
4D theory. The action consists of gravity, scalar filed, Maxwell field and GB term:

\[
S = \frac{1}{16\pi} \int \left\{ R - 2\partial_\mu \phi \partial^\mu \phi - e^{2\phi} \left( F^2 - \alpha L_{GB} \right) \right\} \sqrt{-g} \, d^4x
\]

where Gauss-Bonnet term

\[
L_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}
\]

Chen, Gal'tsov, DO, PRD 75, 084030 (2007), hep-th/0701004
Chen, Gal'tsov, DO, PRD 78, 104013 (2008), 0809.1720
Non-extremal solution

exterior solution

P. Kanti, K. Tamvakis
Coloured black holes in higher Curvature string gravity, hep-th/9609003

T. Torii, H. Yajima, K. Maeda
Dilatonic black holes with Gauss-Bonnet Term, gr-qc/9606034

P. Kanti, N. E. Mavromatos, J. Rizos, K. Tamvakis, E. Winstanley
Dilatonic black holes in higher curvature string gravity, hep-th/9511071

inner solution

S. O. Alexeyev, M. V. Pomazanov
Internal solutions in higher order curvature gravity, gr-qc/9706066

M. V. Pomazanov
On the structure of some typical singularities for implicit ordinary differential equations, math-ph/0007008
Solution with a scalar singularity
(with turning point
by Alexeyev, Pomazanov)

\[ R(r_{ss}) = O\left(\sqrt{R(r_{cs})}\right) \]
**Extremal case**

We consider static spherical symmetric solutions parameterizing the metric as

\[
ds^2 = -W(r)\,dt^2 + \frac{dr^2}{W(r)} + \rho^2(r)\,d\Omega^2_2
\]

where

\[
W(r) = \sigma^2(r)w(r)
\]

There are two common cases of gauge:

1) \( \sigma = 1 \)
2) \( \rho' = 1 \)

For our purpose we choose first one.

The corresponding ansatz for the Maxwell one-form is

\[
A = -f(r)\,dt - q_m \cos \theta \,d\varphi,
\]

Like previously we consider two cases of system - with one charge and dyonic.
Equation of motion have strongly non-linear view, so we have only one way for investigation our model:

- analytic expansion on extremal horizon (amount of free parameters)
- getting global structure of solution by numeric integrating to infinity (we are looking for asymptotically good defined solutions)
- behaviour of solutions
**Analytic expansions**

For general expansion the electric and magnetic charges are related as:

$$ q_e = \frac{\sqrt{4\alpha + q_m^2 \rho_0^2}}{2(2\alpha + q_m^2)} $$

so it’s possible exit an electrically charged and dyonic solution, but not magnetically charged.

Near horizon expansion of solution:

\[
\begin{align*}
    w(r) &= x^2 - \frac{\zeta}{6\alpha^2} \left[ 3(a^2 - 1)q_m^4 + 6\alpha(3a^2 - 2)q_m^2 + 4\alpha^2(5a^2 - 3) \right] x^3 + O(x^4), \\
    \rho(r) &= 1 + \frac{\zeta}{4\alpha^2} \left[ (a^2 - 1)q_m^4 + 2\alpha(3a^2 - 2)q_m^2 + 4\alpha^2(a^2 - 1) \right] x + O(x^2), \\
    \alpha P(r) &= \alpha e^{\alpha \varphi} = \frac{\alpha}{2(2\alpha + q_m^2)} + \zeta x + O(x^2), 
\end{align*}
\]

where \( \zeta = \frac{P_1}{|P_1|} \) - define good behaviour for derivative of metric function \( \rho(r) \) on horizon.
The expansion on space infinity:

\[
\begin{align*}
    w(r) &= 1 - \frac{2M}{r} + \frac{\alpha Q_e^2 + \alpha^{-1} Q_m}{r^2} + O(r^{-3}), \\
    \rho(r) &= r - \frac{D^2}{2r} - \frac{D(2MD - a(\alpha Q_e^2 - \alpha Q_m^2))}{3r^2} + O(r^{-3}), \\
    \phi(r) &= \phi_\infty + \frac{D}{r} + \frac{2DM - \alpha a Q_e^2}{2r^2} + O(r^{-3}).
\end{align*}
\]

where \( Q_e = q_e e^{-a\phi_\infty}, Q_m = q_m e^{a\phi_\infty} \),

\( \phi_\infty \) – the asymptotic value of the dilaton function.
• Numerical analysis for solution with null magnetic charge

Critical values of dilaton coupling constant:

\[ |a_{cr} - 0.488219703| \]

Two type of solution on space infinity:

• asymptotically flat solution  \[ |a| < |a_{cr}| \]

• solution with turning point  \[ |a| > |a_{cr}| \]
Asymptotically flat solution \( |a| < |a_{cr}| \)

The functions \( w(x), \rho'(x), P(x) \), \( x = r - r_H \), for \( \rho_0 = 1 \) and some values of the dilaton coupling constant: \( a = 0.1 \ (P_1 = -0.01) \) — thin line, \( a = 0.4 \ (P_1 = -0.446) \) — normal line, \( a = 0.45 \ (P_1 = -1.18) \) — thick line.
The metric functions $\rho(x), w(\rho), (x = r - r_H)$ for $a > a_{cr}$. Numerical curves are presented for $a = 0.5$ and $\rho_0 = 1$, the corresponding value of $P_1$ being $P_1 = -7.746$. 
Physical Quantities

\[ k_M = \frac{M^2}{q_e}, \quad k_D = \frac{D^2}{q_e}, \quad k_\phi = \frac{e^{2a\phi_\infty}}{q_e} \]
\[ a \to 0 \quad \text{and} \quad a \to a_{cr} \quad \text{we find out, that} \]

\[
\left| \frac{aM}{D} \right| \to 1, \quad \frac{(1 + a^2)M}{Q_e} \to \alpha^{-1/2}
\]

\[
a^2 M^2 + D^2 = \frac{2a^2}{(1 + a^2)} Q_e^2.
\]
Decoupling of dilaton \((a=0)\)

We redefine free parameter of expansion on horizon like:

\[ P_1 = -\frac{a^2 \rho_0 \rho_1}{\alpha} \]

Also we have:

\[ \alpha = \frac{\rho_0^4}{4 q_e^2} \quad \text{and} \quad \alpha = \frac{\rho_0^2}{4 e^{a \phi_H}} \]

In the limit: \(a \to 0, \quad \alpha \to 0\)

expansion on horizon transform to:

\[
\begin{align*}
    w &= \frac{1}{\rho_0^2} (r - r_H)^2 - \frac{2 \rho_1}{\rho_0^3} (r - r_H)^3 + \frac{3 \rho_1^2}{\rho_0^4} (r - r_H)^4 + O(5) \\
    \rho &= \rho_0 + \rho_1 (r - r_H)
\end{align*}
\]

what is expansion for extremal (for some \(\rho_1\) ) RN solution with charge

\[ q_e = \rho_0 \sqrt{e^{2 a \phi_H}} \]
Decoupling of GB term (critical value of a)

When dilaton coupling is tending to critical value, we have growing mass off solution.

Increasing of mass of solution from other hand may be interpreted (from definition of mass), as decreasing of gravitational constant $G$ for the considered system.

But Gauss-Bonnet term doesn’t affected of this changing, so his role in action decreasing and finally neglected in limit of infinite value of mass.
Thermodynamics property of solution

Temperature:

\[ T = \frac{1}{2\pi} \left( \sqrt{g_{rr}} \frac{\partial \sqrt{g_{tt}}}{\partial r} \right) \bigg|_{r=r_H} = \frac{1}{2\pi \rho_0^2} (r - r_H) \bigg|_{r=r_H} = 0 \]

Entropy:

- classical formula (without accounting of GB term)

- Sen’s Entropy function (hep-th/0506177)

simplified Wald’s formula for AdSxS

\[ S = \pi \rho_0^2 \]

\[ S = 2\pi \rho_0^2 \]
• It is exist electrical charged extremal BH solution with a non-zero radius of horizon for D=4 a DGB gravity.

• Entropy for this solution is twice as defined by Berkenstein formula.

• Last result consists with results for a supergravity theory.
**Dyonic solution**

Intervals of critical dilaton coupling constant (from expansion on horizon) is divided by root of equation from denominators of horizon expansions: \[(a^2 - a_j)q_m^4 + \alpha(6a^2 - 4a_j)q_m^2 + 4\alpha^2(a^2(a_j + 2) - a_j) = 0,\]

\[
q_e = \frac{\sqrt{4\alpha + q_m^2\rho_0^2}}{2(2\alpha + q_m^2)}. \]

In the limit of big magnetic charge electrical charge tends to zero, also dilaton field vanishe and solution become RN solution (with finite radius of horizon).
Results of numeric integration

Regions of asymptotically flat solution (black colour) and solution with linear dilaton background behaviour (red colour) for small value magnetical charge and dilaton coupling constant. For LDB case an electric charge belong to background.
• The global parameters of solution (mass, dilaton charge, asymptotic values of dilaton) define from electrical charge like:

$$k_M(a, q_m) = \frac{M^2}{q_e}, \quad k_D = \frac{D^2}{q_e}, \quad k_\phi = \frac{e^{2a\phi_*}}{q_e},$$

where ratios depend only from dilaton constant and magnetic charge.

• Entropy:

$$S = 2\pi q_e \sqrt{q_m^2 + 4\alpha} = \pi \rho_0^2 + \frac{2\alpha \pi \rho_0^2}{2\alpha + q_m^2}$$

but result arises another question about a dependence entropy of surface of black hole.
Near critical points \( (a \rightarrow a_{cr}) \) we have the similar BPS condition for the global parameters of solution:

\[
a^2 M^2 + D^2 = \frac{2a^2}{(1+a^2)} (Q_e^2 + Q_m^2)
\]

Critical value of solution can has connection with transition between black holes and strings

Classical limits:

• dilaton coupling constant tends to zero $a \to 0$, in this case GB term decouple and solution become known dyonic RN solution,

• magnetic charge tends to infinity $q_m \to \infty \ (\alpha \to 0)$ and the role of GB is neglected, and we get RN-solution with infinity magnetic charge for every coupling constant of dilaton except $a^2 = \frac{n(n+1)}{2}$, in the last cases solution is dyonic dilatonic black hole.
Summary and recent results

1. Even first higher order curvature term introduce radically new properties of solution, what allow resolve a puzzle with zero radius and entropy of extremal black hole in the case of a BH solution with one charge and extend range of dilatonic coupling for solution in dyonic case.

2. The critical state of system which satisfy BPS equation on global parameter may have relation to transition between black holes and strings, and have to further invistigation.
• The model was investigated in the Einstein frame, also possible to consider a coupling of GB term in a string frame.

• Including GB term with a exponential of dilaton coupling breaks S-duality of system and entropy doesn’t just depend on surface area of horizon (dyonic case). Possible solution is considering more complicated case, in which coupling of GB term is based on a Dedikind function, which preserves this symetry.

*Gal’tsov, Davydov, 0812.5103*

• For investigated D=4 system AF solution doesn’t exit for a dilaton coupling a=1, what inconsist with known from a string theory. Our recent investigation shows that for D>4 a range of dilaton coupling is extended and also revives some interesting properties, which expect futher consideration.