Bulk locality in higher-spin holography

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- JHEP **1503** (2015) 170 [arXiv:1412.0016 [hep-th]]
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Outline

Bulk locality and Amplitude analyticity

Bulk locality

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which, on the CFT side, corresponds to

(1) a semi-classical limit: a perturbative expansion around a generalized free field theory when $N \gg 1$

(2) a higher-spin gap: all single-trace operators of higher-spins have large conformal dimensions $\Delta_{s>2} \gg \Delta_{s'\leqslant 2}$

Bulk locality and Mellin amplitudes

The properties

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were argued to provide necessary and sufficient conditions for a CFT to possess a local bulk dual.

(Heemskerk, Penedones, Polchinski, Sully, 2009;
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$\Rightarrow$ **What are Mellin amplitudes?**

Mellin amplitudes

Mellin variables: For n prescribed numbers Δ_i, consider the $n(n-3)/2$ variables γ_{ij} corresponding to the independent solutions of

$$\gamma_{ij} = \gamma_{ji}\,, \qquad \gamma_{ii} = -\Delta_i\,, \qquad \sum_{j=1}^{n} \gamma_{ij} = 0\,.$$

Mellin amplitude: Mellin-Barnes representation of conformal correlator of n scalar primary operators

$$\langle\, \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)\,\rangle \,\propto\, \int d\gamma\, M(\gamma_{ij}) \prod_{i<j}^{n} \Gamma(\gamma_{ij})\, |x_{ij}^2|^{-\gamma_{ij}}$$

where each integration contour is a vertical line on the complex plane with fixed real part and running parallel to the imaginary axis.

The Mellin amplitude $M(\gamma_{ij})$ is essentially the Mellin transform over the independent cross ratios of the factor not fixed by conformal symmetry in the n-point correlator.

(Mack, 2009)

Mellin amplitudes

Important properties: For any CFT with a discrete spectrum of conformal dimensions and a large-N limit,

- Mellin amplitudes are meromorphic functions with simple poles determined by the twists of the operators in the OPE
- For single-trace primary-operator correlators, the poles of the Mellin amplitudes can arise only from single-trace operators

when $N \to \infty$.

(Mack, 2009; Penedones, 2010)

Mellin amplitudes

Mandestam invariants: For n prescribed numbers Δ_i and n vectors $p_i \in \mathbb{R}^{D-1,1}$ solutions of the kinematical constraints

$$p_i^2 = -\Delta_i \,, \qquad \sum_{i=1}^{n} p_i = 0 \,,$$

the Mandestam invariants are the $n(n-3)/2$ independent scalar products $s_{ij} = -(p_i + p_j)^2$.

For any solution of the kinematical constraints,
the Mandestam invariants s_{ij} define Mellin variables via

$$\gamma_{ij} = \frac{\Delta_i + \Delta_j - s_{ij}}{2} = p_i \cdot p_j$$

(Mack, 2009)

Mellin amplitudes

Mellin amplitude programme: use Mellin amplitudes to interpret CFT correlation functions as AdS scattering amplitudes.

(Penedones, Fitzpatrick, Kaplan, Raju, van Rees, Paulos, Nandan, Volovich, Wen, ...)

Mellin amplitude dictionary

Flat	AdS
Feynman diagram	Witten diagram
Mandelstam invariants	Mellin variables
Scattering amplitude	Mellin amplitude

Bulk locality and Mellin amplitudes

For individual Feynman/Witten diagrams, the scattering and Mellin amplitudes share common properties as functions of Mandestam invariants vs Mellin variables.

Bulk process	Boundary amplitude
Local contact interaction	Polynomial
Particle exchange	Simple pole

These properties suggest a common

Boundary criterion of bulk locality: *Interactions on flat/AdS spacetime are local iff the amplitudes of the corresponding contact Feynman/Witten diagram are polynomial functions of Mandelstam invariants / Mellin variables.*

Bulk locality and Mellin amplitudes

This explains the addition of the third criteria

(1) a semi-classical limit: $N \gg 1$

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among the conjectured necessary and sufficient conditions for a CFT to possess a local bulk dual.

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Remark: Under these hypotheses, the corresponding bulk dual should possess a simultaneously weakly-coupled & weakly-curved regime.

In fact, the Mellin amplitude programme seems well adapted (but presently restricted) to the holographic reconstruction of bulk theories (or individual scattering processes) possessing a weakly coupled & curved limit.

Mellin amplitude programme

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+ The Mellin amplitude programme of rewriting CFT correlators as Witten diagrams seemingly applies to the large class of strongly-coupled CFTs obeying the criteria (1)-(3).
- However it does not apply directly to the simplest example of CFTs: free ones (or weakly coupled ones). In fact, the Mellin transform of correlators of free CFTs is sometimes not even well defined.

Mellin amplitude programme

Example: The scalar single-trace operator 4-point connected correlator for the free $O(N)$ model

$$\langle\mathcal{O}(y_1)\,\mathcal{O}(y_2)\,\mathcal{O}(y_3)\,\mathcal{O}(y_4)\rangle \propto \frac{1}{(y_{12}^2 y_{34}^2)^{d-2}}\left\{u^{\frac{d-2}{2}}+\left(\frac{u}{v}\right)^{\frac{d-2}{2}}+u^{\frac{d-2}{2}}\left(\frac{u}{v}\right)^{\frac{d-2}{2}}\right\}$$

is a sum of power functions of the two cross ratio u and v

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is a sum of power functions of the two cross ratio u and v.

But a power function $f(x)=x^\alpha$ does not have a well defined Mellin transform

$$M(z)=\int_0^\infty x^z\, f(x)\,\frac{dx}{x}\,.$$

Bulk locality and Mellin amplitudes

The second and third criteria

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among the conjectured necessary and sufficient conditions for a CFT to possess a *local* bulk dual, are violated by the simplest examples of CFTs: free ones.

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among the conjectured necessary and sufficient conditions for a CFT to possess a *local* bulk dual, are violated by the simplest examples of CFTs: free ones.

Higher-spin holography:
In fact, the bulk duals of free CFTs are conjectured to be higher-spin gravity theories.

(Konstein-Vasiliev-Zaikin, Sezgin-Sundell, Sundborg, Witten, Mikhailov, Klebanov-Polyakov, ...)

Higher-spin theories are indeed *non-local* in the restricted sense of locality

Higher-spin gravity and holography

Higher-spin holography

Basic idea behind the conjectured duality:
(Konstein-Vasiliev-Zaikin, Sezgin-Sundell, Sundborg, Witten, Mikhailov)

Free (or integrable) CFTs have an infinite number of higher-order conformal symmetries.

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Their spectrum contains an infinite tower of traceless conserved currents with unbounded spin (including spin two).

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$\overset{\text{AdS/CFT dictionary}}{\Longrightarrow}$

Free (or integrable) CFTs should be dual to "higher-spin gravity" theories whose spectrum contains an infinite tower of gauge fields with unbounded spin (including spin two).

Higher-spin holography: large-N vector model

"Simplest" example: The bulk dual of the *singlet sector* of *vector models* should be minimal higher-spin gravity (Klebanov-Polyakov).

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Boundary spectrum

- $O(N)$-vector
  - **Fundamental conformal scalar fields $\Delta = \frac{d-2}{2}$**

$$\phi^a \qquad (a = 1, 2, \dots, N)$$

- $O(N)$-singlets
  - **Bilinear ("Single-trace") operators**
    - *Scalar* $\Delta_0 = d - 2$: $\qquad \mathcal{O} = \frac{1}{\sqrt{2N}}\phi^a\phi^a$
    - *Conformal currents* $\Delta_s = s + d - 2$:

$$\mathcal{J}_{i_1 \dots i_s} = \phi^a \partial_{(i_1} \dots \partial_{i_s)} \phi^a + \cdots \qquad (s = 2, 4, 6, \dots)$$

Bulk spectrum

- Infinite tower of gauge fields of all even spins $(s = 0, 2, 4, 6, \dots)$

Higher-spin gravity

Higher-spin interactions in four (and higher) dimension appear to be generically

- quasi local in the sense that they possess a perturbative expansion (in powers of fields and their derivatives) where each individual term in the Lagrangian is local.
- non local in the sense that the total number of derivatives is unbounded. This is a corollary of:
  - **Metsaev bounds:** The number of derivatives appearing in an on-shell non-trivial cubic vertex is bounded from
    - above by the sum of the spins involved
    - below by the highest spin involved

    (Metsaev, 1991-2008)
  - **Higher-spin algebra:** The Jacobi identity requires a spectrum with an infinite tower of fields with unbounded spin. (Fradkin-Vasiliev, 1987; Boulanger-Ponomarev-Skvortsov-Taronna, 2013)

Bulk locality and Mellin amplitudes

Tantalising question:
What could be a mild replacement of the second and third criteria in

(1) a semi-classical limit: $N \gg 1$

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that could provide necessary and sufficient conditions for a CFT (including weakly-coupled ones) to possess a (mildly non-local) bulk dual (including higher-spin gravity theories)?

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Tentative answer:

(2') finite number of single-trace operators with conformal dimension below any fixed dimension

(3') analyticity of Mellin amplitudes for contact Witten diagrams

Bulk locality and Mellin amplitudes

The criteria (3') relaxes strict locality and replaces it with the milder requirement that the coefficients in the Taylor series expansion of the amplitude decrease fast enough in order to have an infinite radius of convergence.
In other words, the amplitude can be approximated, for any fixed accuracy, by a polynomial of sufficiently high degree, i.e. by a local interaction of sufficiently higher order.

Criterion of weak locality:
Interactions on flat/AdS spacetime are weakly local iff the amplitudes of the corresponding contact Feynman/Witten diagram are entire functions of Mandelstam invariants / Mellin variables.

Bulk locality of higher-spin gravity

Higher-spin interactions of the bulk theory holographically reconstructed from the $O(N)$ model appear to be

- weakly local in the previous sense (the Mellin amplitude of their contact Witten diagrams are entire functions of the Mellin variables).

This holds at cubic level due to Metsaev upper bound and should hold for the quartic self-interactions of the AdS scalar field due to general facts about Mellin amplitudes.

Remarks:

- These checks at cubic and quartic level appear very generic and therefore weak locality of bulk duals to CFTs satisfying (1)-(2')-(3') looks plausible.

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Remarks:

- It would be nice to be able to compare explicitly this definition of weak locality based on Mellin amplitudes to the recent proposals of
  - (Vasiliev, 2015) based on functional classes of star-product elements,
  - (Skvortsov & Taronna, 2015) based on classes of field redefinitions leaving Witten diagrams invariant.

Tests of bulk locality

Cubic interactions:
Individual cubic higher-spin interactions are local in the sense that any 3-pt contact Witten diagram with fixed external legs is a polynomial function of the Mellin variables.

This follows as a corollary from

Metsaev upper bound: For any triplet of spins, the number of derivatives in any relevant cubic vertex is bounded from above by the sum of the spins.

Tests of bulk locality

Quartic interactions:
The quartic self-interactions of the AdS scalar field dual to a single-trace scalar operator are weakly local in the sense that the 4-pt contact Witten diagram with four scalar external legs is an entire function of the Mellin variables.

This follows from two general facts about the decomposition of scalar single-trace operator 4-pt function in conformal blocks (at leading order in $1/N$):

1. A piece does *not* contain single-trace conformal blocks iff the associated Mellin amplitude *is* an entire function. (Penedones, 2010)

This first fact is a corollary of the properties of such Mellin amplitudes: they are meromorphic functions with simple poles arising only from single-trace operators.

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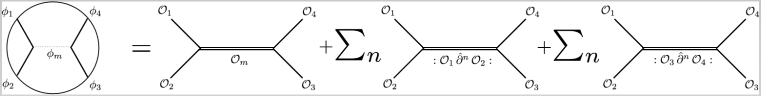
1. A piece does *not* contain single-trace conformal blocks iff the associated Mellin amplitude *is* an entire function. (Penedones, 2010)
2. Any single-trace conformal block can be accounted by a Witten exchange diagrams.

This second fact is based on AdS/CFT standard lore:

Conformal block decomposition

The conformal block decomposition in a given channel of the Witten diagram describing the exchange of a single AdS field in the same channel reproduces the conformal block for the dual single-trace operator, together with additional double-trace conformal blocks. (Hong Liu, 1998; ...)



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The single-trace conformal blocks are accounted by the Witten exchange diagrams, therefore the Witten contact diagram only contains double-trace conformal blocks, hence its Mellin amplitude must be an entire function.

Tests of bulk locality

These facts are expected to generalise (to all spins and to all orders) and suggest that the higher-spin interactions of the bulk dual to the $O(N)$ model might be weakly local.

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This picture is very suggestive but remains somewhat qualitative. The explicit holographic reconstruction of some quartic vertices in higher-spin gravity would provide a more concrete playground to test bulk locality.

Quartic AdS interactions from CFT

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Holographic reconstruction from free CFT:

- Compute the 3 and 4 pt conformal correlators via Wick contraction
- Write the most general ansatz for relevant cubic and quartic vertices
- Compute the corresponding exchange and contact amplitudes
- Fix the coefficients of vertices by matching the correlator with the total amplitude

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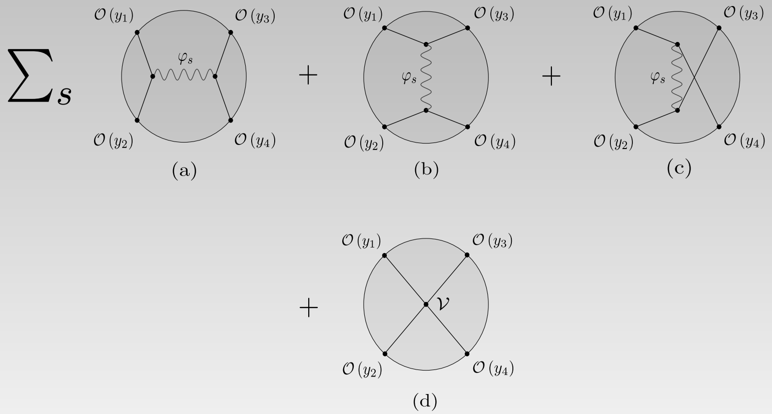
Remark: In principle, it is not guaranteed that a purely holographic reconstruction gives a result compatible with the Noether procedure. However, it is natural to expect that these two perturbative procedures are compatible since the Ward identities of the boundary CFT should be dual to the Noether identities of the AdS theory.

Quartic AdS interactions from CFT

Simplest non-trivial example

Holographic reconstruction of the quartic self-interaction of the AdS_4 scalar field in the higher-spin multiplet dual to the $d = 3$ free $O(N)$ model.



Quartic AdS interactions from CFT

Important technical simplifications for this example:

Scalar field:

- The bulk cubic vertex $s-0-0$ is of Noether type $\varphi_s J_s$:
  Gauge field φ_s $\times$ Conserved current $J_s = \varphi_0 (\nabla)^s \varphi_0 + \ldots$
  (Minkowski: Berends, Burgers, van Dam, 1986;
  Anti de Sitter: XB, Meunier, 2010)

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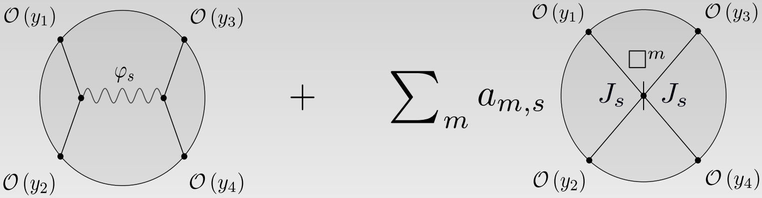
$\Longrightarrow$ The exchange and contact Witten diagrams are of the same type and can be easily compared for each spin s and in each channel.



Quartic AdS interactions from CFT

Important technical simplifications for this example:

Dimension $d = 3$:

A celebrated simplification of higher-spin holography in this case is that the AdS_{d+1} scalar in the higher-spin multiplet is conformal.

(because $d - 2 = \frac{d+1}{2} - 1 \Leftrightarrow d = 3$)

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$\Longrightarrow$ The conserved currents J_s bilinear in the AdS_4 scalar field can be assumed to be on-shell traceless. (Anselmi, 2000)

$\Longrightarrow$ The traces of the exchanged gauge fields in AdS_4 do not contribute to the amplitudes.

Quartic AdS interactions from CFT

Important technical simplifications for this example:

Identical bosonic operators/fields: The sum over the three distinct channels is equivalent to a mere Bose symmetrisation.

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Identical bosonic operators/fields: The sum over the three distinct channels is equivalent to a mere Bose symmetrisation.

$\Longrightarrow$ A formal holographic reconstruction in a single channel may lead to the correct result after suitable symmetrisation.

Quartic AdS interactions from CFT

Strategy:

In order to perform the holographic matching, it is convenient to write both sides in terms of the same building blocks.

Quartic AdS interactions from CFT

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In order to perform the holographic matching, it is convenient to write both sides in terms of the same building blocks.

- Conformal block decomposition
  $\Longleftrightarrow$ decomposition into irreps of conformal group

In the direct channel (12)(34)

$$\langle \mathcal{O}(y_1)\,\mathcal{O}(y_2)\,\mathcal{O}(y_3)\,\mathcal{O}(y_4)\rangle = \frac{1}{\left(y_{12}^2 y_{34}^2\right)^{d-2}}\left\{1+\sum_{\Delta,s} c_{\Delta,s}^2\, G_{\Delta,s}(u,v)\right\}$$

where $c_{\Delta,s}$ is the OPE coefficient in

$$\mathcal{O}\mathcal{O} \sim \mathbb{I} + \sum_{\text{primary}} c_{\Delta,s}\,\mathcal{O}_{\Delta,s} + \text{descendants}$$

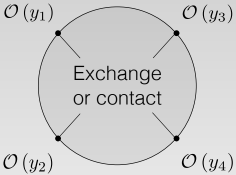
and $G_{\Delta,s}(u,v)$ is the conformal block for the operator $\mathcal{O}_{\Delta,s}$.

Quartic AdS interactions from CFT

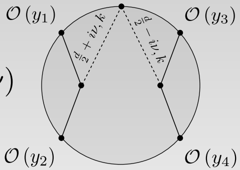
Strategy:

In order to perform the holographic matching, it is convenient to write both sides in terms of the same building blocks.

- Conformal block decomposition
  $\Longleftrightarrow$ decomposition into irreps of conformal group
- Contour integral representation
  The split representation allows to express Witten diagrams only in terms of boundary variables, as a contour integral.



$$= \sum_k \int_{-\infty}^{\infty} d\nu \; g_k(\nu)$$



(split representation of propagator: Leonhardt, Manvelyan, Ruhl, 2003)

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- Conformal block decomposition
  $\Longleftrightarrow$ decomposition into irreps of conformal group
- Contour integral representation
  $\Longrightarrow$ One should obtain the contour integral representation of the conformal block decomposition (also called conformal partial wave expansion) of the 4-point correlator.

$$\langle \mathcal{O}(y_1)\,\mathcal{O}(y_2)\,\mathcal{O}(y_3)\,\mathcal{O}(y_4)\rangle$$
$$= \frac{1}{\left(y_{12}^2 y_{34}^2\right)^{\Delta}} \left\{ 1 + \sum_s \int_{-\infty}^{\infty} d\nu \; f_s(\nu)\, G_{\frac{d}{2}+i\nu,s}(u,v) \right\}$$

(Dobrev, Petkova, Petrova, Todorov, 1976)

Quartic AdS interactions from CFT

Summary: Achieving the holographic reconstruction required to

- build on scattered results in the litterature:
  - Various former results on OPE and conformal block decomposition (Dolan & Osborn, 2001; Diaz & Dorn, 2006)
  - Holographic match of the cubic vertices $s - 0 - 0$ with the corresponding 3-pt correlators
    (Costa, Gonçalves, Penedones, 2014)
  - Basis of quartic vertices $0 - 0 - 0 - 0$ in *flat* spacetime
    (Heemskerk, Penedones, Polchinski, Sully, 2009)
  - Harmonic analysis and split representation of (transverse) traceless part of AdS higher-spin (gauge) field propagators
    (Leonhardt, Manvelyan, Ruhl, 2003;
    Costa, Gonçalves, Penedones, 2014)

Quartic AdS interactions from CFT

Summary: Achieving the holographic reconstruction required to

- overcome various technical hurdles:
  - Split representation of AdS *massless* higher-spin field propagators
  - OPE coefficients for the scalar *double-trace* operators in any d
  - Contour integral form of the conformal block expansion of 4-point
    - Conformal correlator of scalar single-trace operators
    - Exchange Witten diagrams
    - Contact Witten diagram
  - Summation over the three channels ("$\frac{1}{3}$ trick")

Boundary side

Relevant boundary operators

The boundary operators relevant for the present computation are:

- $O(N)$-vector
  - **Fundamental conformal scalar fields $\Delta = \frac{d-2}{2}$**

$$\phi^a \qquad (a = 1, 2, \ldots, N)$$

- $O(N)$-singlets
  - **Single-trace operators**
    - Scalar $\Delta_0 = d - 2$: $\quad \mathcal{O} = \frac{1}{\sqrt{2N}} \phi^a \phi^a$
    - Conformal currents $\Delta_s = s + d - 2$:

$$\mathcal{J}_{i_1 \ldots i_s} = \phi^a \partial_{(i_1} \ldots \partial_{i_s)} \phi^a + \cdots \qquad (s = 2, 4, 6, \ldots)$$

  - **Double-trace operators $\Delta_{n,s} = 2(d-2) + 2n + s$**

$$\mathcal{O}^{(2)}_{n, i_1 \ldots i_s} = \mathcal{O} \Box^n \partial_{(i_1} \ldots \partial_{i_s)} \mathcal{O} + \cdots \qquad (n = 0, 1, 2, \ldots)$$

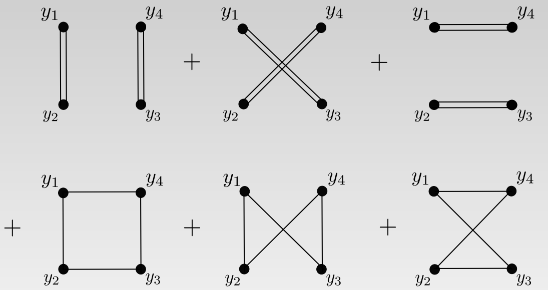
Four-point function of scalar single-trace operators

The full scalar single-trace operator 4-point function

$$\langle \mathcal{O}(y_1)\,\mathcal{O}(y_2)\,\mathcal{O}(y_3)\,\mathcal{O}(y_4)\rangle = \frac{1}{\left(y_{12}^2 y_{34}^2\right)^{d-2}} \times$$

$$\times \left\{ \left(1 + u^{d-2} + \left(\frac{u}{v}\right)^{d-2} \right) + \frac{4}{N} \left(u^{\frac{d}{2}-1} + \left(\frac{u}{v}\right)^{\frac{d}{2}-1} + u^{\frac{d}{2}-1}\left(\frac{u}{v}\right)^{\frac{d}{2}-1} \right) \right\}$$

is obtained via Wick contractions



Four-point function of scalar single-trace operators

The conformal block decomposition of the scalar single-trace operator 4-point function

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$$\times \left\{ 1 + \sum_s c_s^2\, G_{s+d-2,s}(u,v) + \sum_{n,s} c_{n,s}^2\, G_{\Delta_{n,s},s}(u,v) \right\}$$

can be determined from the OPE of the scalar single-trace operator

$$\mathcal{O}\mathcal{O} \;\sim\; \mathbb{I} + \sum_s c_s\, \mathcal{J}_s + \sum_{n,s} c_{n,s}\, \mathcal{O}^{(2)}_{n,s} \;+\; \text{descendants},$$

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The OPE coefficients

- c_s of the conformal current $\mathcal{J}_s$ were known (Dolan & Osborn, 2001; Diaz & Dorn, 2006)
- $c_{n,s}$ of the double-trace operator $\mathcal{O}^{(2)}_{n,s}$ were only known for $d = 4$ (Dolan & Osborn, 2001) so they had to be determined for $d = 3$.

Double-trace OPE coefficients

$$c_{n,s}^2 = \frac{\left[(-1)^s+1\right]2^s\left(\frac{d}{2}-1\right)_n^2(d-2)_{s+n}^2}{s!n!\left(s+\frac{d}{2}\right)_n(d-3+n)_n(2d+2n+s-5)_s\left(\frac{3d}{2}-4+n+s\right)_n}$$
$$\times\left(1+(-1)^n\frac{4}{N}\frac{\Gamma(s)}{2^s\Gamma\left(\frac{s}{2}\right)}\frac{\left(\frac{d}{2}-1\right)_{n+\frac{s}{2}}}{\left(\frac{d-1}{2}\right)_{\frac{s}{2}}(d-2)_{n+\frac{s}{2}}}\right)$$

As a preliminary result, the explicit form of the double-trace operator $\mathcal{O}^{(2)}_{n,s}$ had to be determined and is extremely complicated. The above generic form of the OPE coefficient $c_{n,s}$ remains a conjecture but it

- reproduces known results for
  - $d=4$ and $\forall n$, $\forall s$ (Dolan & Osborn, 2001)
  - $N=\infty$ and $\forall d$, $\forall n$, $\forall s$ (Fitzpatrick & Kaplan, 2011)
- was explicitly computed for
  - $s=0$ and $\forall d$, $\forall n$
  - $n=0,1$ and $\forall d$, $\forall s$

Contour integral representation

For each spin s, find a function $f_s(\nu)$ such that

$$\sum_{\Delta} c^2_{\Delta,s}\, G_{\Delta,s}(u,v) = \int_{-\infty}^{\infty} d\nu\, f_s(\nu)\, G_{\frac{d}{2}+i\nu,s}(u,v)$$

where one closes the contour in the lower-half plane.

It will turn out to be convenient to set

$$f_s(\nu) = p_s(\nu)\, \kappa_s(\nu)$$

where $p_s(\nu)$ is an even function of ν and

$$\kappa_s(\nu) = \frac{2^{-2i\nu+2s-3}\,\Gamma\left(i\nu+\frac{1}{2}\right)\Gamma\left(\frac{2s-2i\nu+1}{4}\right)^2\Gamma\left(\frac{2s+2i\nu+3}{4}\right)^2}{\pi^{5/2}\Gamma(i\nu)(2i\nu+2s+1)}$$

Contour integral representation

For the spin-s conformal current:

$$p_{\mathcal{J}_s}(\nu) = \frac{\pi\, 2^{8-s}}{N} \frac{1}{\nu^2 + (s-\frac{1}{2})^2} \frac{1}{\Gamma\left(\frac{2s-2i\nu+1}{4}\right)^2 \Gamma\left(\frac{2s+2i\nu+1}{4}\right)^2}.$$

For the spin-s double-trace operator contribution:

$$p_{\mathcal{O}^{(2)}_s}(\nu) = \frac{\pi^{\frac{3}{2}}\; 2^{s+4}\Gamma\left(s+\frac{3}{2}\right)}{\Gamma(s+1)\,\Gamma\left(s+\frac{1}{2}+i\nu\right)\Gamma\left(s+\frac{1}{2}-i\nu\right)}$$

$$+\frac{1}{N}\frac{(-1)^{\frac{s}{2}}\,\pi^{\frac{3}{2}}2^{s+4}\Gamma\left(s+\frac{3}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{s}{2}+1\right)\Gamma(s+1)\,\Gamma\left(\frac{3}{4}-\frac{i\nu}{2}\right)\Gamma\left(\frac{3}{4}+\frac{i\nu}{2}\right)\Gamma\left(s+\frac{1}{2}+i\nu\right)\Gamma\left(s+\frac{1}{2}-i\nu\right.}$$

Bulk side

Cubic vertices

Relevant cubic vertex

$$\mathcal{V}^{(3)} = \sum_s g_s \, \mathcal{V}^{(3)}_s$$

expanded in a basis of on-shell non-trivial cubic vertices

$$\mathcal{V}^{(3)}_s \; = \; \varphi_{\mu_1 \cdots \mu_s} \, J^{\mu_1 \cdots \mu_s} \qquad (s \in 2\mathbb{N})$$

where

$$J^{\mu_1 \cdots \mu_s}(x) \; = \; \varphi_0 \nabla_{\mu_1} \cdots \nabla_{\mu_s} \varphi_0 \, + \, \cdots$$

is a basis of on-shell conserved & traceless bilinears in the scalar field φ_0.

Cubic vertices

Compute the amplitude

$$\mathcal{A}^{\text{contact}}_{s+d-2,s}\left(y_1,y_2;y,z\right) \propto \frac{1}{\left(y_{12}^2\right)^{\frac{d}{2}-1}\left(y_{13}^2\right)^{\frac{d}{2}-1}\left(y_{23}^2\right)^{\frac{d}{2}-1}}\left(\frac{y_{13}\cdot z}{y_{13}^2}-\frac{y_{23}\cdot z}{y_{23}^2}\right)^s$$

by means of the bulk-to-boundary propagators, as in (Costa, Gonçalves, Penedones, 2014).

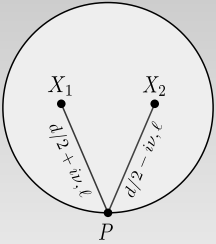
Fix the values of the coefficients g_s by imposing

$$\left\langle \mathcal{O}\left(y_1\right)\mathcal{O}\left(y_2\right)\mathcal{J}_s\left(y_3,z\right)\right\rangle = \mathcal{A}^{\text{contact}}_{s+d-2,s}\left(y_3,y_4;y,z\right)$$

Split representation of propagators

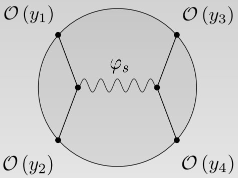
$$\Pi_s\left(X_1,u_1;X_2,u_2\right)\,\propto\,\sum_{k=0}^{\left[\frac{s}{2}\right]}\left(u_1^2\right)^k\left(u_2^2\right)^k\int_{-\infty}^{\infty}d\nu\;g_{s,k}\left(\nu\right)\times$$

$$\int_{\partial\mathsf{AdS}} d^dP\,\Pi_{\frac{d}{2}+i\nu,s-2k}(X_1,u_1;P,\hat{\partial}_z)\,\Pi_{\frac{d}{2}-i\nu,s-2k}\left(X_2,u_2;P,z\right)$$

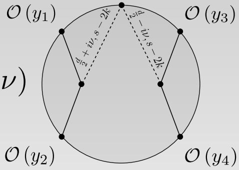


Split representation of exchange Witten diagrams

$$\mathcal{A}_s^{\text{exchange}}(y_1, y_2; y_3, y_4)$$
$$= \sum_{k=0}^{\left[\frac{s}{2}\right]} \int_{-\infty}^{\infty} d\nu \; g_{s,k}(\nu) \int_{\partial \text{AdS}} d^d y \; \mathcal{A}^{\text{contact}}_{\frac{d}{2}+i\nu, s}(y_1, y_2; y, \partial_z) \, \mathcal{A}^{\text{contact}}_{\frac{d}{2}-i\nu, s}(y_3, y_4; y, z)$$



$$= \sum_k \int_{-\infty}^{\infty} d\nu \; g_{s,k}(\nu)$$



Split representation of exchange Witten diagrams

d=3: only the term $k = 0$ in the sum

$$\mathcal{A}_s^{\text{exchange}}(y_1, y_2; y_3, y_4)$$
$$= \int_{-\infty}^{\infty} d\nu \, g_s(\nu) \int_{\partial \text{AdS}} d^3y \, \mathcal{A}^{\text{contact}}_{\frac{3}{2}+i\nu,s}(y_1, y_2; y) \, \mathcal{A}^{\text{contact}}_{\frac{3}{2}-i\nu,s}(y_3, y_4; y)$$
$$= \frac{1}{y_{12}^2 y_{34}^2} \int_{-\infty}^{\infty} d\nu \, \frac{1}{\nu^2 + (s-\frac{1}{2})^2} \, \kappa_s(\nu) \, G_{\frac{3}{2}+i\nu,s}(u,v)$$

Quartic vertices

Relevant quartic vertex

$$\mathcal{V} = \sum_{m,s} a_{m,s}\, \mathcal{V}_{m,s}$$

expanded in a basis of on-shell non-trivial quartic vertices

$$\mathcal{V}_{m,s} \;=\; J_{\mu_1\cdots\mu_s}\,\Box^m\, J^{\mu_1\cdots\mu_s} \qquad (s=2k,\quad k\geq m\geq 0,\quad k,m\in\mathbb{N})$$

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If we relax the bound $m \leqslant k$ (as will turn out to be convenient technically), then the set of vertices $\mathcal{V}_{m,s}$ remains a generating set of quartic vertices but they are no more independent on-shell.

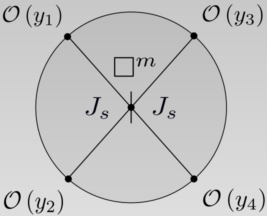
Remark: In principle, one should rewrite the final expression in terms of the genuine basis in order to compute the corresponding coefficients (similarly to the recent analysis of cubic vertices arising from Vasiliev equations by Boulanger, Kessel, Skvortsov and Taronna).

Split representation of contact Witten diagrams

$$\mathcal{A}_s^{\text{contact}}(y_1, y_2; y_3, y_4) = \sum_m a_{m,s}\mathcal{A}_{m,s}^{\text{contact}}(y_1, y_2; y_3, y_4)$$

where

$$\mathcal{A}_{m,s}^{\text{cont.}}(y_1, y_2; y_3, y_4) =$$



$$= \frac{1}{y_{12}^2 y_{34}^2}\int_{-\infty}^{\infty} d\nu\,\left(\nu^2 + s + \tfrac{9}{4}\right)^m \kappa_s(\nu)\, G_{\frac{3}{2}+i\nu,s}(u,v)$$

Quartic AdS interactions from CFT

Both the 4-point

1. correlator of scalar single-trace operators, and the
2. amplitudes of the previous s-channel Witten diagrams

have been expressed in the contour integral representation in terms of given spin conformal blocks in the direct channel (12)(34).

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Remaining obstacle:
The total amplitude is the sum over all channels (s, t and u).

It is very hard to reexpress this total amplitude in terms of a single channel. (Being able to rewrite a conformal block into another channel is essentially equivalent to solving the conformal bootstrap.)

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Trick: Perform a formal holographic matching in a single channel and make sure that the total amplitude after symmetrisation gives the correct result.

Quartic AdS interactions from CFT

Final solution: $\mathcal{V} = \sum_{s\in\mathbb{N}} \mathcal{V}_s$ with

$$\mathcal{V}_s \;=\; J_{\mu_1\cdots\mu_s}\, a_s\left(\Box\right)\; J^{\mu_1\cdots\mu_s}$$

where the generating functions

$$a_s\left(\nu^2+s+\tfrac{9}{4}\right) = \sum_{m=0}^{\infty} a_{m,s}\,\left(\nu^2+s+\tfrac{9}{4}\right)^m$$

$$\propto \frac{2^{8-s}}{\nu^2+(s-\frac{1}{2})^2}\left[\frac{\pi}{\Gamma\left(\frac{2s-2i\nu+1}{4}\right)^2\Gamma\left(\frac{2s+2i\nu+1}{4}\right)^2}-\frac{1}{\Gamma\left(s\right)^2}\right]$$

$$-\frac{(-1)^{\frac{s}{2}}\,\pi^{\frac{3}{2}}2^{s+5}\Gamma\left(s+\frac{3}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{s}{2}+1\right)\Gamma\left(s+1\right)\Gamma\left(\frac{3}{4}-\frac{i\nu}{2}\right)\Gamma\left(\frac{3}{4}+\frac{i\nu}{2}\right)\Gamma\left(s+\frac{1}{2}+i\nu\right)\Gamma\left(s+\frac{1}{2}-i\nu\right)}$$

are entire functions (though it may not be manifest).

Conclusions, results and perspectives

Conclusions

1. Feynman/Witten diagram-ology suggests to
   - weaken the definition of locality

   in order to
   - encompass interactions on flat/AdS spacetime for which the amplitudes of the corresponding contact Feynman/Witten diagrams are entire functions of Mandelstam invariants / Mellin variables.
2. At the level of 4-point scalar single-trace correlators and quartic bulk vertices, this
   - weak locality criterion

   is equivalent to the property that
   - Witten exchange diagrams account for all single-trace conformal blocks in the decomposition of the correlators.

   Therefore,
   - In the higher-spin gravity dual to the free $O(N)$ model, the quartic self-interaction of the AdS scalar field should be weakly local.

Main result

Explicit holographic reconstruction of the latter quartic vertex

Based on various technical intermediate results

- Split representation of AdS gauge fields propagators
- Holographic reconstruction of cubic vertices $s-0-0$
- Generating set of quartic vertices $0-0-0-0$
- OPE coefficients for the scalar double-trace operators
- Contour integral form of the conformal block expansion of four-point
    - Conformal correlator of scalar single-trace operators
    - Exchange Witten diagrams
    - Contact Witten diagram
- Summation over the three channels

Perspectives

1. Extend the holographic reconstruction to
   - spin $s \neq 0$ (use twistors)
   - boundary dimension $d \neq 3$
2. Compare explicitly with
   - Vasiliev higher-spin gravity
   - Mellin amplitude programme