Solutions in Extended Field Theories

Queen Mary University of London

based on
1403.7198, 1409.6314, 1412.2768
with Felix Rudolph

Key Points

- Introductions to DFT and EFT
- Solutions to Equations of Motions $\rightarrow$ supergravity objects
- Different view of string dualities

References for DFT

Origins, developments and contributions by many groups:
Imperial, MIT, Munich and Korea

Outline

Motivation

Defining Features

- Extended geometry by including dual directions
- Coordinates for momentum and winding modes of string
- Dualities become manifest
- Gauge fields are geometrized

Extended Field Theories

- Double Field Theory $O(D,D)$ - T duality
- Exceptional Field Theory E_d - U duality

Motivation

Kaluza-Klein Theory

- Massless, uncharged state in full theory
- States in reduced theory have mass and charge
- Given by momentum in KK direction

Example

- Null wave solution in M-theory gives D0-brane
- D0-brane is momentum mode in 11th direction
- Mass and charge given by momentum - BPS state

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Introduction to Double Field Theory

What does it achieve?

- Makes T-duality a manifest symmetry of the action
- Metric and B-field on equal footing - geometric unification
- Diffeos and gauge transformations $\rightarrow$ generalized diffeos (generated by generalized Lie derivative)

Geometric Framework

- Double dimensions to $2D$ by including winding coordinates
- Need strong constraint to pick D dimensions

Double Field Theory

Generalized coordinates

- Combine x^μ and $\tilde{x}_\mu$ into

$$X^M = (x^\mu, \tilde{x}_\mu)$$

- $\mu = 1, \ldots, D$ and $M = 1, \ldots, 2D$

Generalized metric

- Combine metric $g_{\mu\nu}$ and Kalb-Ramond field $B_{\mu\nu}$ into

$$\mathcal{H}_{MN} = \begin{pmatrix} g_{\mu\nu} - B_{\mu\rho} g^{\rho\sigma} B_{\sigma\nu} & B_{\mu\rho} g^{\rho\nu} \\ -g^{\mu\sigma} B_{\sigma\nu} & g^{\mu\nu} \end{pmatrix}$$

- Rescale the dilaton $\quad e^{-2d} = \sqrt{g} e^{-2\phi}$

The Doubled Space

$O(D,D)$ structure on doubled space

$$\eta_{MN} = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$$

Generalized metric parametrizes coset $O(D,D)/O(D) \times O(D)$

$$\mathcal{H}_{MK}\, \eta^{KL}\, \mathcal{H}_{LN} = \eta_{MN}$$

Strong constraint

$$\eta^{MN}\, \partial_M \Phi\, \partial_N \Psi = 0$$

The DFT Action

The action integral

$$S = \int \mathrm{d}^{2D} X e^{-2d} R$$

The "Ricci" scalar

$$\begin{aligned} R = &\frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_K\mathcal{H}_{NL} \\ &+ 4\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} \\ &- 4\mathcal{H}^{MN}\partial_M d\partial_N d + 4\partial_M\mathcal{H}^{MN}\partial_N d \end{aligned}$$

Equations of Motion

Since $\mathcal{H}$ is constrained, get projected EoMs

$$P_{MN}{}^{KL} K_{KL} = 0$$

where

$$K_{MN} = \delta R / \delta \mathcal{H}^{MN}$$

$$P_{MN}{}^{KL} = \frac{1}{2}\left(\delta_M{}^{(K}\delta_N{}^{L)} - \mathcal{H}_{MP}\eta^{P(K}\eta_{NQ}\mathcal{H}^{L)Q}\right)$$

Dilaton equation

$$R = 0$$

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The DFT Wave Solution

$$X^M = (t, z, y^m, \tilde{t}, \tilde{z}, \tilde{y}_m)$$

Generalized metric

$$\begin{aligned} \mathrm{d}s^2 &= \mathcal{H}_{MN}\mathrm{d}X^M\mathrm{d}X^N \\ &= (H-2)\left[\mathrm{d}t^2 - \mathrm{d}z^2\right] - H\left[\mathrm{d}\tilde{t}^2 - \mathrm{d}\tilde{z}^2\right] \\ &\quad + 2(H-1)\left[\mathrm{d}t\mathrm{d}\tilde{z} + \mathrm{d}\tilde{t}\mathrm{d}z\right] \\ &\quad + \delta_{mn}\mathrm{d}y^m\mathrm{d}y^n + \delta^{mn}\mathrm{d}\tilde{y}_m\mathrm{d}\tilde{y}_n \end{aligned}$$

Rescaled dilaton

$$d = const.$$

The DFT Wave Solution

Harmonic Function H

$$H(r) = 1 + \frac{h}{r^{D-4}}$$

$$r^2 = \delta_{mn} y^m y^n, \qquad h = const.$$

The DFT Wave Solution

Properties

- No mass, null-like [Blair '15, Park et al '15, Naseer '15]
- Carries momentum in $\tilde{z}$ direction
- Interprete as null wave in DFT
- Smeared over dual directions $\rightarrow$ obeys strong constraint

The Supergravity Picture

KK-Ansatz to remove dual directions

- Get fundamental string solution (F1-string)
- Extended along z
- Mass and charge given by momentum in $\tilde{z}$

If z and $\tilde{z}$ are exchanged

- Get pp-wave in z direction
- Expected as wave and string are T-dual

Key Result

The fundamental string is a massless wave in doubled space with momentum in a dual direction.

Goldstone Mode Analysis

[Cederwall et al. '98]

Zero modes

- Symmetry breaking
- Moduli $\rightarrow$ collective coordinates
- Generated by gauge transformations / diffeos
- Make local on worldvolume $\rightarrow$ get zero modes

Number of modes

- String: $D-2$ modes
- Doubled wave/string: ???

Constructing the Zero Modes

Transformations of $\mathcal{H}$ and d

$$h_{MN} = \mathcal{L}_\xi \mathcal{H}_{MN} \qquad \lambda = \mathcal{L}_\xi d$$

- gauge parameter $\xi^M = (0, H^\alpha \hat{\phi}^m, 0, H^\beta \hat{\tilde{\phi}}_m)$
- $\hat{\phi}^m$ and $\hat{\tilde{\phi}}_m$ are constant moduli

Allow dependece on $x^a = (t, z)$ to get zero modes

$$\hat{\phi}^m \to \phi^m(x) \qquad \hat{\tilde{\phi}}_m \to \tilde{\phi}^m(x)$$

Equations of motion

- Insert into DFT EoMs (two derivatives, first order)
- Find $\Box\phi = 0$ and $\Box\tilde{\phi} = 0$
- Also get self-duality relation for $\Phi^M = (0, \phi^m, 0, \tilde{\phi}_m)$

$$\mathcal{H}_{MN}\mathrm{d}\Phi^N = \eta_{MN} \star \mathrm{d}\Phi^N$$

Duality symmetric string in doubled space [Tseytlin '91]

- Can be written as (anti-)chiral equation for $\psi_\pm = \phi \pm \tilde{\phi}$

$$\mathrm{d}\psi_\pm = \pm \star \mathrm{d}\psi_\pm$$

Singularity at the core

Singularities are frame dependent [Duff, Khuri, Lu '92 & '95]

- Fundamental string is singular at core
- Solitonic fivebrane is non-singular
- pp-wave is smooth everywhere
- DFT wave also smooth

Is there a preferred frame?

DFT wave - $X^A = (t, z, \tilde{t}, \tilde{z})$

$$\mathcal{H}_{AB} = \begin{pmatrix} H-2 & 0 & 0 & H-1 \\ 0 & 2-H & H-1 & 0 \\ 0 & H-1 & -H & 0 \\ H-1 & 0 & 0 & H \end{pmatrix}$$

Generalized metric is fibred over r

- Diagonalize to find preferred/large dimensions
- For large r: no preferred set of coordinates (t, z) or $(\tilde{t}, \tilde{z})$
- Near the core: twisted light-cone coordinates $(\tilde{t} - z, t + \tilde{z})$ and $(t - \tilde{z}, \tilde{t} + z)$

The DFT Monopole Solution

$$X^M = (z, y^i, x^a, \tilde{z}, \tilde{y}_i, \tilde{x}_a), \quad i = 1, 2, 3$$

Generalized metric

$$\begin{aligned} ds^2 &= \mathcal{H}_{MN} dX^M dX^N \\ &= H(1 + H^{-2}A^2)dz^2 + H^{-1}d\tilde{z}^2 + 2H^{-1}A_i[dy^i d\tilde{z} - \delta^{ij}d\tilde{y}_j dz] \\ &\quad + H(\delta_{ij} + H^{-2}A_iA_j)dy^i dy^j + H^{-1}\delta^{ij}d\tilde{y}_i d\tilde{y}_j \\ &\quad + \eta_{ab}dx^a dx^b + \eta^{ab}d\tilde{x}_a d\tilde{x}_b \end{aligned}$$

Rescaled dilaton

$$e^{-2d} = H$$

The DFT Monopole Solution

Harmonic Function H

$$H(r) = 1 + \frac{h}{r}$$

Magnetic Potential

$$\partial_{[i} A_{j]} = \frac{1}{2} \epsilon_{ij}{}^{k} \partial_k H$$

$$r^2 = \delta_{ij} y^i y^j, \qquad h = const.$$

The DFT Monopole Solution

Properties

- Hopf fibration $S^2 \times S^1$
- Monopole circle in $\tilde{z}$ direction
- Interprete as monopole in DFT
- Smeared over dual directions $\rightarrow$ obeys strong constraint

The Supergravity Picture

KK-Ansatz to remove dual directions

- Get solitonic fivebrane solution (NS5-brane)
- Delocalized in z (infinte periodic array)
- Couples magnetically to $B_{iz} = A_i$

If z and $\tilde{z}$ are exchanged

- Get KK-monopole with S^1 in z direction
- Expected as monopole and fivebrane are T-dual

Key Result

The solitonic fivebrane is a monopole in doubled space with the monopole circle in a dual direction.

T Duality in DFT

Strong Constraint vs Section Condition

- Strong constraint: depend only on half the coordinates
- Section condition: set of coordinates for physical spacetime

Ambiguity in choice

- Isometries: fewer coordinate dependencies than required by strong constraint
- Different sections (frame choices) possible
- Get the different T-duality frames in supergravity

Solutions in DFT

DFT Wave = Fundamental Solution

- F1-string and pp-wave

DFT Monopole = Solitonic Solution

- NS5-brane and KK-monopole

Duality between Fundamental and Solitonic Solution

- Need electro-magentic or S-duality

$$\text{DFT} \rightarrow \text{EFT}$$

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Exceptional Field Theory

[Hohm, Samtleben '13]

Features

- M-Theory analogue of DFT
- Exceptional group E_d - U duality manifest
- Split 11-dim. supergravity

$$M^{11} = M^{11-d} \times M^{d} \longrightarrow M^{11-d} \times M^{\dim E_d}$$

- Extend by including membrane and fivebrane wrappings

$$TM^d \oplus \Lambda^2 T^* M^d \oplus \Lambda^5 T^* M^d \oplus (T^* M^d \otimes \Lambda^7 T^* M^d)$$

Exceptional Field Theory

Work with $d = 7$: E_7

- Fundamental representation $\mathbf{56}$: Y^M
- Adjoint representation $\mathbf{133}$: t_α
- Invariant symplectic form of $Sp(56) \supset E_7$

$$\Omega_{MN} = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$$

Field content

$$\{g_{\mu\nu}, \mathcal{M}_{MN}, \mathcal{A}_\mu{}^M, B_{\mu\nu\ \alpha}, B_{\mu\nu\ M}\}$$

$g_{\mu\nu}$	${\mathcal{A}_\mu}^N$
$\cdot$	$\mathcal{M}_{MN}$

External sector

- 4-dim. spacetime: metric $g_{\mu\nu}$

Internal sector

- 56-dim. exceptional extended space: generalized metric $\mathcal{M}_{MN}$
- parametrizes coset $E_7/SU(8)$

Cross-terms

- EFT vector potential ${\mathcal{A}_\mu}^M$ with self-dual field strength ${\mathcal{F}_{\mu\nu}}^M$

$$e = \sqrt{\det g_{\mu\nu}}$$

Action

$$S = \int \mathrm{d}^4x \mathrm{d}^{56}Y e \Big[\hat{R} + \frac{1}{48} g^{\mu\nu} \mathcal{D}_\mu \mathcal{M}^{MN} \mathcal{D}_\nu \mathcal{M}_{MN} + e^{-1}\mathcal{L}_{\mathrm{top}} - \frac{1}{8}\mathcal{M}_{MN}\mathcal{F}^{\mu\nu\ M}\mathcal{F}_{\mu\nu}{}^{N} - V(\mathcal{M}_{MN}, g_{\mu\nu})\Big]$$

Twisted self-duality

$$\mathcal{F}_{\mu\nu}{}^{M} = \frac{1}{2} e \epsilon_{\mu\nu\rho\sigma} \Omega^{MN} \mathcal{M}_{NK} \mathcal{F}^{\rho\sigma\ K}$$

Embedding Supergravity in EFT

Section constraint

- $(t_\alpha)^{MN}\partial_M\partial_N\Phi = 0$ and $(t_\alpha)^{MN}\partial_M\Phi\partial_N\Psi = 0$
- $\Omega^{MN}\partial_M\Phi\partial_N\Psi = 0$

Solution to section constraint

- Decomposition $\mathbf{56} \to \mathbf{7} + \mathbf{21} + \mathbf{7} + \mathbf{21}$
- Coordinates $Y^M = (y^m, y_{mn}, y_m, y^{mn})$
- Dependence $\partial^{mn} \to 0\,, \quad \partial^m \to 0\,, \quad \partial_{mn} \to 0$

Generalized metric

$$\mathcal{M}_{MN}(g_{mn}) = g^{1/2}\mathrm{diag}[g_{mn}, g^{mn,kl}, g^{-1}g^{mn}, g^{-1}g_{mn,kl}]$$

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The Self-dual EFT Solution

$$H(r) = 1 + \frac{h}{r}$$

External spacetime

- Coordinates: $x^\mu = (t, w^i)$ and $r^2 = \delta_{ij} w^i w^j$
- Metric: $g_{\mu\nu} = \text{diag}[-H^{-1/2}, H^{1/2}\delta_{ij}]$

Internal exceptional extended space

- Coordinates: $Y^M = (y^m, y_{mn}, y_m, y^{mn})$
- Generalized metric:

$$\mathcal{M}_{MN} = \text{diag}[H^{3/2}, H^{1/2}\delta_6, H^{-1/2}\delta_6, H^{1/2}\delta_{15}, \\ H^{-3/2}, H^{-1/2}\delta_6, H^{1/2}\delta_6, H^{-1/2}\delta_{15}]$$

The Self-dual EFT Solution

Vector Potential $\mathcal{A}_\mu{}^M$

$$\mathcal{A}_t{}^M = \frac{H-1}{H} a^M \qquad \mathcal{A}_i{}^M = A_i \tilde{a}^M$$

Dual vectors a^M and $\tilde{a}^M$ - give direction

$$\hat{a}^M = \Omega^{MN} \mathcal{M}_{NK} \hat{\tilde{a}}^K$$

Other fields trivial

$$B_{\mu\nu\ \alpha} = 0 \qquad B_{\mu\nu\ M} = 0$$

Properties

Charge in KK-theory

- Electric charge: $q_e = n/R_e$ $\qquad n, k \in \mathbb{Z}$
- Margnetic charge: $q_m = kR_m$
- Self-duality: $q_e = q_m \implies n/k = R_e R_m$

Radii given in terms of a^M and $\tilde{a}^M$

- $R_e = |a^M| = H^{3/4}$
- $R_m = |\tilde{a}^M| = H^{-3/4}$
- $R_e R_m = 1 \implies n = k$

The Supergravity Picture

Get supergravity solutions in $4+7$ split

- $\mathcal{M}_{MN}$ gives internal metric g_{mn}
- ${\mathcal{A}_t}^M$ and ${\mathcal{A}_i}^M$ give C_{mnk} and $C_{m_1 \dots m_6}$ or KK-vector
- $g_{\mu\nu}$ carries over to external metric

Pick a direction for a^M

- y^m: pp-wave — y_m: KK-monopole
- y_{mn}: M2-brane — y^{mn}: M5-brane

Example

The M2-brane

- Let a^M be in the y_{12} direction
- Then $\mathcal{A}_t{}^M$ becomes $\mathcal{A}_{t\ 12} = C_{ty^1y^2} = -(H^{-1} - 1)$
- And $\mathcal{A}_i{}^M$ becomes $\mathcal{A}_i{}^{12} = \frac{1}{5!}\epsilon^{1\ldots7}C_{iy^3\ldots y^7} = A_i$
- Also $\mathcal{M}_{MN}$ gives $g_{mn} = H^{1/3}\mathrm{diag}[H^{-1}\delta_2, \delta_5]$
- Combine with $g_{\mu\nu}$ to get

$$\mathrm{d}s^2 = H^{-2/3}[-\mathrm{d}t^2 + (\mathrm{d}y^1)^2 + (\mathrm{d}y^2)^2] + H^{1/3}[\mathrm{d}\vec{w}^2_{(3)} + \mathrm{d}\vec{y}^2_{(5)}]$$

The 1/2 BPS Spectrum of Supergravity

theory	solution	orientation	EFT vector	${\mathcal{A}_t}^M$	${\mathcal{A}_i}^M$
$D=11$	WM	y^m	${\mathcal{A}_\mu}^m$	KK-vector	dual graviton
	M2	y_{mn}	$\mathcal{A}_{\mu\ mn}$	C_3	C_6
	M2/M5	$*$	$*$	$C_3 \oplus C_6$	$C_6 \oplus C_3$
	M5	y^{mn}	${\mathcal{A}_\mu}^{mn}$	C_6	C_3
	KK7	y_m	$\mathcal{A}_{\mu\ m}$	dual graviton	KK-vector
$D=10$ Type IIA	WA	$y^{\bar{m}}$	${\mathcal{A}_\mu}^{\bar{m}}$	KK-vector	dual graviton
	D0	y^θ	${\mathcal{A}_\mu}^\theta$	C_1	C_7
	D2	$y_{\bar{m}\bar{n}}$	$\mathcal{A}_{\mu\ \bar{m}\bar{n}}$	C_3	C_5
	F1	$y_{\bar{m}\theta}$	$\mathcal{A}_{\mu\ \bar{m}\theta}$	B_2	B_6
	KK6A	$y_{\bar{m}}$	$\mathcal{A}_{\mu\ \bar{m}}$	dual graviton	KK-vector
	D6	y_θ	$\mathcal{A}_{\mu\ \theta}$	C_7	C_1
	D4	$y^{\bar{m}\bar{n}}$	${\mathcal{A}_\mu}^{\bar{m}\bar{n}}$	C_5	C_3
	NS5	$y^{\bar{m}\bar{\theta}}$	${\mathcal{A}_\mu}^{\bar{m}\theta}$	B_6	B_2
$D=10$ Type IIB	WB	$y^{\bar{m}}$	${\mathcal{A}_\mu}^{\bar{m}}$	KK-vector	dual graviton
	F1 / D1	$y_{\bar{m}\ a}$	$\mathcal{A}_{\mu\ \bar{m}\ a}$	B_2 / C_2	B_6 / C_6
	D3	$y_{\bar{m}\bar{n}\bar{k}}$	$\mathcal{A}_{\mu\ \bar{m}\bar{n}\bar{k}}$	C_4	C_4
	NS5 / D5	$y^{\bar{m}\ a}$	${\mathcal{A}_\mu}^{\bar{m}\ a}$	B_6 / C_6	B_2 / C_2
	KK6B	$y_{\bar{m}}$	$\mathcal{A}_{\mu\ \bar{m}}$	dual graviton	KK-vector

Wave vs Monopole

Combine à la Kaluza-Klein

$$\mathcal{H}_{\hat{M}\hat{N}} = \begin{pmatrix} g_{\mu\nu} + \mathcal{A}_\mu{}^M \mathcal{A}_\nu{}^N \mathcal{M}_{MN} & \mathcal{A}_\mu{}^M \mathcal{M}_{MN} \\ \mathcal{M}_{MN} \mathcal{A}_\nu{}^N & \mathcal{M}_{MN} \end{pmatrix}$$

Find wave and monopole sector

$$\mathcal{H}_{\hat{M}\hat{N}} = \begin{pmatrix} H^{1/2} \mathcal{H}^{\text{wave}}_{AB} & 0 \\ 0 & H^{-1/2} \mathcal{H}^{\text{mono}}_{\bar{A}\bar{B}} \end{pmatrix}$$

Summary

Solutions in DFT

- Wave in winding direction gives string
- Monopole in winding direction gives fivebrane
- Isometries $\to$ ambiguity $\to$ T duality

Solutions in EFT

- One self-dual solution gives full 1/2 BPS spectrum
- Orientation in extended space determines supergravity object
- Isometries $\to$ ambiguity $\to$ U duality

Bound States

M2/M5 System

- EFT solution in mixed direction - parameter ξ

$$a^M_{(\mathrm{M2/M5})} = \sin\xi\, a^M_{(\mathrm{M2})} + \cos\xi\, a^M_{(\mathrm{M5})}$$

- Recover metric and C-fields for bound state

[Papadopoulos, Townsend '96]

Winding coordinate dependence

Monopole localised in winding space

- On $\mathbb{R}^3$: $H(r) = 1 + \frac{h}{r}$
- On $\mathbb{R}^4$: $H(r,z) = 1 + \frac{h}{r^2+z^2}$
- On $\mathbb{R}^3 \times S^1$:

$$H(r,z) = 1 + \frac{h}{2Rr}\frac{\sinh r/R}{\cosh r/R - \cos z/R}$$
$$= 1 + \frac{h}{2Rr}\sum_{k=-\infty}^{\infty} e^{-|k|\frac{r}{R}+ik\frac{z}{R}}$$

[Gregory, Harvey, Moore '97, Jensen '11]

Membrane winding modes?

On $\mathbb{R}^5$:

$$H(r, z_1, z_2) = 1 + \frac{h}{[r^2 + z_1^2 + z_2^2]^{3/2}}$$

On $\mathbb{R}^3 \times T^2$:

$$H(r, z, \bar{z}; \tau) = 1 + \frac{h}{r} \sum_{m,n} \sum_{\pm} \exp\left\{ \frac{\pi}{\tau_2} \Big[z(m + \bar{\tau} n) - \bar{z}(m + \tau n) \pm i r \sqrt{2(m + n\tau)(m + \bar{\tau} n)} \Big] \right\}$$

where $\tau = \tau_1 + i\tau_2$ and $z = z_1 + \tau z_2$