Overview of numerical relativity

- Setting ‘realistic’ or ‘physically motivated’ initial conditions
- Solving Einstein’s equations
- Solving gauge conditions
- Solving the constraint equations
- Locating BH (solving AH finder)
- BH excision
- Extracting GWs

Main loop

GR-HD
GR-MHD
GR-Rad(M)HD

Microphysics
- EOS
- weak processes

July 28 - August 3, 2011

APCTP International school on NR and GW July 28-August 3, 2011
Scope of this lecture

Solving the constraint equations

Setting ‘realistic’ or ‘physically motivated’ initial conditions

Locating BH (solving AH finder)

Extracting GWs

Solving Einstein’s equations

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Solving source filed equations

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Main loop

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Microphysics
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The main goal that we are aiming at:

“To Derive Einstein’s equations in BSSN formalism”
Notation and Convention

- The signature of the metric: $(−+++)$
- (We will use the abstract index notation)
  - e.g. Wald (1984); see also Penrose, R. and Rindler, W. Spinors and spacetime vol.1, Cambridge Univ. Press (1987)
- Geometrical unit $c=G=1$
- symmetric and anti-symmetric notations

\[
T_{(a_1...a_n)} = \frac{1}{n!} \sum_{\pi} T_{a_{\pi(1)}...a_{\pi(n)}}, \quad T_{[a_1...a_n]} = \frac{1}{n!} \sum_{\pi} \text{sgn}(\pi) T_{a_{\pi(1)}...a_{\pi(n)}}
\]

\[
T_{(ab)} = \frac{1}{2} (T_{ab} + T_{ba}), \quad T_{[ab]} = \frac{1}{2} (T_{ab} - T_{ba})
\]
Overview of numerical relativity

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Solving Einstein’s equations on computers

- Einstein’s equations in full covariant form are a set of coupled partial differential equations
  - The solution, metric $g_{ab}$, is not a dynamical object and represents the full geometry of the spacetime just as the metric of a two-sphere does
  - To reveal the dynamical nature of Einstein’s equations, we must break the 4D covariance and exploit the special nature of time
- One method is 3+1 decomposition in which spacetime manifold and its geometry (gravitational fields) are divided into a sequence of ‘instants’ of time
- Then, Einstein’s equations are posed as a Cauchy problem which can be solved numerically on computers
3+1 decomposition of spacetime manifold

- Let us start to introduce foliation or slicing in the spacetime manifold $M$

- **Foliation** $\{\Sigma\}$ of $M$ is a family of slices (spacelike hypersurfaces) which do not intersect each other and fill the whole of $M$

- In a globally hyperbolic spacetime, each $\Sigma$ is a Cauchy surface which is parameterized by a global time function, $t$, as $\Sigma_t$

- Foliation is characterized by the gradient one-form $\Omega^a = \nabla^a t$

\[
\Omega_a = \nabla_a t, \quad \nabla_{[a} \Omega_{b]} = 0
\]
The lapse function

- The norm of $\Omega_a$ is related to a function called "lapse function", $\alpha(x^a)$, as:

$$g^{ab}\Omega_a\Omega_b = g^{ab}\nabla_a t\nabla_b t = -\frac{1}{\alpha^2}$$

- As we shall see later, the lapse function characterize the proper time between the slices

- Also let us introduce the normalized one-form:

$$n_a = -\alpha \Omega_a, \quad g^{ab}n_an_b = -1$$

- the negative sign is introduced so that the direction of $n$ corresponds to the direction to which $t$ increases

- $n^a$ is the unit normal vector to $\Sigma$
The spatial metric of $\Sigma : \gamma_{ab}$

- The **spatial metric** $\gamma_{ab}$ induced by $g_{ab}$ onto $\Sigma$ is defined by

\[
\gamma_{ab} = g_{ab} + n_a n_b \\
\gamma^{ab} = g^{ac} g^{bd} \gamma_{cd} = g^{ab} + n^a n^b
\]

- Using this 'induced' metric, a tensor on $M$ is decomposed into two parts: components **tangent** and **normal to** $\Sigma$

- The **tangent-projection operator** is defined as

\[
\perp_b^a = \delta_b^a + n^a n_b
\]

- The **normal-projection operator** is $N_b^a = -n^b n_a = \delta_b^a - \perp_b^a$

- Then, projection of a tensor into $\Sigma$ is defined by

\[
\perp T^{a_1 \ldots a_r b_1 \ldots b_s} \equiv \perp_{c_1}^{a_1} \ldots \perp_{c_r}^{a_r} \perp_{d_1}^{b_1} \ldots \perp_{d_s}^{b_s} T^{c_1 \ldots c_r d_1 \ldots d_s}
\]

- It is easy to check

\[
\perp g_{ab} = \perp_c^c \perp_d^d g_{cd} = \gamma_{ab}
\]
Covariant derivative associated with $\gamma_{ab}$

- **Covariant derivative** acting on spatial tensors is defined by

$$D_e T^{a_1...a_r}_{b_1...b_s} \equiv \downarrow \nabla_e T^{a_1...a_r}_{b_1...b_s}$$

$$= \downarrow^f \downarrow^{a_1}_{c_1} \ldots \downarrow^{a_r}_{c_r} \downarrow^{b_1}_{d_1} \ldots \downarrow^{b_s}_{d_s} \nabla_f T^{c_1...c_r}_{d_1...d_s}$$

- The covariant derivative must satisfy the following conditions
  - It is a linear operator : (obviously holds from linearity of $\nabla$)
  - Torsion free : $D_a D_b f = D_b D_a f$, (easy to check by direct calculation)
  - Compatible with the metric : $D_c g_{ab} = 0$, (easy to check also)
  - Leibnitz’s rule holds :

$$D_a (v^c w_c) = \downarrow^b_a \nabla_b (v^c w_c) = \downarrow^b_a v^d \delta^c_d \nabla_b w_c + \downarrow^b_a w_d \delta^d_c \nabla_b v^c$$

$$= \downarrow^b_a v^d (\downarrow^c_d + N^c_d) \nabla_b w_c + \downarrow^b_a w_d (\downarrow^d_c + N^d_c) \nabla_b v^c$$

$$= v^c D_a w_c + w_d D_a v^d + \downarrow^b_a (N^c_d v^d \nabla_b w_c + N^d_c w_d \nabla_b v^c)$$

$$= v^c D_a w_c + w_d D_a v^d \quad \text{for} \quad N^c_a v^d = N^d_c w_d = 0$$
The Riemann tensor for the slice $\Sigma$ is defined by

$$(D_a D_b - D_b D_a) v_c = v_d R^d_{abc}$$

An other curvature tensor, the extrinsic curvature for $\Sigma$ is defined by

$$K_{ab} \equiv - \nabla_{(a} n_{b)} , \quad n^a : \text{unit normal to } \Sigma$$

extrinsic curvature provides information on how much the normal direction changes and hence, how $\Sigma$ is curved

the antisymmetric part vanish due to Frobenius's theorem : “For unit normal $n^a$ to a slice $n_{[a} \nabla_b n_{c]} = 0$ “

$$0 = \nabla_{[a} n_{b] n_{c]} = - \nabla_{[b} n_{c]}$$
The other expressions of $K_{ab}$

- First, note that

$$\nabla_a n_b = \delta^c_a \delta^d_b \nabla_c n_d = (\perp^c_a + \mathbf{N}^c_a)(\perp^d_b + \mathbf{N}^d_b) \nabla_c n_d = \perp \nabla_a n_b - n_a \perp a_b = -K_{ab} - n_a a_b$$

- where $a_b$ is the acceleration of $n^b$ which is purely spatial

$$n^b a_b = n^a n^b \nabla_a n_b = n^a \nabla_a (n_b n^b) = 0$$

- Because the extrinsic curvature is symmetric, we have

$$K_{ab} = -\nabla_{(a} n_{b)} - n_{(a} a_{b)} = -\frac{1}{2} \mathbf{L}_n (g_{ab} + n_a n_b) = -\frac{1}{2} \mathbf{L}_n \gamma_{ab}$$

- where $\mathbf{L}_n$ is the Lie derivative with respect to $n$

- Also, we simply have

$$K_{ab} = -\perp \nabla_{(a} n_{b)} = -\frac{1}{2} \perp \mathbf{L}_n g_{ab}$$

- Thus the extrinsic curvature is related to the “velocity” of the spatial metric $\gamma_{ab}$
3+1 decomposition of 4D Riemann tensor

- Geometry of a slice $\Sigma$ is described by $\gamma_{ab}$ and $K_{ab}$
  - $\gamma_{ab}$ and $K_{ab}$ represent the "instantaneous" gravitational fields in $\Sigma$
- In order that the foliation $\{\Sigma\}$ to "fits" the spacetime manifold, $\gamma_{ab}$ and $K_{ab}$ must satisfy certain conditions known as Gauss, Codazzi, and Ricci relations
  - They are related to 3+1 decomposition of Einstein's equations
- These equations are obtained by taking the projections of the 4D Riemann tensor

$$\perp 4R_{abcd}, \perp 4R_{abcd}n^d, \perp 4R_{abcd}n^b n^d$$
Gauss relation: spatial projection to \( \Sigma \)

- Let us calculate the spatial Riemann tensor

\[
D_a D_b w_c = \nabla_a (\nabla_b w_c)
\]

\[
= \nabla_a \nabla_b w_c + \nabla (\nabla_b w_d)(\nabla_a n^d) + \nabla (\nabla_d w_c)(\nabla a n^d)
\]

\[
= \nabla_a \nabla_b w_c - \nabla K_{ac} K_b^d w^d - \nabla K_{ab} n^d \nabla_d w_c
\]

- where we used (also note that \( \nabla n \) vanishes if \( n \) is uncontracted)

\[
\nabla_a n^d = n_a (\nabla_b n^d) + n^d (\nabla_a n_b) = -n_b (K^d_{a} + n_a a^d) - n^d (K_{ab} + n_a a_b)
\]

- Then we obtain the Gauss relation

\[
(D_a D_b - D_b D_a)w_c = R^{\quad d}_{abcd} w_d = \nabla^4 R^{\quad d}_{abcd} w_d - K_{ac} K^d_b w_d + K_{bc} K^d_a w_d
\]

\[
\nabla^4 R_{abcd} = R_{abcd} + K_{ac} K_{bd} - K_{ad} K_{bc}
\]

- The contracted Gauss relations are

\[
\nabla^4 R_{ac} + \nabla^4 R_{abcd} n^b n^d = R_{ac} + KK_{ac} - K_{ab} K_{c}^b
\]

\[
R + 2^4 R_{ab} n^a n^b = R + K^2 - K_{ab} K^{ab}
\]
Codazzi relation: *mixed projection to* $\Sigma$ *and* $n$

- Next, let us consider the “mixed” projection

$$\downarrow^4 R_{abc}^d n_d = \downarrow (\nabla_a \nabla_b n_c - \nabla_b \nabla_a n_c)$$

where the right hand side is calculated as

$$\downarrow \nabla_a \nabla_b n_c = \downarrow \nabla_a (-K_{bc} - n_b a_c) = -D_a K_{bc} - \downarrow a_c \nabla_a n_b = -D_a K_{bc} + a_c K_{ab}$$

- Then we obtain the Codazzi relation

$$\downarrow^4 R_{abc}^d n_d = D_b K_{ac} - D_a K_{bc}$$

- The contracted Codazzi relation is

$$\downarrow^4 R_{ab} n^b = D_a K - D_b K^b_a$$
Gauss and Codazzi relations

- Note that the *Gauss and Codazzi relations depend only on the spatial metric* $\gamma_{ab}$, the extrinsic curvature $K_{ab}$, and their spatial derivatives.

- This implies that the **Gauss-Codazzi relations represent integrability conditions** that $\gamma_{ab}$ and $K_{ab}$ must satisfy for any slice to be embedded in the spacetime manifold.

- The Gauss-Codazzi relations are directly associated with the constraint equations of Einstein’s equation.
Ricci relation (1)

- Let us start from the following equation

\[ \downarrow^4 R_{abcd} n^b n^d = \downarrow n^b (\nabla_a \nabla_b n_c - \nabla_b \nabla_a n_c) \]

\[ = \downarrow n^b [-\nabla_a (K_{bc} + n_b a_c) + \nabla_b (K_{ac} + n_a a_c)] \]

\[ = \downarrow [K_{bc} \nabla_a n^b + \nabla_a a_c + n^b \nabla_b K_{ac} + a_c a_a + n_a n^b \nabla_b a_c] \]

\[ = \downarrow [K_{bc} (-K_a - n_a a^b) + \nabla_a a_c + n^b \nabla_b K_{ac} + a_c a_a + n_a n^b \nabla_b a_c] \]

\[ = - K_{bc} K_a^b + D_a a_c + \downarrow n^b \nabla_b K_{ac} + a_c a_a \]

- The Lie derivative of \( K_{ac} \) is

\[ \downarrow L_n K_{ac} = \downarrow (n^b \nabla_b K_{ac} + K_{ab} \nabla_c n^b + K_{cb} \nabla_a n^b) = \downarrow n^b \nabla_b K_{ac} - K_{ab} K_c^b - K_{cb} K_a^b \]

- Then we obtain the Ricci relation

\[ \downarrow^4 R_{abcd} n^b n^d = \downarrow L_n K_{ac} + K_{ab} K_c^b + D_a a_c + a_c a_a \]
Ricci relation (2)

- Note that the Lie derivative of $K_{ab}$ is purely spatial, as
  \[ n^a L_n K_{ab} = n^a n^c \nabla_c K_{ab} + n^a K_{ac} \nabla_b n^c + n^a K_{bc} \nabla_a n^c = -K_{ab} a^a + K_{bc} a^c = 0 \]

- Thus the Ricci relation is
  \[ 4 R_{abcd} n^b n^d = L_n K_{ac} + K_{ab} K^b_c + D_a a_c + a_c a_a \]
Ricci relation (3)

- The acceleration $a^b$ is related to the lapse function $\alpha$, as

\[
a^b = n^c \nabla_c n^b = 2 n^c \nabla_{[c} n^b] = -2 n^c \nabla_{[c} \alpha \Omega_{b]} = -n^c (\Omega_{b} \nabla_c \alpha - \Omega_c \nabla_b \alpha) = n^c n_b \nabla_c \ln \alpha + \delta^c_b \nabla_c \ln \alpha = D_b \ln \alpha
\]

where we have used the fact that $\Omega$ is closed one-form.

- Then the Ricci relation can be written as

\[
\mp^4 R_{abcd} n^b n^d = L_n K_{ac} + K_{ab} K^b_c + D_a D_c \ln \alpha + D_a \ln \alpha D_a \ln \alpha = L_n K_{ac} + K_{ab} K^b_c + \frac{1}{\alpha} D_a D_c \alpha
\]

- Furthermore, using the contracted Gauss relation

\[
\mp^4 R_{ac} + \mp^4 R_{abcd} n^b n^d = R_{ac} + KK_{ac} - K_{ab} K^b_c
\]

we obtain

\[
\mp^4 R_{ac} = -L_n K_{ac} - \frac{1}{\alpha} D_a D_c \alpha + R_{ac} + KK_{ac} - 2K_{ab} K^b_c
\]
“Evolution vector” and $\alpha n^a$

- What is the natural “evolution” vector?
  - As stated before, the foliation is characterized by the closed one-form $\Omega$
  - **Dual vectors** $t^a$ to $\Omega$ will be the evolution vector: $\Omega_a t^a = 1$
  - One simple candidate is $t^a = \alpha n^a$
  - Note that $n^a$ is not the natural evolution vector because

\[
L_{n_a} \perp_b = n^c \nabla_c \perp_b - \perp_c \nabla_c n^a + \perp^a \nabla_b n^c = n^c \nabla_c (n^a n_b) + K_b^a - (K_b^a + n_b a^a)) = n^a a_b \neq 0
\]

- This means that the Lie derivative with respect to $n^a$ of a tensor tangent to $\Sigma$ is NOT a tensor tangent to $\Sigma$
- On the other hand, $L_{\alpha n} \perp_b = 0$ and any tensor field tangent to $\Sigma$ is Lie transported by $\alpha n^a$ to a tensor field tangent to $\Sigma$
The shift vector

- We have a degree of freedom to add any spatial vector, called "shift vector", \( \beta^a \) to \( \alpha n^a \) because \( \Omega_a \beta^a = 0 \)
- Therefore the general evolution vector is: \( t^a = \alpha n^a + \beta^a \)
- This freedom in the definition of the evolution time vector stems from the general covariance of Einstein's equations

- It is convenient to rewrite the Ricci relation in terms of the Lie derivative of the evolution time vector, as

\[
\perp 4 \mathbf{R}_{ac} = - \frac{1}{\alpha} (L_t - L_\beta) K_{ac} - \frac{1}{\alpha} D_a D_c \alpha + R_{ac} + \mathbf{K} \mathbf{K}_{ac} - 2 K_{ab} K^b_c
\]

- where we have used

\[
L_t K_{ab} = L_{\alpha n} K_{ab} + L_\beta K_{ab} = \alpha n^c \nabla_c K_{ab} + K_{ac} \nabla_b (\alpha n^c) + K_{bc} \nabla_a (\alpha n^c) + L_\beta K_{ab} = \alpha L_n K_{ab} + L_\beta K_{ab}
\]
3+1 decomposition of Einstein’s equations (1)

- Decomposition of $T_{ab}$

Now we proceed 3+1 decomposition of Einstein’s equations using the Gauss, Codazzi, and Ricci relations.

To do it, let us decompose the stress-energy tensor as

$$T_{ab} = E n_a n_b + 2 P_{(a} n_{b)} + S_{ab}$$

where $E \equiv n_a n_b T_{ab}$, $P_a \equiv -\perp (n^b T_{ab})$, and $S_{ab} \equiv \perp T_{ab}$ are the energy density, momentum density/momentum flux, and stress tensor of the source field measured by the Eulerian observer.

The trace is $T = S - E$

We shall also use Einstein’s equations in the form of

$$4 R_{ab} = 8\pi \left( T_{ab} - \frac{1}{2} g_{ab} T \right)$$
We first project Einstein’s equation into the direction perpendicular to $\Sigma$ to obtain

$$4R_{ab}n^a n^b + \frac{1}{2} 4R = 8\pi E$$

For the left-hand-side, we use the contracted Gauss relation

$$4R + 2 4R_{ab}n^a n^b = R + K^2 - K_{ab}K^{ab}$$

We finally obtain the Hamiltonian constraint

$$R + K^2 - K_{ab}K^{ab} = 16\pi E$$

This is a single elliptic equation which must be satisfied everywhere on the slice
3+1 decomposition of Einstein’s equations (3)

- **Momentum constraint**

- Similary, “**mixed” projection** of Einstein’s equations gives
  \[ \downarrow^4 R_{ab} n^b = -8\pi P_a \]

- Using the contracted Codazzi relation
  \[ \downarrow^4 R_{ab} n^b = D_a K - D_b K^b \]

- We reach the **momentum constraint**
  \[ D_b K^b_a - D_a K = 8\pi P_a \]

- includes 3 elliptic equations
3+1 decomposition of Einstein’s equations (4)

- Evolution equations

The evolution part of Einstein’s equations is given by the full projection onto $\Sigma$ of Einstein’s equations:

$$
\perp \nabla^4 R_{ab} = \perp 8\pi \left( S_{ab} + 2n_a P_b - \frac{1}{2} \gamma_{ab} (S - E) + \frac{1}{2} n_a n_b (S + E) \right) = 8\pi \left( S_{ab} - \frac{1}{2} \gamma_{ab} (S - E) \right)
$$

- Using a version of the Ricci relation

$$
\perp \nabla^4 R_{ac} = -\frac{1}{\alpha} \left( \mathcal{L}_\alpha - \mathcal{L}_\beta \right) K_{ac} - \frac{1}{\alpha} D_a D_c \alpha + R_{ac} + KK_{ac} - 2K_{ab}K^b_c
$$

We obtain the evolution equation for $K_{ab}$

$$
(\mathcal{L}_\alpha - \mathcal{L}_\beta) K_{ab} = -D_a D_b \alpha + \alpha [R_{ab} + KK_{ab} - 2K_{ac}K^c_b] - 8\pi \alpha \left( S_{ab} - \frac{1}{2} \gamma_{ab} (S - E) \right)
$$

- The evolution equation for $\gamma_{ab}$ is given by an expression of $K_{ab}$

$$
(\mathcal{L}_\alpha - \mathcal{L}_\beta) \gamma_{ab} = -2\alpha K_{ab}
$$
Summary of 3+1 decomposition

- Einstein's equations are 3+1 decomposed as follows

\[ G_{ab} n^a n^b = 8\pi T_{ab} n^a n^b \] \hspace{1cm} Gauss rel.

\[ \nabla_b \nabla_n n^b = 8\pi \nabla_b T_{ab} n^b \] \hspace{1cm} Codazzi rel.

\[ \nabla_b \nabla_n = 8\pi \nabla_b T_{ab} \] \hspace{1cm} Ricci rel.

\[ K_{ab} \equiv -\nabla_{(a} n_{b)} \] \hspace{1cm} Definition of \( K_{ab} \)

**Hamiltonian constraint**

\[ R + K^2 - K_{ab} K^{ab} = 16\pi E \]

**Momentum constraint**

\[ D_b K^b_a - D_a K = 8\pi P_a \]

**Evolution Eq. of \( K_{ab} \)**

\[ (L_t - L_\beta) K_{ab} = -D_a D_b \alpha + \alpha [R_{ab} + K K_{ab} - 2K_{ac} K^{c}_b] - 4\pi\alpha (2S_{ab} - \gamma_{ab} (S - E)) \]

**Evolution Eq. of \( \gamma_{ab} \)**

\[ (L_t - L_\beta) \gamma_{ab} = -2\alpha K_{ab} \]

**3+1 decomposition of the stress-energy tensor**

\[ T_{ab} = E n_a n_b + 2P_{(a} n_{b)} + S_{ab} \]

\[ E \equiv n_a n_b T_{ab}, \quad P_a \equiv -\nabla (n^b T_{ab}), \quad S_{ab} \equiv \nabla T_{ab} \]
Similarity with the Maxwell’s equations

\[ \nabla_b F^{ab} = 4\pi J^a = 4\pi (\rho^e n_a + J^a) \]
\[ \nabla [a F_{bc}] = 0 \iff \nabla_a \star F^{ab} = 0 \]

- **Normal projection**
  \[ n_a \nabla_b F^{ab} = n_a (4\pi J^a) \]
  \[ n_a \nabla_b \star F^{ab} = 0 \]

- **Spatial projection**
  \[ \perp \nabla_b F^{ab} = \perp (4\pi J^a) \]
  \[ \perp \nabla_b \star F^{ab} = 0 \]

\[ F^{ab} = n^a E^b - n^b E^a + \varepsilon^{abc} B_c \]
\[ \star F^{ab} = n^a B^b - n^b B^a - \varepsilon^{abc} E_c \]
\[ \varepsilon^a n_a = 0, \quad B^a n_a, \quad J^a n_a = 0 \]

**Gauss’s law, No monopole**
no time derivatives of \( E, B \)

**Constraint equations**

\[ D_a E^a = 4\pi \rho^e \]
\[ D_a B^a = 0 \]

**Faraday’s law, Ampere’s law**

\[ \left( \partial_t - \nabla_\beta \right) E^a = \varepsilon^{abc} D_b (\alpha B_c) - 4\pi \alpha J^a + \alpha KE^a \]
\[ \left( \partial_t - \nabla_\beta \right) B^a = -\varepsilon^{abc} D_b (\alpha E_c) + \alpha KB^a \]

**Evolution equations**
It can be shown that the "evolution" equations for the Hamiltonian ($C_H$) and Momentum ($C_M$) constraints becomes

$$
\left( \partial_t - L_\beta \right) C_H = -D_k (\alpha C_M^k) - C_M^k D_k \alpha + \alpha K (2C_H - F) + \alpha K^{ij} F_{ij}
$$

$$
\left( \partial_t - L_\beta \right) C_M^i = -D_j (\alpha F^{ij}) + 2\alpha K_j^i C_M^j + \alpha K C_M^i + \alpha D^k (F - C_H) + (F - 2H) D^i \alpha
$$

Where $F_{ij}$ is the spatial projection: the evolution equation

$$
F_{ab} \equiv \perp \left[ 4 R_{ab} - 8\pi \left( T_{ab} - \frac{1}{2} T g_{ab} \right) \right]
$$

The evolution equations for the constraints show that the constraints are "preserved" or "satisfied", if

- They are satisfied initially ($C_H = C_M = 0$)
- The evolution equation is solved correctly ($F_{ab} = 0$)
Coordinate-basis vectors

- Let us choose the *coordinate basis vectors*
- First, we choose the evolution timelike vector $t^a$ as the time-basis vector: $t^a = (e_0)^a$
- The spatial basis vectors are chosen such that
  - The spatial basis vectors are Lie transported along $t^a$:
    \[
    L_t(e_i)^a = t^b \nabla_b (e_i)^a - (e_i)^b \nabla_b t^a = [t, e_i]^a = 0
    \]
  - $(e_i)^a$ remains purely spatial because
    \[
    L_t(\Omega_a(e_i)^a) = (L_t\Omega_a)(e_i)^a - \Omega_a L_t(e_i)^a = (L_t\Omega_a)(e_i)^a
    \]
    \[
    = (t^b \nabla_b \Omega_a + \Omega_b \nabla_a t^b)(e_i)^a = 2t^b \nabla_{[b} \Omega_a] (e_i)^a = 0
    \]
- $(e_\mu)^a$ constitute the commutable coordinate basis
- Then $L_t = \partial_t$
- We define the dual basis vectors by
  \[
  (\xi^{\mu})_a : (\xi^{\mu})_a (e_\mu)^a
  \]
Components of geometrical quantities (1)

- Now we have set the coordinate basis we proceed to calculate the components of geometrical quantities.
- Because the evolution time vector is the time-coordinate basis we have \( t^a = t^\mu (e_\mu)^a = (e_0)^a \Rightarrow t^\mu = [1000] \).
- From the property of the spatial basis, we have \( n_i = 0, \therefore 0 = \Omega_a (e_i)^a = \Omega_\mu \delta_i^\mu = \alpha n_i \).
- Then, 0th contravariant components of spatial tensors vanish.
- From the definition of the time vector and normalization condition of \( n^a \), we obtain \( t^a = \alpha n^a + \beta^a \Rightarrow n^\mu = [\alpha^{-1} - \alpha^{-1} \beta^i] \).
- \( n^a n_a = -1 \Rightarrow n_\mu = [-\alpha 0 0 0] \).
From the definition of spatial metric, we have

\[ g^{ab} = \gamma^{ab} - n^a n^b \Rightarrow g^{\mu\nu} = \begin{bmatrix} -\alpha^{-2} & \alpha^{-2} \beta^i \\ \alpha^{-2} \beta^i & \gamma^{ij} - \alpha^{-2} \beta^i \beta^j \end{bmatrix} \]

\[ g_{ab} = \gamma_{ab} - n_a n_b \Rightarrow g_{ij} = \gamma_{ij} \]

We here note that from the spatial component of the following equation, we have

\[ \gamma^{\mu\sigma} g_{\sigma\nu} = (g^{\mu\sigma} + n^\mu n^\sigma) g_{\sigma\nu} = \delta^\mu_\nu + n^\mu \delta^0_\nu \Rightarrow \gamma^{ik} \gamma_{kj} = \delta^i_j \]

This means that the indices of spatial tensors can be lowered and raised by the spatial metric.

Then, from the inverse of \( g^{ab} \), we obtain

\[ g_{\mu\nu} = \begin{bmatrix} -\alpha^2 + \beta_k \beta^k & \beta_i \\ \beta_i & \gamma_{ij} \end{bmatrix}, \quad ds^2 = -\alpha^2 \, dt^2 + \gamma_{ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt) \]
An intuitive interpretation

The lapse function measures proper time between two adjacent slices.

The shift vector gives relation of the spatial origin between slices.

\[ ds^2 = -\alpha^2 dt^2 + \gamma_{ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt) \]
The importance of the **conformal decomposition** in the time evolution problem was first noted by York \((PRL 26, 1656 (1971); PRL 28, 1082 (1972))\).

He showed that the two degrees of freedom of the gravitational field are carried by the conformal equivalence classes of 3-metric, which are related each other by the conformal transformation:

\[
\gamma_{ij} = \psi^4 \widetilde{\gamma}_{ij}
\]

In the initial data problems, the conformal decomposition is a powerful tool to solve the constraint equations, as studied by York and O’Murchadha \((J. Math. Phys. 14, 456 (1973); PRD 10, 428 (1974))\) (see for reviews, e.g., Cook, G.B., Living Rev. Rel. 3, 5 (2000); Pfeiffer, H. P. gr-qc/0412002).

In the following, we shall derive conformal decomposition of Einstein’s equations.
“Conformal” decomposition of Ricci tensor (1)

- The covariant derivative associated with the conformal metric is characterized by
  \[ \tilde{D}_c \tilde{\gamma}_{ab} = 0 \]

- The two covariant derivatives are related by (e.g. Wald)
  \[ D_k T^i_j = \tilde{D}^i_j + C^i_{kl} T^l_j - C^l_{kj} T^i_j \]

  where \( C^i_{jk} \) is a tensor defined by difference of Christoffel symbols
  \[ C^k_{ij} = \Gamma^k_{ij} - \tilde{\Gamma}^k_{ij} = \frac{1}{2} \gamma^{kl} (\tilde{D}_i \gamma_{lj} + \tilde{D}_l \gamma_{ij} - \tilde{D}_l \gamma_{ij}) \]
  \[ = 2(\delta^k_i D_j \ln \psi + \delta^k_j D_i \ln \psi - \tilde{\gamma}_{ij} \tilde{D}^k \ln \psi) \]

- By a straightforward calculation, we can show (e.g. Wald)
  \[ R_{ij} v^j = (D_j D_i - D_i D_j) v^j = (\tilde{D}_j \tilde{D}_i - \tilde{D}_i \tilde{D}_j) v^j + (\tilde{D}_k C^k_{ij} - \tilde{D}_l C^k_{kj} + C^l_{lk} C^k_{ij} - C^k_{il} C^l_{kj}) v^j \]
  \[ = \tilde{R}_{ij} v^j + (\tilde{D}_k C^k_{ij} - \tilde{D}_l C^k_{kj} + C^l_{lk} C^k_{ij} - C^k_{il} C^l_{kj}) v^j \]
“Conformal” decomposition of Ricci tensor (2)

- Thus the Ricci tensor is decomposed into two parts, one which is the Ricci tensor associated with the conformal metric and one which contains the conformal factor \( \psi \).

- More explicitly one can show (see e.g. Wald (1984))

\[
R_{ij} = \tilde{R}_{ij} - 2\tilde{D}_i \tilde{D}_j \ln \psi - 2\tilde{\gamma}_{ij} \tilde{D}_k \tilde{D}^k \ln \psi
+ 4(\tilde{D}_i \ln \psi)(\tilde{D}_j \ln \psi) - 4\tilde{\gamma}_{ij} (\tilde{D}_k \ln \psi)(\tilde{D}^k \ln \psi)
\]

\[
\equiv \tilde{R}_{ij} + R^\phi_{ij}
\]

- Then, the Ricci scalar is decomposed as

\[
R = \psi^{-4} [\tilde{R} - 8(\tilde{D}_k \tilde{D}^k \ln \psi + (\tilde{D}_k \ln \psi)(\tilde{D}^k \ln \psi))]
= \psi^{-4} \tilde{R} - 8\psi^{-5} \tilde{D}_k \tilde{D}^k \psi
\]
The first step is to decompose $K_{ij}$ into trace ($K$) and traceless ($A_{ij}$) parts as

$$K_{ij} = A_{ij} + \frac{1}{3} \gamma_{ij} K, \quad K^{ij} = A^{ij} + \frac{1}{3} \gamma^{ij} K$$

Then, we perform the conformal decomposition of the traceless part as

$$A_{ij} = \psi^4 \tilde{A}_{ij}, \quad A^{ij} = \gamma^{ik} \gamma^{jl} A_{kl} = \psi^{-4} \tilde{A}_{ij}$$

Under these conformal decompositions of the spatial metric and the extrinsic curvature, let us consider the conformal decomposition of Einstein's equation
“Conformal” decomposition of the evolution equations (0) – an additional constraint

- In the following, with BSSN reformulation in mind, we set the determinant of the conformal metric to be unity:
  \[ \tilde{\gamma} = \text{det} \tilde{\gamma}_{ij} = 1 \]

- with this setting, the conformal factor becomes
  \[ \ln \psi = \frac{1}{12} \ln \gamma \]

- In the BSSN formulation, the conformal factor is defined by \( \phi = \ln \psi \) so that \( \phi = \ln \gamma / 12 \)

- In the case that we do not impose the above condition to the background conformal metric, the equations derived in the following are modified slightly (for this, see Gourgoulhon, E., gr-qc/0703035)
“Conformal” decomposition of the evolution equations (1): the conformal factor

- Let us start from the evolution equation of the spatial metric $\gamma_{ij}$:
  \[(L_t - L_\beta)\gamma_{ab} = L_{an}\gamma_{ab} = -2\alpha K_{ab}\]

- Taking the trace of this equation, we have
  \[\gamma^{ab}L_{an}\gamma_{ab} = -2\alpha K\]

- Now we use an identity for any matrix $A$:
  \[\det[\exp A] = \exp[\tr A]\]

- By setting $\gamma_{ij} = \exp A$ and taking the Lie derivative, we obtain
  \[L_{an}\gamma = \exp[\tr (\ln \gamma_{ij})]L_{an}(\tr (\ln \gamma_{ij})) = \gamma^{ij}L_{an}\gamma_{ij}\]

- Now we can derive the evolution equation for the conformal factor:
  \[\gamma^{ij}L_{an}\gamma_{ij} = L_{an}\ln \gamma = 12L_{an}\ln \psi = -2\alpha K\]
  \[(L_t - L_\beta)\ln \psi = -\frac{1}{6}\alpha K\]
“Conformal” decomposition of the evolution equations (2): the conformal metric

- Again, we start from the evolution equation for $\gamma_{ij}$:
  \[
  (L_t - L_\beta) \gamma_{ab} = L_\alpha \gamma_{ab} = -2\alpha K_{ab}
  \]

- Substituting the decomposition of $\gamma_{ij}$ and $K_{ij}$, we obtain
  \[
  \psi^4 (L_t - L_\beta) \tilde{\gamma}_{ij} + 4\psi^3 \tilde{\gamma}_{ij} (L_t - L_\beta) \psi = -2\alpha \left( \psi^4 \tilde{A}_{ij} + \frac{1}{3} \psi^4 \tilde{\gamma}_{ij} K \right)
  \]
  \[
  (L_t - L_\beta) \tilde{\gamma}_{ij} + 4\tilde{\gamma}_{ij} (L_t - L_\beta) \ln \psi = -2\alpha \left( \tilde{A}_{ij} + \frac{1}{3} \tilde{\gamma}_{ij} K \right)
  \]

- Now, we shall use the evolution equation for the conformal factor, and finally, we get
  \[
  (L_t - L_\beta) \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij}
  \]
“Conformal” decomposition of the evolution equations (3): the inverse conformal metric

- For the later purpose, let us derive the evolution equation for the inverse of the conformal metric.
- It is easily obtained from the evolution equation for the conformal metric, as

\[
\begin{align*}
\tilde{\gamma}^{ik} \tilde{\gamma}^{jl} (L_t - L_\beta) \tilde{\gamma}_{kl} &= -2\alpha \tilde{A}^{ij} \\
\tilde{\gamma}^{ik} \left[ (L_t - L_\beta) \delta^j_k - \tilde{\gamma}_{kl} (L_t - L_\beta) \tilde{\gamma}^{jl} \right] &= -2\alpha \tilde{A}^{ij} \\
(L_t - L_\beta) \tilde{\gamma}^{ij} &= 2\alpha \tilde{A}^{ij}
\end{align*}
\]
“Conformal” decomposition of the evolution equations (4a): the trace of the extrinsic curvature

- We start from the evolution equation for $K_{ij}$:
  \[
  
  \mathcal{L}_\alpha K_{ij} = -D_i D_j \alpha + \alpha [R_{ij} + K K_{ij} - 2 K_{ik} K_{kj}] + 4\pi\alpha (\gamma_{ij} (S - E) - 2S_{ij})
  \]

- We first simply take the trace of this equation:
  \[
  \mathcal{L}_\alpha K - K_{ij} \mathcal{L}_\alpha \gamma^{ij} = -D_i D^i \alpha + \alpha [R + K^2 - 2 K_{ij} K^{ij}] + 4\pi\alpha (3(S - E) - 2S)
  \]

- Here, let make use of the evolution equation for the inverse of the spatial metric,
  \[(L_t - L_{\beta}) \gamma^{ab} = 2\alpha K^{ab}\]
  then we obtain
  \[
  \mathcal{L}_\alpha K = -D_i D^i \alpha + \alpha [R + K^2] + 4\pi\alpha (S - 3E)
  \]

- Finally, using the Hamiltonian constraint, we obtain
  \[
  (L_t - L_{\beta}) K = -D_i D^i \alpha + \alpha [K_{ij} K^{ij} + 4\pi(E + S)]
  \]
  \[R + K^2 - K_{ab} K^{ab} = 16\pi E\]
“Conformal” decomposition of the evolution equations (4b): the trace of the extrinsic curvature

- For convenience, let us express the right-hand-side in terms of the conformal quantities, as well as give a suggestion how to evaluate the derivative term:

\[
D_k D^k \alpha = \frac{1}{\sqrt{\gamma}} \partial_k \left( \sqrt{\gamma} D^k \alpha \right) = \psi^{-6} \partial_k \left( \psi^6 \gamma^{jk} D_j \alpha \right) = \psi^{-6} \partial_k \left( \psi^2 \tilde{\gamma}^{jk} \partial_j \alpha \right)
\]

\[
K_{ij} K^{ij} = A_{ij} A^{ij} + \frac{1}{3} K^2 = \tilde{A}_{ij} \tilde{A}^{ij} + \frac{1}{3} K^2
\]
“Conformal” decomposition of the evolution equations (5a) : the traceless part of the extrinsic curvature

- We start from the Lie derivative of $K_{ij}$ :
  \[
  L_{an} K_{ij} = L_{an} A_{ij} + \frac{1}{3} \gamma_{ij} L_{an} K + \frac{1}{3} KL_{an} \gamma_{ij}
  \]

- Substituting the following equations into this yields
  \[
  L_{an} K_{ij} = -D_i D_j \alpha + \alpha [R_{ij} + KK_{ij} - 2K_{ik} K^k_j] + 4\pi\alpha (\gamma_{ij} (S - E) - 2S_{ij})
  \]
  \[
  L_{an} K = -D_i D^i \alpha + \alpha [R + K^2] + 4\pi\alpha (S - 3E)
  \]
  \[
  L_{an} \gamma_{ij} = -2\alpha K_{ij}
  \]

- Substituting the following equations into this yields
  \[
  L_{an} A_{ij} = -(D_i D_j \alpha)^{TF} + \alpha (R_{ij}^{TF} - 8\pi S_{ij}^{TF}) + \alpha \left[ \frac{5}{3} KK_{ij} - 2K_{ik} K^k_j - \frac{1}{3} K^2 \gamma_{ij} \right]
  \]

- where TF denotes the trace free part : $T_{ij}^{TF} = T_{ij} - (1/3)\gamma_{ij}(\text{tr}T)$

- The terms that involve $K$ in the right-hand-side can be written as
  \[
  \frac{5}{3} KK_{ij} - 2K_{ik} K^k_j - \frac{1}{3} K^2 \gamma_{ij} = \frac{1}{3} KA_{ij} - 2A_{ik} A^k_j = \psi^4 \left[ \frac{1}{3} K\tilde{A}_{ij} - 2\tilde{A}_{ik} \tilde{A}^k_j \right]
  \]
"Conformal" decomposition of the evolution equations (5b): the traceless part of the extrinsic curvature

- We further proceed to decompose the left-hand-side:

\[ L_{\alpha\alpha}A_{ij} = \psi^4 \left[ L_{\alpha\alpha} \tilde{A}_{ij} + 4 \tilde{A}_{ij} L_{\alpha\alpha} \ln \psi \right] = \psi^4 \left[ L_{\alpha\alpha} \tilde{A}_{ij} - \frac{2}{3} \alpha K\tilde{A}_{ij} \right] \]

- Combining all of the result, we finally reach

\[ (L_t - L_\beta)\tilde{A}_{ij} = \psi^{-4} \left[ -(D_i D_j \alpha)^{TF} + \alpha (R_{ij}^{TF} - 8\pi S_{ij}^{TF}) \right] + \alpha \left[ K\tilde{A}_{ij} - 2\tilde{A}_{ik} \tilde{A}_{j}^{k} \right] \]

- We here note that the second-order covariant derivative of the lapse function may be calculated as

\[
D_i D_j \alpha = D_i \partial_j \alpha = \tilde{D}_i \partial_j \alpha - C_{ij}^{k} \partial_k \alpha \\
= \left[ \partial_i \partial_j \alpha - \tilde{\Gamma}_{ij}^{k} \partial_k \alpha \right] - 2\left[ 2\partial_{(i} \ln \psi \partial_j) \alpha - \tilde{\gamma}_{ij}^{kl} \partial_k \ln \psi \partial_l \alpha \right]
\]

- NOTE: there is the same 2\textsuperscript{nd} order derivative in \( R_{ij}^{\phi} \)
“Conformal” decomposition of the constraint equations

- Let us turn now to consider the conformal decomposition of the constraint equations
- **Hamiltonian constraint**
  \[ R + K^2 - K_{ab}K^{ab} = 16\pi E \]
  \[ K_{ij}K^{ij} = \tilde{A}_{ij}\tilde{A}^{ij} + K^2 / 3 \]
  \[ R = \psi^{-4}\tilde{R} - 8\psi^{-5}\tilde{D}_k\tilde{D}^k\psi \]
  \[ \tilde{D}_i\tilde{D}^i\psi - \frac{1}{8}\tilde{R}\psi + \left(\frac{1}{8}\tilde{A}_{ij}\tilde{A}^{ij} - \frac{1}{12}K^2 + 2\pi E\right)\psi^5 = 0 \]
- **Momentum constraint**
  \[ D_bK^{ab} - D^aK = 8\pi P^a \]
  \[ \tilde{D}_j\tilde{A}^{ij} + 6\tilde{A}^{ij}\tilde{D}_j\ln\psi - \frac{2}{3}\tilde{D}^iK = 8\pi\psi^4P^i \]
  \[ D_jK^{ij} = D_jA^{ij} + D^iK / 3 \]
  \[ D_jA^{ij} = \tilde{D}_jA^{ij} + C^i_jA^{kj} + C^i_jA^{ik} \]
  \[ = \tilde{D}A^{ij} + 10A^{ij}\tilde{D}_j\ln\psi \]
  \[ = \psi^{-4}\left[\tilde{D}_j\tilde{A}^{ij} + 6\tilde{A}^{ij}\tilde{D}_j\ln\psi\right] \]
Summary of conformal decomposition

- With the conformal decomposition defined by
  \[ \gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}, \quad K_{ij} = \psi^4 A_{ij} + \frac{1}{3} \gamma_{ij} K, \quad \tilde{\gamma} = \det \tilde{\gamma}_{ij} = 1 \]

- The 3+1 decomposition (ADM formulation) of Einstein’s equations becomes

\[ \begin{align*}
\tilde{D}_i \tilde{D}^i \psi - & \frac{1}{8} \tilde{R} \psi + \left( \frac{1}{8} \tilde{A}_{ij} \tilde{A}^{ij} - \frac{1}{12} K^2 + 2\pi E \right) \psi^5 = 0 \\
\tilde{D}_j \tilde{A}^{ij} + & 6 \tilde{A}^{ij} \tilde{D}_j \ln \psi - \frac{2}{3} \tilde{D}^i K = 8\pi \psi^4 P^i
\end{align*} \]

**Constraint equations**

**Evolution equations**

\[ \begin{align*}
(L_t - L_\beta) \ln \psi &= -\frac{1}{6} \alpha K \\
(L_t - L_\beta) \tilde{\gamma}_{ij} &= -2\alpha \tilde{A}_{ij} \\
(L_t - L_\beta) K &= -D_i D^i \alpha + \alpha [K_{ij} K^{ij} + 4\pi (E + S)] \\
(L_t - L_\beta) \tilde{A}_{ij} &= \psi^{-4} \left[ -(D_i D_j \alpha)^{TF} + \alpha (R_{ij}^{TF} - 8\pi S_{ij}^{TF}) \right] + \alpha [K \tilde{A}_{ij} - 2\tilde{A}_{ik} \tilde{A}_{j}^k]
\end{align*} \]
Lie derivatives of tensor density

- A tensor density of weight $w$ is an object which is a tensor times $\gamma^{w/2} \cdot T_{ij}$.

- One should be careful because the Lie derivative of a tensor density is different from that of a tensor, as

$$\mathbf{L}_\beta T_{b_1 \ldots b_r}^{a_1 \ldots a_s} = \left[ \beta^c \partial_c T_{b_1 \ldots b_r}^{a_1 \ldots a_s} - \sum_{i=1}^s T_{b_1 \ldots b_r}^{a_1 \ldots c \ldots a_s} \partial_c \beta^a_i - \sum_{i=1}^r T_{b_1 \ldots c \ldots b_r}^{a_1 \ldots a_s} \partial_i \beta^c \right] + w T_{b_1 \ldots b_r}^{a_1 \ldots a_s} \partial_k \beta^k$$

$$= \left[ \mathbf{L}_\beta T_{b_1 \ldots b_r}^{a_1 \ldots a_s} \right]_{w=0} + w T_{b_1 \ldots b_r}^{a_1 \ldots a_s} \partial_k \beta^k$$

$$\mathbf{L}_\beta T_j^i = \mathbf{L}_\beta (\gamma^{w/2} T_j^i) = \gamma^{w/2} \left[ \beta^k \partial_k T_j^i - T_j^i \beta^k \partial_k \beta^i + T_k^i \partial_j \beta^k \right] + T_j^i (w/2) \gamma^{w/2-1} \mathbf{L}_\beta \gamma$$

$$= \left[ \beta^k \partial_k (\gamma^{w/2} T_j^i) - T_j^i \beta^k \partial_k \gamma^{w/2} - T_j^i \partial_k \beta^i + T_k^i \partial_j \beta^k \right] + T_j^i (w/2) \gamma^{w/2-1} \left[ \gamma \gamma^{ij} \mathbf{L}_\beta \gamma_{ij} \right]$$

$$= \left[ \beta^k \partial_k T_j^i - T_j^i w \gamma^{(w-1)/2} \beta^k \partial_k \gamma^{1/2} - T_j^i \partial_k \beta^i + T_k^i \partial_j \beta^k \right] + T_j^i w \gamma^{w/2} D_k \beta^k$$

$$= \left[ \beta^k \partial_k T_j^i - T_j^i w \gamma^{(w-1)/2} \beta^k \partial_k \gamma^{1/2} - T_j^i \partial_k \beta^i + T_k^i \partial_j \beta^k \right] + T_j^i w \gamma^{w/2} \left[ \gamma^{-1/2} \partial_k (\gamma^{1/2} \beta^k) \right]$$

$$= \left[ \beta^k \partial_k T_j^i - T_j^i \partial_k \beta^i + T_k^i \partial_j \beta^k \right] + w T_j^i \gamma^{w/2} \partial_k \beta^k$$
Lie derivatives in conformal decomposition

- The weight factor of the conformal factor $\psi = \gamma^{1/12}$ is $1/6$
- Thus the weight factor of the conformal metric and the conformal extrinsic curvature is $-2/3$, so that
- Note that the Lie derivative along $t^a$ is equivalent to the partial derivative along the time direction
- Thus

\[
(L_t - L_\beta) \ln \psi = (\partial_t - \beta^k \partial_k) \ln \psi - \frac{1}{6} \partial_k \beta^k
\]

\[
(L_t - L_\beta) \tilde{\gamma}_{ij} = (\partial_t - \beta^k \partial_k) \tilde{\gamma}_{ij} - \tilde{\gamma}_{ik} \partial_j \beta^k - \tilde{\gamma}_{jk} \partial_i \beta^k + \frac{2}{3} \tilde{\gamma}_{ij} \partial_k \beta^k
\]

\[
(L_t - L_\beta) \tilde{A}_{ij} = (\partial_t - \beta^k \partial_k) \tilde{A}_{ij} - \tilde{A}_{ik} \partial_j \beta^k - \tilde{A}_{jk} \partial_i \beta^k + \frac{2}{3} \tilde{A}_{ij} \partial_k \beta^k
\]
Evolution of constraints

- It can be shown that the “evolution” equations for the Hamiltonian ($C_H$) and Momentum ($C_M$) constraints becomes

\[
\left( \partial_t - \mathbf{L}_\beta \right) C_H = -D_k (\alpha C_M^k) - C_M^k D_k \alpha + \alpha K (2C_H - F) + \alpha K^{ij} F_{ij}
\]

\[
\left( \partial_t - \mathbf{L}_\beta \right) C_M^i = -D_j (\alpha F^{ij}) + 2\alpha K_j^i C_M^j + \alpha K C_M^i + \alpha D^k (F - C_H) + (F - 2H) D^i \alpha
\]

- Where $F_{ij}$ is the spatial projection of the evolution equation

\[
F_{ab} \equiv \perp \left[ 4R_{ab} - 8\pi \left( T_{ab} - \frac{1}{2} T g_{ab} \right) \right]
\]

- The evolution equations for the constraints show that the constraints are “preserved” or “satisfied”, if

  - They are satisfied initially ($C_H = C_M = 0$)
  - The evolution equation is solved correctly ($F_{ab} = 0$)
Numerical-relativity simulations based on the 3+1 decomposition is unstable!!

- It is known that simulations based on the 3+1 decomposition (ADM formulation), unfortunately crash in a rather short time.
- This crucial limitation may be captured in terms of notions of hyperbolicity (e.g. see textbook by Alcubierre (2008)).

Consider the following first-order system

\[ \partial_t U + A^i \cdot \partial_i U = 0 \]

- The system is called
  - **Strongly Hyperbolic**, if a matrix representation of A has real eigenvalues and complete set of eigenvectors.
  - **Weakly Hyperbolic**, if A has real eigenvalues but not a complete set of eigenvectors.

- The key property of strongly and weakly hyperbolic systems:
  - Strongly hyperbolic system is well-posed, and hence, the solution for the finite-time evolution is bounded.
  - Weakly hyperbolic system is ill-posed and the solution can be unbounded.
Numerical-relativity simulations based on the 3+1 decomposition is unstable!!

- It is known that the ADM formulation is only weakly hyperbolic ([Alcubierre (2008)])
- Consequently, the ADM formulation is ill-posed and the numerical solution can be unbounded, leading to termination of the simulation
- We need formulations for the Einstein’s equation which is (at least) strongly hyperbolic
- Let us consider Maxwell’s equations in flat spacetime to capture what we should do to obtain a more stable system

\[
\begin{align*}
\partial_i E^i &= 4\pi \rho_e \\
\partial_i B^i &= 0 \\
\partial_t E_i &= \varepsilon_{ijk} \partial_j B^k - 4\pi j_i \\
\partial_t B_i &= -\varepsilon_{ijk} \partial^j E_k
\end{align*}
\]

\[
\begin{align*}
A^\mu &= (\Phi, A^k) \\
B_i &= \varepsilon_{ijk} \partial^j A^k \\
\partial_i E^i &= 4\pi \rho_e \\
\partial_t E_i &= D_i D^j A_j - D^j D_j A_i - 4\pi j_i \\
\partial_t A_i &= -E_i - D_i \Phi
\end{align*}
\]

Note the similarity of these equations to those in the ADM formulation.
Consideration in Maxwell’s equations

First of all, let us note the similarity of the Maxwell’s equations with the ADM equations (for simplicity in vacuum)

\[ \partial_i E^i = 0 \text{ (constraint eq.)} \]
\[ \partial_i E_i = D_i D^j A_j - D^j D_j A_i \]
\[ \partial_i A_i = -E_i - D_i \Phi \]

(constraint eqs.)

\[ 2L_{\alpha n} K_{ij} = -2\alpha R_{ij} + \cdots \]
\[ = \alpha \left( \gamma^{kl} \partial_i \partial_j \gamma_{ij} + \gamma^{kl} \partial_j \partial_k \gamma_{ij} - \gamma^{kl} \partial_k \partial_l \gamma_{ij} \right) + \cdots \]
\[ L_{\alpha n} \gamma_{ij} = -2\alpha K_{ij} \]

wave-like part

“mixed-derivative” part

Second, the Maxwell’s equations are ‘almost’ wave equation

\[ -\partial^2 A_i + D^k D_k A_i - D_i D^j A_j = D_i \partial^i \Phi \]

Recall that in the Coulomb gauge \( D_j A^j = 0 \), the longitudinal part (associated with divergence part) of the electric field \( E \) does not obey a wave equation but is described by a Poisson equation (see a standard textbook, e.g., Jakson)
Reformulating Maxwell’s equations (1)
- Introducing auxiliary variables

- A simple but viable approach is to introduce independent auxiliary variables to the system
- Let us introduce a new independent variable defined by
  \[ F = D^k A_k \]
- The evolution equation for this is
  \[ \partial_t F = \partial_t D^k A_k = -D^i E_i - D_k D^k \Phi \]
- Then, the Maxwell’s equations for the vector potential become a wave equation in the form:
  \[ -\partial^2_t A_i + D^k D_k A_i = D_i \partial_t \Phi + D_i F \]
Reformulating Maxwell’s equations (2)

- **Imposing a better gauge**

  - A second approach is to impose a good gauge condition
  - In the Lorenz gauge, the Maxwell’s equations in the flat spacetime are wave equations
    \[
    \partial_\mu \partial^\mu A_\nu = 0
    \]
  - Alternatively, by introducing a source function, one may “generalize” the Coulomb gauge condition so that Poisson-like equations do not appear
    \[
    D^k A_k = H(x^\mu)
    \]
  - Recall again, that in the Coulomb gauge \( D_f A^i = 0 \), the longitudinal part (associated with divergence part) of the electric field \( E \) is described by a Poisson-type equation
Reformulating Maxwell’s equations (3)

- *Using the constraint equations*

- A third approach is to use the constraint equations
- To see this, let us back to the example considered in “introducing auxiliary variables”

\[
D_i E^i = 4\pi \rho_e \\
\partial_t E_i = D_i F - D^k D_k A_i - 4\pi j_i \\
\partial_t A_i = -E_i - D_i \Phi \\
\partial_t F = -D^i E_i - D_k D^k \Phi
\]

- The constraint equation can be used to rewrite the evolution equation for the auxiliary variable
- Seen as the first-order system, the hyperbolic properties of the two system is different: the hyperbolicity could be changed!

- It is important and sometimes even crucial to use the constraint equations to change the hyperbolic properties of the system
Reformulating Einstein’s equations

- The lessons learned from the Maxwell’s equations are
  - Introducing new, independent variables
    - **BSSN** (Shibata & Nakamura PRD 52, 5428 (1995); Baumgarte & Shapiro PRD 59, 024007 (1999))
    - Kidder-Scheel-Teukolsky (Kidder et al. PRD 64, 064017 (2001))
    - Bona-Masso (Bona et al. PRD 56, 3405 (1997))
    - Nagy-Ortiz-Reula (Nagy et al. PRD 70, 044012 (2004))
  - **Choosing a better gauge**
    - Generalized harmonic gauge (Pretorius, CQG 22, 425 (2005))
    - Z4 formalism (Bona et al. PRD 67, 104005 (2003))
  - **Using the constraint equations to improve the hyperbolicity**
    - adjusted ADM/BSSN (Shinkai & Yoneda, gr-qc/0209111)
  - **BSSN outperforms** (Alcubierre (2008))!
  - Exact reason is not clear
Let first analyze the conformal Ricci tensor

By noting that $2\tilde{\Gamma}^k_{ik} = \partial_i \ln \tilde{\gamma} = 0$ the conformal Ricci tensor is

$$\tilde{R}_{ij} = \partial_j \tilde{\Gamma}^l_{ik} - \partial_i \tilde{\Gamma}^l_{jk} + \tilde{\Gamma}^l_{ij} \tilde{\Gamma}_{kl} - \tilde{\Gamma}^l_{il} \tilde{\Gamma}_{kj}$$

$$= -\frac{1}{2} \tilde{\gamma}^{kl} \left( \partial_k \partial_l \tilde{\gamma}_{ij} - \partial_i \partial_k \tilde{\gamma}_{lj} - \partial_j \partial_k \tilde{\gamma}_{il} \right) + \text{(terms with } \partial \gamma \partial \gamma \text{)}$$

If we divide the conformal metric formally as $\tilde{\gamma}^{ij} = \delta^{ij} + f^{ij}$, we have

$$W_{ij} = \partial_j \partial^k \tilde{\gamma}_{ij} - \left( \partial_i \partial^k \gamma_{kj} + \partial_j \partial^k \gamma_{ki} \right) + \text{(terms with } f \partial f \text{)}$$

Thus we can eliminate the “mixed derivative” terms by introducing new auxiliary variable (Shibata & Nakamura (1995))

$$F_i \equiv \delta^{jk} \partial_k \tilde{\gamma}_{ij} = \partial^j \tilde{\gamma}_{ij}$$

$$W_{ij} = \partial_k \partial^k \tilde{\gamma}_{ij} - \left( \partial_i F_j + \partial_j F_i \right) + \text{(terms with } f \partial f \text{)}$$
BSSN formalism (2)

- Baumgarte and Shapiro introduced the slightly different auxiliary variables

\[ \Gamma^i = -\partial_j \tilde{\gamma}^{ij} \]

- In this case, the mixed-second-derivative terms are encompassed as

\[ \tilde{R}_{ij} = -\frac{1}{2} \left( \tilde{\gamma}^{kl} \partial_k \partial_l \tilde{\gamma}_{ij} - \tilde{\gamma}_{ik} \partial_j \Gamma^k - \tilde{\gamma}_{jk} \partial_i \Gamma^k \right) + \text{(terms with } \partial \gamma \partial \gamma) \]

- In linear regime, SN and BS are equivalent
Finally let us consider the evolution equation for the auxiliary variables (giving only a rough sketch of derivation)

Let us start from the momentum constraint equation

\[
\tilde{D}_k (\tilde{\gamma}^{jk} \tilde{A}_{ij}) + 6 \tilde{A}_{ij} \tilde{\gamma}^{jk} \tilde{D}_k \ln \psi - \frac{2}{3} \tilde{D}_i K = 8\pi \psi^4 P_i
\]

\[
\tilde{D}_j \tilde{A}^{ij} + 6 \tilde{A}^{ij} \tilde{D}_j \ln \psi - \frac{2}{3} \tilde{D}^i K = 8\pi \psi^4 P^i
\]

Substituting the evolution equation for the conformal extrinsic curvature

\[
(L_t - L_\beta) \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij}
\]

\[
(L_t - L_\beta) \tilde{\gamma}^{ij} = 2\alpha \tilde{A}^{ij}
\]

We obtain the evolution equations for \( F_i \) and \( \Gamma^i \), respectively

It can be seen from the above sketch of derivation, the evolution equation for \( \Gamma^i \) is slightly simpler
The explicit forms of the evolution equations are

\[
\begin{align*}
\left( \partial_t - \beta^k \partial_k \right) F_i &= -16\pi\alpha P_i + 2\alpha \left[ f^{jk} \partial_k \tilde{A}_{ij} + \tilde{A}_{ij} \partial_k \tilde{\gamma}^{jk} - \frac{1}{2} \tilde{A}^{jk} \partial_i \tilde{\gamma}_{jk} + 6\tilde{A}_i \partial_j \ln \psi - \frac{2}{3} \partial_i K \right] \\
+ \delta^{jk} \left[ -2\tilde{A}_{ij} \partial_k \alpha + (\partial_k \beta^l) (\partial_l \tilde{\gamma}_{ij}) + \partial_k \left( \tilde{\gamma}_{il} \partial_j \beta^l + \tilde{\gamma}_{jl} \partial_i \beta^l - \frac{2}{3} \tilde{\gamma}_{ij} \partial_l \beta^l \right) \right]
\end{align*}
\]

\[
\begin{align*}
\left( \partial_t - \beta^k \partial_k \right) \Gamma^i &= -16\pi\alpha P^i + 2\alpha \left[ \tilde{\Gamma}^i_{jk} \tilde{A}^{jk} + 6\tilde{A}^i_j \partial_j \ln \psi - \frac{2}{3} \tilde{\gamma}^{ij} \partial_j K \right] - 2\tilde{A}^{ij} \partial_j \alpha \\
+ \beta^j \partial_j \Gamma^i - \Gamma^j \partial_j \beta^i + \frac{2}{3} \Gamma^i \partial_j \beta^j + \frac{1}{3} \tilde{\gamma}^{ij} \partial_j \partial_k \beta^k + \tilde{\gamma}^{jk} \partial_j \partial_k \beta^i
\end{align*}
\]
BSSN formalism: summary (1)

\[ \tilde{D}_i \tilde{D}^i \psi - \frac{1}{8} \tilde{R} \psi + \left( \frac{1}{8} \tilde{A}_j \tilde{A}^{ij} - \frac{1}{12} K^2 + 2\pi E \right) \psi^5 = 0 \]

\[ \tilde{D}_j \tilde{A}^{ij} + 6 \tilde{A}^{ij} \tilde{D}_j \ln \psi - \frac{2}{3} \tilde{D}^i K = 8\pi \psi^4 P^i \]

\[
\left( \partial_t - \beta^k \partial_k \right) \ln \psi = -\frac{1}{6} \alpha K + \frac{1}{6} \partial_k \beta^k \\
\left( \partial_t - \beta^k \partial_k \right) \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} + \tilde{\gamma}_{ik} \partial_j \beta^k + \tilde{\gamma}_{jk} \partial_i \beta^k - \frac{2}{3} \tilde{\gamma}_{ij} \partial_k \beta^k \\
\left( \partial_t - \beta^k \partial_k \right) K = -D_i D^i \alpha + \alpha [ K_{ij} K^{ij} + 4\pi (E + S) ] \\
\left( \partial_t - \beta^k \partial_k \right) \tilde{A}_{ij} = \psi^{-4} \left[ - (D_i D_j \alpha)^{TF} + \alpha (R^{TF}_{ij} - 8\pi S^{TF}_{ij}) \right] + \alpha [ K\tilde{A}_{ij} - 2\tilde{A}_{ik} \tilde{A}^k_j ] \\
\quad + \tilde{A}_{ik} \partial_j \beta^k + \tilde{A}_{jk} \partial_i \beta^k - \frac{2}{3} \tilde{A}_{ij} \partial_k \beta^k 
\]

*Hamiltonian constraint is used*
\[
(\partial_t - \beta^k \partial_k) F_i = -16\pi \alpha P_i + 2\alpha \left[ f^{jk} \partial_k \tilde{A}_{ij} + \tilde{A}_{ij} \partial_k \tilde{\gamma}^{jk} - \frac{1}{2} \tilde{A}^{jk} \partial_i \tilde{\gamma}^{jk} + 6\tilde{A}^j_i \partial_j \ln \psi - \frac{2}{3} \partial_i K \right] \\
+ \delta^{jk} \left[ -2\tilde{A}_{ij} \partial_k \alpha + (\partial_k \beta^l) (\partial_i \tilde{\gamma}_{ij}) + \partial_k \left( \tilde{\gamma}_{il} \partial_j \beta^l + \tilde{\gamma}_{jl} \partial_i \beta^l - \frac{2}{3} \tilde{\gamma}_{ij} \partial_i \beta^l \right) \right]
\]

**Momentum constraint is used**

\[
(\partial_t - \beta^k \partial_k) \Gamma^i = -16\pi \alpha P^i + 2\alpha \left[ \tilde{\Gamma}^i_{jk} \tilde{A}^{jk} + 6\tilde{A}^{ij} \partial_j \ln \psi - \frac{2}{3} \tilde{\gamma}^{ij} \partial_j K \right] - 2\tilde{A}^{ij} \partial_j \alpha \\
+ \beta^j \partial_j \Gamma^i - \Gamma^j \partial_j \beta^i + \frac{2}{3} \Gamma^i \partial_j \beta^j + \frac{1}{3} \tilde{\gamma}^{ij} \partial_j \partial_k \beta^k + \tilde{\gamma}^{jk} \partial_j \partial_k \beta^i
\]
Overview of numerical relativity

Solving the constraint equations

Setting ‘realistic’ or ‘physically motivated’ initial conditions

Locating BH (solving AH finder)

Solving Einstein’s equations

Extracting GWs

Solving gauge conditions

Solving source filed equations

Solving the constraint equations

Main loop

GR-HD
GR-MHD
GR-Rad(M)HD

Microphysics
- EOS
- weak processes

BH excision
Gauge conditions

- Associated directly with the general covariance in general relativity, there are degrees of freedom in choosing coordinates (gauge freedom)
  - **Slicing condition** is a prescription of choosing the lapse function
  - **Shift condition** is that of choosing the shift vector
- Einstein’s equations say nothing about how the gauge conditions should be imposed
- As we have seen in the reformulation of the ADM system, choosing “good” gauge conditions are very important to achieve stable and robust numerical simulations
  - An improper slicing conditions in a stellar-collapse problem will lead to appearance of (coordinate and physical) singularities
  - Also, the shift vector is important in resolving the frame dragging effect in simulations of e.g. compact binary merger
The covariant derivative of a timelike unit vector \( z^a \) can be decomposed as

\[
\nabla_a z_b = \omega_{ab} + \sigma_{ab} + \frac{1}{3} h_{ab} \theta - z_a \zeta_b
\]

Where, the deformation of the congruence of the timelike vector is characterized by these tensors:

- \( h_{ab} \equiv g_{ab} + z_a z_b \), (induced metric)
- \( \omega_{ab} \equiv \nabla_{(a} z_{b)} \), (twist)
- \( \sigma_{ab} \equiv \nabla_{(a} z_{b)}^{\text{TF}} \), (shear)
- \( \theta \equiv \nabla_c z^c \), (expansion)
- \( \zeta^a \equiv z^c \nabla_c z^a \), (acceleration)

For the unit normal vector to \( \Sigma \), \( n^a \) we have:

- The expansion is \(-K\)
- The shear is \(-A_{ab}\)
- The twist vanishes
Geodesic slicing $\alpha=1, \ \beta^i=0$

- In the geodesic slicing, the evolution equation of the trace of the extrinsic curvature is
  \[ \partial_t K = K_{ij} K^{ij} + 4\pi (E + 3S) \]
- For normal matter (which satisfies the strong energy condition), the right-hand-side is positive
- Thus the expansion of time coordinate ($-K$) decreases monotonically in time
- In terms of the volume element $\gamma^{1/2}$, this means that the volume element goes to zero, as
  \[ \partial_t \ln \gamma^{1/2} = \frac{1}{2} \gamma^{ij} \partial_i \gamma_{ij} = -\alpha K + D_k \beta^k \Rightarrow -K \]
- This behavior results in a coordinate singularity
- As can be seen in this example, how to impose a slicing condition is closely related to the trace of the extrinsic curvature
Maximal slicing

- Because the decrease in time of the volume element results in a coordinate singularity, let us maximize the volume element.
- We take the volume of a 3D-domain $S$: $V[S] = \int_S \sqrt{\gamma} d^3x$
  and consider a variation along the time vector $t^a = \alpha n^a + \beta^a$
- At the boundary of $S$, we set $\alpha = 1, \beta^i = 0$
  
  $$L_t V[S] = \int_S d^3x \left[ -\alpha K \sqrt{\gamma} + \partial_i (\sqrt{\gamma} \beta^i) \right] = -\int_S \alpha K \sqrt{\gamma} d^3x$$

- Thus if $K = 0$ on a slice, the volume is extremal (maximal).
- We shall demand that this **maximal slicing condition** holds for all slices.
  
  $$0 = (L_t - L_\beta)K = -D_i D^i \alpha + \alpha [K_{ij} K^{ij} + 4\pi (E + S)]$$

- The maximal slicing has a **strong singularity avoidance** property
  (E.g. Estabrook & Wahlquist PRD 7, 2814 (1973); Smarr & York, PRD 17, 1945/2529 (1978))

- However this is a **elliptic equation** and is computationally expensive.
(K-driver)/(approximate maximal) condition

- As a generalization of the maximal slicing condition, let us consider the following condition with a positive constant $c$

  \[ \partial_t K = -cK \]

- This (elliptic) condition drives $K$ back to zero even when $K$ deviates from zero due to some error or insufficient convergence

- Balakrishna et al. (CQG 13, L135 (1996)) and Shibata (Prog. Theor. Phys. 101, 251 (1999)) converted this equation into a parabolic one by adding a “time” derivative of the lapse:

  \[ \partial_\lambda \alpha = D_i D^i \alpha - \alpha (K_{ij} K^{ij} + 4\pi (E + S)) - \beta^i D_i K + cK \]

- If a certain degree of the “convergence” is achieved and the lapse relaxes to a “stationary state”, it suggests $\partial_t K = -cK$

- This condition is called K-driver or approximate maximal slicing condition

  \[ \partial_t \alpha = -\varepsilon (\partial_t K + cK), \quad \lambda = \varepsilon t \]
Harmonic slicing

- The harmonic gauge condition $\nabla_c \nabla^c x^a = 0$ have played an important role in theoretical developments (Choquet-Bruhat’s textbook).
- Existence and uniqueness of the solution of the Cauchy problem of Einstein’s equations (somewhat similar to Lorenz gauge in EM).
- The harmonic slicing condition is defined by

  $$\nabla_c \nabla^c t = 0 \iff \partial_\mu (\sqrt{-g} g^{\mu 0}) = 0$$

- Note that $\sqrt{-g} = \alpha \sqrt{\gamma}$
- The harmonic slicing condition can be written as

  $$\left( \partial_t - \partial_k \beta^k \right) \alpha = -\alpha^2 K$$

- This is an evolution equation.
- It is known that the harmonic slicing condition has some singularity avoidance property, although weaker than that of the maximal slicing (e.g. Cook & Scheel PRD 56, 4775 (1997), Alcubierre’s textbook).
Generalized harmonic slicing

- Bona et al. (PRL 75, 600 (1995)) generalized the harmonic slicing condition to

\[
  \left( \partial_t - \partial_k \beta^k \right) \alpha = -\alpha^2 f(\alpha) K
\]

- This family of slicing includes the geodesic slicing \((f=0)\), the harmonic slicing \((f=1)\), and formally the maximal slicing \((f=\infty)\)

- The choice \(f(\alpha) = 2/\alpha\), which is called \(1+\log\) slicing, has stronger singularity avoidance properties than the harmonic slicing (Anninos et al. PRD 52, 2059 (1995))

- The \(1+\log\) slicing has been widely used and has proven to be a successful and robust slicing condition
Minimal distortion (shift) condition

- Smarr and York ([PRD 17, 1945/2529 (1978)](http://doi.org/)) proposed a well motivated shift condition called the **minimal distortion condition**

- As seen in the preliminary, the “distortion” part of the congruence is contained in the shear tensor

- They define a distortion functional by

- here the distortion tensor is defined by

\[
\Sigma_{ab} = \frac{1}{2} \left[ L_t \gamma_{ab}^{TF} - K_{ab}^{TF} \right] = \nabla_{(a} n_{b)}^{TF}
\]

- The resulting shift condition is

\[
D_a \Sigma^{ab} = 0
\]

- Beautiful and physical but vector elliptic equations (computationally expensive)

\[
D_c D^c \beta^a + D_a D_c \beta^c + R_{ab} \beta^b = D^b \left[ 2\alpha A_{ab} \right] = 2 A^{ab} D_b \alpha + \alpha \left( \frac{4}{3} \gamma^{ab} D_b K + 16\pi P_a \right)
\]
Γ-Freezing and approximate minimal condition

- With some calculations, one can show that the minimal distortion condition is written as
  \[ \tilde{D}^j (\psi^6 \partial_t \tilde{\gamma}_{ij}) = 0 \]
  - The conformal factor is coupled!


  - E.g., Nakamura et al. proposed instead to solve the decoupled pseudo-minimal distortion condition:
  \[ \tilde{D}^j (\tilde{\gamma}_{ij}) = 0 \]

- Alcubierre and Brugmann (*PRD* 63, 104006 (2001)) proposed an approximate minimal distortion condition called Gamma-Freezing:
  \[ \tilde{D}_j (\partial_t \tilde{\gamma}^{ij}) = \partial_t \Gamma^i = 0 \]

- Anyway, these conditions are elliptic-type!
Gamma-Driver condition

- Alcubierre and Brugmann (PRD 63, 104006 (2001)) converted the Gamma-freezing elliptic condition into a parabolic one by adding a time derivative of the shift (somewhat similar to the K-driver)

\[ \partial_t \beta^i = k \partial_t \Gamma^i \]

- Alcubierre et al. (PRD 67, 084023 (2003)) and others (Lindblom & Scheel PRD 67, 124005 (2003); Bona et al. PRD 72, 104009 (2005)) extended the Gamma-freezing condition to hyperbolic conditions

- There are several alternative conditions

- Shibata (ApJ 595, 992 (2003)) proposed a hyperbolic shift condition

\[ \partial_t \beta^i = \tilde{\gamma}^{ij} \left( F_i + \Delta t_{\text{step}} \partial_t F_i \right), \quad \Delta t_{\text{step}} : \text{time-step used in simulation} \]

- To date, the above two families of shift conditions are known to be robust
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3+1 decomposition of \( \nabla_a T^{ab} = 0 \)

- Energy Conservation Equation (1)

- First, substitute the \textbf{3+1 decomposition of} \( T_{ab} \) to obtain

\[
0 = \nabla_b T^b_a = \nabla_b (E n^b n_a + P^b n_a + P_a n^b + S^b_a) \\
= n_a n^b \nabla_b E + E a_a - K E n_a - P^b K^a_b + n_a \nabla_b P^b_a + n^b \nabla_b P_a - K P_a + \nabla_b S^b_a
\]

- Then, let us project it onto normal direction to \( \Sigma \).

Noting that \( P^a, K^{ab}, \) and \( a^b \) is purely spatial, we obtain

\[
- n^b \nabla_b E + K E - \nabla_a P^a_a + n_a n^b \nabla_b P_a + n^a \nabla_b S^b_a = 0
\]

- Because \( n^a S_{ab} = 0 \), we have

\[
n^a \nabla_b S^b_a = - S^b_a \nabla_b n^a = S^b_a (K^a_b + n_b a^a) = S^{ab} K_{ab}
\]

- Similarly,

\[
n^a n^b \nabla_b P_a = - P_a n^b \nabla_b n^a = - P_a a^b
\]

- The divergence term of \( P^a \) is

\[
D_a P^a = \nabla_a P^a = (\delta^b_a + n_a n^b) \nabla_b P^a = \nabla_a P^a - P_a a^a
\]
3+1 decomposition of $\nabla_a T^{ab} = 0$

- Energy Conservation Equation (2)

- Combining altogether, we reach the energy conservation equation

\[ n^b \nabla_b E + D_b P^b + 2P^b a_b - KE - K_{ab} S^{ab} = 0 \]

\[ (\partial_t - \beta^k D_k) E + \alpha [D_b P^b - KE - K_{ab} S^{ab}] + 2P^b D_b \alpha = 0 \]

\[ \partial_t E + D_k (\alpha P^k - E \beta^k) + E (D_k \beta^k - \alpha K) - \alpha K_{ab} S^{ab} + P^b D_b \alpha = 0 \]

- where we have used

\[ n^b \nabla_b E = L_n E = \alpha^{-1} (L_t - L_\beta) E = \alpha^{-1} (\partial_t - \beta^k D_k) E \]

\[ a_b = D_b \ln \alpha \]

- The last equation will be used to derive the conservative forms of the energy equation
3+1 decomposition of $\nabla_a T^{ab} = 0$

- **Momentum Conservation Equation (1)**

- To this turn, let us project the equation onto $\Sigma$ to obtain

$$E a_a - P^b_k b_a + \perp_a n^b \nabla_b P_c - K P_a + \perp_a \nabla_b S_c^b = 0$$

- The spacetime-divergence term of $S_c^b$ can be replaced by the spatial-divergence by

$$D_b S_c^b = \perp_b \perp_c \nabla_d S_e^b = \perp_c (\delta^d_b + n_b n^d) \nabla_d S_e^b = \perp_c \nabla_b S_e^b - S_c^d a_d$$

- The projection term with the covariant derivative of $P_c$ is

$$\perp_a n^b \nabla_b P_c = \alpha^{-1} \perp_a (\alpha n^b) \nabla_b P_c = \alpha^{-1} \perp_a (L_{\alpha n} P_c - P_d \nabla_c (\alpha n^d))$$

- Note that $(\alpha n)$-Lie derivative of any spatial tensor is spatial, and

$$\nabla_b (\alpha n^a) = n^a \nabla_b \alpha + \alpha \nabla_b n^a = n^a \nabla_b \alpha - \alpha (K^a_b + n_b a^a)$$

- so that

$$\perp_a n^b \nabla_b P_c = \alpha^{-1} L_{\alpha n} P_a + K_{ab} P^b$$
3+1 decomposition of \( \nabla_a T^{ab} = 0 \)

- Momentum Conservation Equation (2)

Combining altogether, we obtain the momentum conservation equation:

\[
(L_t - L_\beta) P_a + \alpha[D_b S_a^b + S_a^b a_b - KP_a + Ea_a] = 0
\]

\[
(\partial_t - L_\beta) P_a + \alpha[D_b S_a^b - KP_a] + S_a^b D_b \alpha + ED_a \alpha = 0
\]

\[
(\partial_t - \beta^c D_c) P_a + \alpha D_b S_a^b + (D_c \beta^c - \alpha K) P_a + S_a^b D_b \alpha + ED_a \alpha = 0
\]

\[
\partial_t P_a + D_c (\alpha S_a^c - \beta^c P_c) + (D_c \beta^c - \alpha K) P_a - P_c D_a \beta^c + ED_a \alpha = 0
\]

Where we have expressed the Lie derivative by spatial covariant derivative

The last equation will be used in conservative reformulation

NOTE: In York (1979), because he used \( P^a \) instead of \( P_a \), an extra term appears in the equation.
Now we will show the energy and momentum conservation equations can be recast to conservative form

\[ \partial_t E + D_c (\alpha P^c - E \beta^c) + (D_c \beta^c - \alpha K) E - \alpha K_{ab} S^{ab} + P^b D_b \alpha = 0 \]
\[ \partial_t P_a + D_c (\alpha S^c_a - \beta^c P_a) + (D_c \beta^c - \alpha K) P_a - P_c D_a \beta^c + E D_a \alpha = 0 \]

First, by taking the trace of evolution eq. of \( \gamma_{ab} \), we get

\[ \gamma^{ab} (\partial_t \gamma_{ab} - D_a \beta_b - D_b \beta_a) = -2\alpha K \Rightarrow D_a \beta^a - \alpha K = \frac{1}{2} \gamma^{ij} \partial_t \gamma_{ij} = \frac{1}{\sqrt{\gamma}} \partial_t \sqrt{\gamma} \]

Second, note that for any rank-(1,1) spatial tensor,

\[ D_k T_i^k = \partial_k T_i^k + \Gamma_j^k T_i^j - \Gamma_i^j T_j^k = \partial_k T_i^k + (\partial_j \ln \sqrt{\gamma}) T_i^j - \Gamma_i^j T_j^k \]
\[ = \frac{1}{\sqrt{\gamma}} \partial_k (\sqrt{\gamma} T_i^k) - \Gamma_i^j T_j^k \]
3+1 decomposition of  $\nabla_a T^{ab} = 0$

- Conservative Formulation (2)

- Using the equations derived the above, we can finally reach the **conservative forms of the energy and momentum equations**

\[
\begin{align*}
\partial_t \left( \sqrt{\gamma} E \right) + \partial_k \left( \sqrt{\gamma} (\alpha P^k - E \beta^k) \right) &= \sqrt{\gamma} \left( \alpha K_{ij} S^{ij} - P^k D_k \alpha \right) \\
\partial_t \left( \sqrt{\gamma} P_i \right) + \partial_k \left( \sqrt{\gamma} (\alpha S^k_i - \beta^k P_i) \right) &= \sqrt{\gamma} \left[ P_k D_i \beta^k - ED_i \alpha + \Gamma^k_{ij} (\alpha S^j_k - \beta^j P_k) \right]
\end{align*}
\]

- For the perfect fluid, for instance, these equations may be solved by high resolution shock capturing schemes
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July 28 - August 3, 2011

APCTP International school on NR and GW  July 28-August 3, 2011
Locating the apparent horizon (1)

- **Apparent horizon (e.g. Wald (1984))**: the apparent horizon is the boundary of the (total) trapped region
  - **Trapped region**: the trapped region is collections of points where the expansion of the null geodesics is negative or zero

Thus, to locate the apparent horizon, we must calculate the expansion of the null geodesics and determine the points where the expansion vanishes.

Recall that the expansion is related to the trace of the extrinsic curvature: \( K \Leftrightarrow \) expansion

So that let us first define the extrinsic curvature of a null surface \( N \) generated by an outgoing null vector on a slice \( \Sigma \):
Locating the apparent horizon (2)

- Let $\mathcal{S}$ to be an intersection of the slice $\Sigma$ and the null surface $\mathcal{N}$
  - We denote the unit normal of $\mathcal{S}$ in $\Sigma$, as $s^a$
- Then the outgoing ($k^a$) and ingoing ($l^a$) null vectors on $\mathcal{S}$ are

Using $k^a$ and $l^a$, the metric on $\mathcal{S}$ induced by $g_{ab}$ is given by

\[ \chi_{ab} = g_{ab} + k_a l_b + k_b l_a \]
\[ = g_{ab} + n_a n_b - s_a s_b \]

Thus we can define the projection operator to $\mathcal{S}$:

\[ P_b^a = \delta_b^a + n^a n_b - s^a s_b \]
Locating the apparent horizon (3)

- Using the projection operator, the extrinsic curvature for $\mathcal{N}$ is defined by

$$\kappa_{ab} = -P^c_a P^d_b \nabla_{(c} k_{d)}$$

- Because $k^a$ is the outgoing null vector on $S$, the 2D-surface $S$ is the apparent horizon if $\text{tr}[\kappa] = \kappa_a^a = \kappa = 0$

- This condition can be written in terms of $s^a$ as

$$D_k s^k - K + K_{ij} s^i s^j = 0$$

- This is a single equation for the three unknown “functions” $s^k$!

- However, the condition that $S$ is closed 2-sphere and that $s^a$ is a unit normal vector bring two additional relation to $s^k$

- For detail, see (e.g. Bowen, J. M. & York, J. W., PRD 21, 2047 (1980); Gundlach, C. PRD 57, 863 (1998))
Energy and Momentums
The Lagrangian density of gravitational field in General Relativity is (e.g. Wald (1984))

$$L_G \equiv \sqrt{-g} \, 4^R$$

Because the 4D Ricci scalar is written as

$$4^R = 2(G_{ab} n^a n^b - 4R_{ab} n^a n^b)$$

$$= \alpha \sqrt{\gamma} (R + K_{ab} K^{ab} - K^2) + (\text{Divergence terms})$$

Noting that the extrinsic curvature is

$$K_{ab} = \frac{1}{2\alpha} (\gamma_{ab} - D_a \beta^b - D_b \beta^a)$$

The conjugate momentum $\pi^{ab}$ is defined by

$$\pi^{ab} \equiv \frac{\partial L_G}{\partial \dot{\gamma}_{ab}} = \sqrt{\gamma} (K^{ab} - K \gamma^{ab})$$
Canonical Formulation (2)

Now we obtain the Hamiltonian density as

$$H_G \equiv \pi^{ab} \dot{\gamma}_{ab} - L_G$$

$$= \sqrt{\gamma}\left[ \frac{\alpha}{\gamma} \left( -\gamma R + \pi^{ab} \pi_{ab} - \frac{1}{2} \pi^2 \right) - 2 \beta_b D_a \left( \gamma^{-1/2} \pi^{ab} \right) \right] + \text{(Divergence terms)}$$

The Hamiltonian is defined by

$$H_G \equiv \int H_G dx^3$$

The constraint equations are derived by taking the variations with respect to the lapse and the shift, respectively, as

$$C_H \equiv -R + \gamma^{-1} \pi_{ab} \pi^{ab} - \frac{1}{2} \gamma^{-1} \pi^2 = 0 \quad : \text{Hamiltonian constraint}$$

$$C_M^b \equiv D_a \left( \gamma^{-1/2} \pi^{ab} \right) = 0 \quad : \text{Momentum constraint}$$

where we have dropped the surface term.
The evolution equations are derived by taking the variations with respect to the canonical variables (e.g. Wald (1984)):

\[
\dot{\gamma}_{ab} \equiv \frac{\delta H_G}{\delta \pi_{ab}} = 2\alpha \gamma^{-1/2} \left[ \pi_{ab} - \frac{1}{2} \gamma_{ab} \pi \right] + 2D_{(a} \beta_{b)} \equiv B_{ab}
\]

\[
\dot{\pi}_{ab} \equiv \frac{\delta H_G}{\delta \gamma_{\alpha\beta}} = -\alpha \gamma^{1/2} \left[ R - \frac{1}{2} R \gamma_{ab} \right] + \frac{1}{2} \alpha \gamma^{-1/2} \gamma_{ab} \left[ \pi^{cd} \pi_{cd} - \frac{1}{2} \pi^2 \right] - 2\alpha \gamma^{-1/2} \left[ \pi^{ac} \pi_c^b - \frac{1}{2} \pi \pi_{ab} \right]
\]

\[
+ \gamma^{1/2} (D^a D^b \alpha - \gamma^{ab} D_c D^c \alpha) + \gamma^{1/2} D_c \left( \gamma^{-1/2} \beta^c \pi_{ab} \right) - 2\pi^{c(a} D_c \beta^{b)} \equiv A^{ab}
\]

again, we here dropped the divergence terms
Energy for Asymptotically Flat spacetime (1)

- Let us consider the energy of gravitational field in the asymptotically flat spacetime
  - Although there is no unique definition of 'local' gravitational energy in General Relativity, we can consider the total energy in the asymptotically flat spacetime
  - Asymptotically flat spacetime represent ideally isolated spacetime, and hence, there will be the conserved energy

- A simple consideration based on the Hamiltonian density,
  \[ H_G = \sqrt{\gamma} \left[ \alpha C_H - 2 \beta_b C^b_M \right] + (\text{Divergence terms}) \]
  may lead to a conclusion that the energy of any spacetime is zero when the constraint equations are satisfied!

- This “contradiction” stems from the wrong treatment of the divergence (surface) terms (which we have dropped)
Energy for Asymptotically Flat spacetime (2)

- The boundary conditions to be imposed are not fixed ones \( \delta Q \mid_{\text{boundary}} = 0 \) where \( Q \) denotes relevant geometrical variables, but the “asymptotic flatness”:

\[
\alpha = 1 + O(r^{-1}), \quad \beta^i = O(r^{-1}), \quad \gamma_{ij} - \delta_{ij} = O(r^{-1}), \quad \pi^{ij} = O(r^{-2})
\]

- Keeping the divergence terms, the variation of the Hamiltonian now becomes (Regge & Teitelboim, Ann. Phys 88. 286 (1974))

\[
\delta H_G = -\int M^{ijkl} \left[ \alpha D_k (\delta \gamma_{ij}) - (D_k \alpha) \delta \gamma_{ij} \right] d\sigma_l \\
-\int \left[ 2\beta_k \delta \pi^{kl} + (2\beta^k \pi^{jl} - \beta^l \pi^{jk}) \delta \gamma_{jk} \right] d\sigma_l
\]

- where we have assumed that the constraint equations and the evolution equations are satisfied, \( d\sigma_l \) is the volume element of the boundary sphere and \( M^{ijkl} \) is defined as

\[
M^{ijkl} = \frac{1}{2} \sqrt{\gamma} [\gamma^{ik} \gamma^{jl} + \gamma^{il} \gamma^{jk} - 2\gamma^{ij} \gamma^{kl}]
\]
Energy for Asymptotically Flat spacetime (3)

- Under the boundary conditions of the asymptotic flatness, the non-zero contribution of the surface terms is,

\[-\int M^{ijkl} D_k (\delta \gamma_{ij}) \, d\sigma_l = -\delta \int \gamma^{ij} \gamma^{kl} (\partial_j \gamma_{ik} - \partial_k \gamma_{ij}) \, d\sigma_l\]

- Thus, we define the Hamiltonian of the asymptotically flat spacetime as

\[H_{G}^{\text{asympt.flatt}} \equiv H_G + 16\pi E_G [\gamma_{ij}]\]

\[E_G [\gamma_{ij}] \equiv \frac{1}{16\pi} \int \sqrt{\gamma} \gamma^{ij} \gamma^{kl} (\partial_j \gamma_{ik} - \partial_k \gamma_{ij}) \, d\sigma_l\]

- Then, the energy of the gravitational fields is not zero but \(E[\gamma_{ij}]\)

- The overall factor is determined by the requirement that the energy of an asymptotically flat spacetime is \(M\)

\[ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(\delta_{ij} + \frac{2Mx_i x_j}{r^3}\right) dx^i dx^j\]
Momentum for Asymptotically Flat spacetime (1)

- The action for the region $V \ (t_1 < t < t_2)$ is
  \[ S_G = \int_V L_G dx^4 = \int_{t_1}^{t_2} dt \int d^3 x \left[ \pi^{ij} \dot{\gamma}_{ij} - H_G \right] \]

- Taking the variation, we obtain (Regge & Teitelboim, Ann. Phys 88. 286 (1974))
  \[ \delta S_G = \int_{t_1}^{t_2} dt \frac{d}{dt} \int d^3 x \left[ \pi^{ij} \delta \gamma_{ij} + \text{terms vanishing by EOM} \right] \]

- When there is a Killing vector $\xi^a$, the action is invariant under the Lie transport by $\xi^a$

- Making use of \( \delta \gamma_{ij} = -L_{\xi} \gamma_{ij} = -(D_i \xi_j + D_j \xi_i) \), we obtain
  \[ \delta_{\xi} S_G = \int_{t_1}^{t_2} dt \frac{d}{dt} \int d^3 x \left[ -D_i \left( 2\pi^{ij} \xi_j \right) + 2 \xi_i D_j \pi^{ij} \right] = 0 \]

- Note that the second term in the integrand vanished thanks to the momentum constraint
Finally the variation of the action is reduced to
\[ \partial S_G = \left[ \int d\sigma \left( -2\pi^k_l \xi_k \right) \right]_{t_1}^{t_2} = 0 \]

Because the Killing vector approaches at the boundary (spacelike infinity) to a constant translation vector field \( \tau_a \), we have
\[ \tau_k \left[ P_G^k (t_2) - P_G^k (t_1) \right] = 0, \quad P_G^k [\gamma_{ij}] \equiv -\frac{1}{8\pi} \int d\sigma \pi^k l \]

This equation means that \( P^k_G \) represent the total linear momentum.

Similarly, the generator of the rotational Lie transport approaches \( \varepsilon_{ijk} \phi^j x_k \) (\( \phi \) is a constant vector field, \( \varepsilon \) is the totally anti-symmetric tensor), we may define the total angular momentum by
\[ \phi^k \left[ L_k^G (t_2) - L_k^G (t_1) \right] = 0, \quad L_k^G [\gamma_{ij}] \equiv \frac{1}{8\pi} \int d\sigma \varepsilon_{ijk} \pi^{jl} x_k \]
Energy and Momentums: summary

To summarize, we define the energy, the linear momentum, and the angular momentum in the asymptotically flat spacetime by

\[
E_G[\gamma_{ij}] \equiv \frac{1}{16\pi} \int \sqrt{\gamma} \gamma^{ij} \gamma^{kl} (\partial_j \gamma_{ik} - \partial_k \gamma_{ij}) d\sigma_l
\]

\[
P_G^k[\gamma_{ij}] \equiv -\frac{1}{8\pi} \int d\sigma_l \pi^{kl}
\]

\[
L_G^k[\gamma_{ij}] \equiv \frac{1}{8\pi} \int d\sigma_l \varepsilon_{ijk} \pi^{jl} x^k
\]

A number of examples of the actual calculation will be found in a textbook (Baumgarte & Shapiro (2010)).
Overview of numerical relativity

- Setting ‘realistic’ or ‘physically motivated’ initial conditions
- Locating BH (solving AH finder)
- Solving the constraint equations
- Solving Einstein’s equations
- Solving gauge conditions
- Solving source filed equations
- Extracting GWs
- BH excision

Main loop

GR-HD
GR-MHD
GR-Rad(M)HD

Microphysics
- EOS
- weak processes
References

- **General Reference on General Relativity**

- **The classics on the basis of Numerical Relativity**
References

- **Recent Textbooks on Numerical Relativity**
    - Results of Numerical Relativity Simulations
    - Boundary Conditions
  - Gourgoulhon E., *3+1 Formalism and Bases of Numerical Relativity*, to be published in Lecture Notes in Physics, (gr-qc/0703035)

- **Einstein’s equations as the Cauchy Problem**
References

- **Locating the Horizons**

- **ADM mass, and Some useful calculations**

- **Initial Data Problems**
  - *Pfeiffer, H. P.*, *Initial value problem in numerical relativity*, *gr-qc/0412002*
References

General Relativistic Numerical Hydrodynamics


Extracting Gravitational Waves