Generalized Painlevé-Gullstrand metrics

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Paul Painlevé (1863–1933)  
mathematician and politician:  
Grand Prix des Sciences Mathématiques (1890)  
Prix Bordin (1894), Prix Poncelet (1896)  
Member, geometry section of the Académie des Sciences  
Prime minister of France  
(the 84th: 12 Sept. – 16 Nov. 1917;  
& the 92nd: 17 Apr. – 28 Nov. 1925)  

Allvar Gullstrand (1862–1930)  
Nobel Prize (1911)  
(in Physiology or Medicine)  
(for his work in in Physical & Physiological Optics)  
member, Nobel Physics Committee (1911-1929), and its Chairman (1923-1929).  

"Einstein must never receive a Nobel Prize, even if the whole world demands it"
Generalized Painlevé-Gullstrand (PG) metrics

General spherically symmetric metric

\[ ds^2 = -N(R, T)dT^2 + f^{-1}(R, T)dR^2 + R^2 d\Omega^2 \]

Note: Lorentzian signature \( \implies N(R, T)/f(R, T) = e^{\Phi(R, T)} > 0 \).

PG time-coordinate: \( dt_p \equiv e^{I(T, r)}(dT + \beta dr) \), \( g(r) \equiv R(r)/L \); 

\[ ds^2 = -Ne^{-2I}dt_p^2 + 2\beta Ne^{-I}dt_p dr + \left( L^2 f^{-1}g'^2 - \beta^2 N \right) dr^2 + L^2 g^2(r)d\Omega^2 \]

with \( g'(r) \equiv \frac{dg(r)}{dr} \)

For \( dt_p \) to remain a perfect differential, introduce integrating factor \( e^I \), with the requirement \( \partial_r e^I = \partial_T (e^I \beta) \); uniquely specifies \( e^I \) given its initial value \( e^I(r, T_0) \).

Choice of \( \beta = L(Nf)^{-\frac{1}{2}} \sqrt{g'^2 - f} \implies \)

Spatial metric on constant-\( t_P \) 3-dim. hypersurfaces is: \( L^2 \left( dr^2 + g^2(r)d\Omega^2 \right) \); 
And 4-dim. spacetime metric (Generalized PG metric):

\[
\begin{align*}
    ds_{GPG}^2 &= -N f^{-1}g'^2 e^{-2I}dt_p^2 + \left( g'^{-1} dR + dt_p N^{\frac{1}{2}} f^{-\frac{1}{2}} e^{-I} \sqrt{g'^2 - f} \right)^2 + R^2 d\Omega^2 \\
    &= -N f^{-1}g'^2 e^{-2I}dt_p^2 + \left( Ldr + dt_P N^{\frac{1}{2}} f^{-\frac{1}{2}} e^{-I} \sqrt{g'^2 - f} \right)^2 + L^2 g^2(r)d\Omega^2
\end{align*}
\]
\[ ds^2 = -N f^{-1} g'^2 e^{-2I} dt_p^2 + \left( g'^{-1} dR + dt_p N^\frac{1}{2} f^{-\frac{1}{2}} e^{-I} \sqrt{g'^2 - f} \right)^2 + R^2 d\Omega^2 \]

Remarks:

- **Note**: \( N(R, T)/f(R, T) = e^\Phi \implies \) metric can be regular even when \( f \to 0 \) at the "horizon(s)".

- Solutions of Einstein’s Eqs. with \( N(R) = f(R) \); \( e^I = 1 \)
  Conventional choice of \( g(r) : \frac{R}{L} \equiv g(r) = r; g'(r) = 1 \implies dt_p = dT + f^{-1} \sqrt{1 - f} dR \)
  \( \iff \) “usual” PG metric with (spatially flat) 3-dim. hypersurfaces:
  \[ ds^2_{PG} = -dt_p^2 + \left[ dR + dt_p \sqrt{1 - f} \right]^2 + R^2 d\Omega^2 \]

- Special case: Schwarzschild solution: \( f(R) = 1 - \frac{2GM}{R} \) (original PG metric).

- A particular parametrization is: \( f(R, T) = 1 - \frac{2GM(R, T)}{R} \) with Misner-Sharp mass.

- **FOLK THEOREM**: Always possible to choose spatially flat slicings i.e.
  \( g(r) \equiv R/L = r, g'(r) = 1. \) Works even when \( N(R, T) \neq f(R, T) \)
  i.e.
  A spherically symmetric metric can always be reduced to the spatially flat PG form.

  "Proof": look at the metric above.

- What’s the problem with the folk theorem?
Lorentz boost

\[ ds^2 = -N(R,T)dt^2 + f^{-1}(R,T)dR^2 + R^2d\Omega^2 \]
\[ = -N f^{-1} g'^2 e^{-2I} dt_P^2 + \left( g'^{-1} dR + dt_p N^{\frac{1}{2}} f^{-\frac{1}{2}} e^{-I} \sqrt{g'^2 - f} \right)^2 + R^2 d\Omega^2 \]
\[ = \eta_{AB} e^A e^B \]

Vierbein 1-forms are related by a local radial Lorentz boost

\[
\begin{pmatrix}
e^0 = g' e^{-I} dt_P \\
e^3 = g'^{-1} dR + \sqrt{g'^2 - f} e^{-I} dt_P
\end{pmatrix}
\left(GPG\right)
= \begin{pmatrix}
\cosh \xi & \sinh \xi \\
\sinh \xi & \cosh \xi
\end{pmatrix}
\begin{pmatrix}
e^0 = N^{1/2} dT \\
e^3 = f^{-1/2} dR
\end{pmatrix}_{\text{Standard}}

\[ \tanh \xi = g'^{-1} \sqrt{g'^2 - f} \quad \left(f \to 0 \quad 1\right) \]

\( \xi = \text{rapidity of the boost.} \)

**Criterion** for REAL PG variables and PHYSICAL Lorentz boosts is: \( g'^2 - f \geq 0 \)

Cannot be satisfied for generic spherically symmetric metrics if we also demand spatially flat slicings \( g' = 1 \). (\( \exists \text{obstruction} \) to spatial flatness).
Explicit examples of the "problem" with spatial flatness. Consider time-indpt. case.

\[ ds^2 = -N f^{-1} g'^2 dt_p^2 + \left( g'^{-1} dR + dt_p N^\frac{1}{2} f^{-\frac{1}{2}} \sqrt{g'^2 - f} \right)^2 + R^2 d\Omega^2 \]

\[ \frac{R}{L} \equiv g(r) = r, g' = 1; N(R) = f(R) \Rightarrow \]

\[ ds^2 = -dt_p^2 + \left[ dR + dt_p \sqrt{1 - f} \right]^2 + R^2 d\Omega^2 \]

- Schwarzscild-anti-deSitter \( N = f = 1 - \frac{2GM}{R} - \frac{\Lambda}{3} R^2, \quad \Lambda < 0. \)

\[ ds^2 = -dt_p^2 + \left[ dR + dt_p \sqrt{\frac{2GM}{R} + \frac{\Lambda R^2}{3}} \right]^2 + R^2 d\Omega^2 \]

"Unphysical" (complex) variables for \( R > R_c = 3\sqrt{\frac{6GM}{-\Lambda}} \)

- Reissner-Nordström \( N = f = 1 - \frac{2GM}{R} + \frac{Q^2}{R^2}, \)

\[ ds^2 = -dt_p^2 + \left( dR + dt_p \sqrt{\frac{2GM}{R} - \frac{Q^2}{R^2}} \right)^2 + R^2 d\Omega^2, \]

"Unphysical" (complex) variables for \( R < R_c = \frac{Q^2}{2GM} \)

- "Misner-Sharp mass function" becomes negative

- Note: No problem for original PG metric \( N = f = 1 - \frac{2GM}{R} \) for Schwarzschild soln.

- Resolution: Spatial flatness is too strong a demand! Give up spatial flatness. Criterion \( (g'^2 - f) \geq 0 \quad \forall R \) can be always satisfied by choosing appropriate \( g(r) \) to give trouble-free GENERALIZED PG metrics which are however NOT always spatially flat.
Generalizations beyond spatially flat slicings: (explicit examples)

- Schwarzschild-(anti-)deSitter metrics $N = f = 1 - \frac{2GM}{R} - \frac{\Lambda}{3}R^2$

$$\frac{R}{L} \equiv g(r) = \begin{cases} 
\sin r & (k = +1, \text{ elliptic}) \\
r & (k = 0, \text{ flat}) \\
\sinh r & (k = -1, \text{ hyperbolic}) 
\end{cases}$$

$$g'(r) = \sqrt{1 - kL^{-2}R^2}$$

$$ds^2 = -\left(1 - kL^{-2}R^2\right)dt^2_p + \left(1 - kL^{-2}R^2\right)^{-1/2}dR + dt_p\sqrt{\frac{2GM}{R} + \left(\frac{\Lambda}{3} - \frac{k}{L^2}\right)R^2} + R^2d\Omega^2$$

**Constant curvature slicings** with 3-dim. Ricci scalar $3R = 6k/L^2$

Criterion $(g'^2 - f) = \frac{2GM}{R} + \left(\frac{\Lambda}{3} - \frac{k}{L^2}\right)R^2 \geq 0$ can be guaranteed $\forall R$ iff $\frac{\Lambda}{3} \geq \frac{k}{L^2}$.

$\Rightarrow$ for $\Lambda \geq 0$ spatially flat $(k = 0)$ slicings can be attained, but for the anti-deSitter case, hyperbolic $(k = -1)$ 3-geometry is needed. Note that for $\Lambda > 0$, all spatial topologies $k = 0, \pm 1$ are allowed, but for $k = 1$ the range of $R$ is governed by $\frac{R^2}{L^2} = \sin r \leq 1$, yielding $R \leq L$ which can be as large as needed.

- RN metric with $f = 1 - \frac{2GM}{R} + \frac{Q^2}{R^2}$, can choose $g(r) = \sqrt{r^2 - L^{-2}O^2} = R/L$

$$ds^2 = -\left(1 + \frac{O^2}{R^2}\right)dt^2_p + \left(\frac{R}{\sqrt{R^2 + O^2}}\right)dR + dt_p\sqrt{\frac{2GM}{R} + \frac{O^2 - Q^2}{R^2}} + R^2d\Omega^2,$$

Criterion holds for $O^2 > Q^2$. Constant-$t_p$ 3-dim. hypersurfaces characterized by eigenvalues of 3-dim. Ricci tensor $3R_{ij}^i : \lambda_{i=1,2,3} = (0, 0, \frac{2O^2}{R^4})$. ($O$ parametrizes deviation from spatial flatness).
Some applications/remarks:

- Classical GR: Classical solutions and their Generalized PG forms.

- \{Spherical symmetric metrics\} ⊂ \{Axially symmetric metrics e.g. Kerr-Newmann-(anti)deSitter\} ... \implies\ PG form of axially symmetric metrics must go beyond spatially flat slicings as well.

- Canonical 3 + 1 formulation and canonical quantization: Spherically symmetric sector cannot be trivialized as spatially flat by choosing PG coordinates. Rather, spatial metric on constant-$t_P$ 3-dim. Cauchy hypersurfaces (even with choice of (Gen.)PG coordinates) is: $L^2 \left( dr^2 + g^2(r)d\Omega^2 \right)$.

- Elimination of spurious contributions in computation of Hawking radiation as tunneling via Parikh-Wilczek method.

Parikh-Wilczek treatment of Hawking radiation:
Hawking radiation treated as tunneling across the horizon from $R_i$ to $R_f$ of massless semiclassical s-wave emission with energy $\omega$, and the black hole with initial mass parameter $M$ shrinks by an amount $\omega$ maintaining energy conservation (assume simple back reaction: $M$ to $M - \omega$ and the form of the metric is preserved).

Use Gen. PG metric satisfying Einstein’s equations with $N(R) = f(R)$:
outgoing particles follow null geodesic: $\dot{R} \equiv \frac{dR}{dt_P} = g' \left( g' - \sqrt{g''^2 - f} \right)$.
Decay rate comes from the imaginary part of the particle action which is associated with

$$I = \int_{R_i}^{R_f} p_R dR = \int_{R_i}^{R_f} \left( \int_0^{p_R} dp_R \right) dR = \int_{R_i}^{R_f} \int_{H_0 - \omega}^{H_0} \frac{dH}{\dot{R}} dR.$$ 

In the last step Hamilton’s equation, $\frac{dH}{dp_R} \bigg|_R = \dot{R}$, for the semiclassical process is invoked.
Switching the order of integration, together with \( dH = -d\omega \), yields

\[
I = \int_0^\omega \int_{R_i}^{R_f} \frac{dR}{R} (-d\omega')
\]

\[
= \int_0^\omega \int_{R_i}^{R_f} \frac{g' + \sqrt{g'^2 - f}}{g'f} dR (-d\omega'),
\]

\( f \) in the integrand is evaluated at \( M - \omega' \); and the pole is located at the horizon through which the tunneling occurs i.e. at \( R_h \) with \( f(R_h)|_{M-\omega'} = 0 \).

Integral over \( R \) is defined by deforming the contour to go through infinitesimal semicircle \( R = R_h + \epsilon e^{i\theta} \) around the pole\(^a\)

Imaginary part is then

\[
\text{Im} \int_{R_i}^{R_f} dR \frac{g' + \sqrt{g'^2 - f}}{g'f} = \lim_{\epsilon \to 0} \int_{2\pi}^0 d\theta e^{i\theta} \frac{g' + \epsilon e^{i\theta} \partial_R g' + \sqrt{g'^2 - f} + \epsilon e^{i\theta} \partial_R (2g' - f)}{g'f + \epsilon e^{i\theta} (\partial_R g' + g' \partial_R f)} \bigg|_{R_h(\omega')} = -\frac{2\pi}{\partial_R f(R)} \bigg|_{R_h(\omega')},
\]

provided \( \sqrt{g'^2 - f} \) remains real. Otherwise, \( \exists \) spurious contributions whenever \( g'^2 - f \geq 0 \) is violated.

Final result

\[
\text{Im} I = \int_0^\omega d\omega' \frac{2\pi}{\partial_R f(R)} \bigg|_{R_h(\omega')}
\]

governing the decay rate is, remarkably, independent of \( g(r) \), and hence "universal".

\(^a\)A positive decay rate is associated with clockwise traversal of the semicircle in the contour.
Change of the Bekenstein-Hawking entropy from $\Delta S = -2\text{Im} I$ (note $I(\omega)$) yields, at the lowest order, the temperature from the first law $T_{\text{eff}}.\Delta S = -\omega$ agrees with the Hawking temperature $T_H \equiv \frac{\kappa}{2\pi} = \frac{1}{4\pi} \partial_R f|_{R_h}$; BUT exists deviations from pure thermal physics indicated by higher order corrections in $\text{Im} I$ of (1) are displayed below.

<table>
<thead>
<tr>
<th></th>
<th>Schwarzschild</th>
<th>Reissner-Nordström</th>
<th>Schwarzschild (anti-)deSitter</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$f = 1 - \frac{2GM}{R}$</td>
<td>$f = 1 - \frac{2GM}{R} + \frac{Q^2}{R^2}$</td>
<td>$f = 1 - \frac{2GM}{R} - \frac{\Lambda R^2}{3}$</td>
</tr>
<tr>
<td>$2 \frac{d\text{Im} I}{d\omega}$</td>
<td>$8\pi G (M - \omega)$</td>
<td>$2\pi \left( G(M-\omega) + \sqrt{G^2(M-\omega)^2 - Q^2} \right)^2 \frac{\sqrt{G^2(M-\omega)^2 - Q^2}}{\sqrt{G^2(M-\omega)^2 - Q^2}}$</td>
<td>$8\pi \Lambda^{-1/2} \sin \left( \frac{1}{3} \arcsin \left( 3G(M-\omega)\sqrt{\Lambda} \right) \right) - 1 + 2 \cos \left( \frac{2}{3} \arcsin \left( 3G(M-\omega)\sqrt{\Lambda} \right) \right)$</td>
</tr>
</tbody>
</table>

Table 1: In the table, values of $\frac{d\text{Im} I}{d\omega}$ for various spherically symmetric spacetimes are shown for tunneling through the outer horizon of RN metric, and through the black hole horizon for the others. For the SAdS case, the expression $\sqrt{\Lambda}$ should be understood as $i\sqrt{-\Lambda}$. Note also that $\Delta S = -2\text{Im} I$ as evaluated from (1) coincides with the computation from the area law $\Delta S = \frac{\Delta A}{4G} = \frac{\pi \Delta R^2}{G}$.  
