Towards Weakly Constrained Double Field Theory

An associative subsector of full WDFT

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Duality and Novel Geometry in M-theory

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Based on arXiv:1509.06973 and ongoing work

- Consider $\mathbb{R}^{2,2}$ with metric signature $++--$. For a spin 0 field, massless equation is given by

$$\Box f = 0$$

Momentum vector should satisfy null condition $p_\mu p^\mu = 0$.

- Lorentz transform is given by SO$(2,2)$ and it is locally isomorphic with SL$(2,\mathbb{R}) \times$ SL$(2,\mathbb{R})$. Then all the SO$(2,2)$ vectors can be replaced by spinor indices using gamma matrices. Null momentum p_μ is represented by two real spinors

$$p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}\,, \qquad p_\mu p^\mu = \det p = 0\,.$$

- Fourier transform

$$f(x^\mu) = \int d^4p \tilde{f}(p_\mu) e^{ip\cdot x} \delta(p^2) = \int d^2\lambda d^2\tilde{\lambda}\; \tilde{f}(\lambda, \tilde{\lambda}) e^{i\lambda x \tilde{\lambda}}$$

- For $\tilde{f}(\lambda, \tilde{\lambda})$ introduce a Fourier transform with respect to $\tilde{\lambda}$ only [Witten, 2004]

$$\tilde{f}(\lambda, \tilde{\lambda}) = \int \mathrm{d}^2\mu \tilde{f}(\lambda, \mu) e^{i\tilde{\lambda}_{\dot{a}}\mu^{\dot{a}}}$$

the (λ, μ) are twistor variables.

- Then the massless field is written as

$$f(x^\mu) = \int \mathrm{d}^2\lambda \mathrm{d}^2\mu \tilde{f}(\lambda, \mu)\delta^2(\mu + x\lambda)$$

This equation is so called Penrose transformation . Here the $\mu + x\lambda = 0$ is called incidence relation, which defines 2-dimensional null surfaces within $\mathbb{R}^{2,2}$

- Grassmannian: The space of all k-planes in an n-dimensional space $\mathbb{R}^n$ is called the Grassmannian Gr(k,n). For the $\mathbb{R}^{2,2}$ with $O(2,2)$ metric signature we can define null Grassmannian Gr$_0(k,n) \subset$ Gr(k,n), which is the space of all null planes.
- We can specify a k-plane in n dimensions by giving $k \times n$ matrices ${\Pi_i}^I \in \mathbb{R}^{k\times n}$, whose span defines the plane.
- For Gr$_0(2,4)$, null-planes in 4-dimension, ${\Pi_i}^\mu$ is determined by twistor variable λ_a as

$$
{\Pi_i}^\mu (\gamma_\mu)^{\dot{a}a} \longrightarrow {\Pi_{\dot{b}}}^{\dot{a}a} = {\delta_{\dot{b}}}^{\dot{a}} \lambda^a
$$

- Generalization to d-dimensional plane on a T^{2d} with $O(d,d)$ metric is straightforward.

- Comparison between DFT and Penrose transform

DFT	Penrose transform
weakly constrained fields	massless fields
level matching constraint	wave equation
section condition	light cone

Difficulties

- Physical states should satisfy level matching constraint

$$(L_0 - \bar{L}_0)|\text{phys}\rangle = 0\,,$$

$$4\alpha'\left(n_a w^a + N - \bar{N}\right) = 0\,.$$

- Assume that $N = \bar{N} = 1$.

$$\partial_i\tilde{\partial}^i(f \cdot g) = \partial_i f\tilde{\partial}^i g + \partial_i g\tilde{\partial}^i f \neq 0$$

- Prescription: Requiring strong constraint.

- Tachyon vertex operator with winding modes is

$$V_{k_L,k_R}(z,\bar{z}) =: e^{i\left(k_L\cdot X_L(z,\bar{z})+k_R\cdot\tilde{X}_R(z,\bar{z})\right)}:$$

with OPE

$$V_{k_L,k_R}(z_1,\bar{z}_1)V_{k'_L,k'_R}(z_2,\bar{z}_2) \sim z_{12}^{\alpha' k_L k'_L/2}\bar{z}_{12}^{\alpha' k_R k'_R/2}V_{(k+k')_L,(k+k')_R}(z_2,\bar{z}_2)$$

Under the interchange $1 \leftrightarrow 2$ and momentum $k \leftrightarrow k'$, the lefthand side is symmetric but a sign factor arises on the righthand side

$$\exp[\pi i(nw' + wn')]$$

- Vertex operator requires an additional sign factor

$$C(k,\hat{P}) = \exp[\pi i(k_L - k_R)(\hat{P}_L + \hat{P}_R)\alpha'/4]$$

- When we take a field theory limit, this factor should be disappeared!

- Main issue of this talk

$$\left.\begin{array}{l}(1)\ \text{Level matching constraint}\\(2)\ \text{Cocycle factor}\end{array}\right\}\Longrightarrow \text{[K.L 2015]}$$

(3) Consistent field theory

Radon (X-ray) transform on a torus

- Consider a doubled torus T^{2d} with periodic coordinates X^I

$$X^I \sim X^I + 1\,, \qquad I = 1, 2, \cdots, 2d$$

$$X^I = \begin{pmatrix} x^i \\ \tilde{x}_i \end{pmatrix}\,, \qquad i = 1, 2 \cdots, d$$

- $I, J, \cdots$ are $\mathbf{O}(d, d)$ vector indices with a $\mathbf{O}(d, d)$ metric

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- A closed d-dimensional plane $\mathcal{D}(X^I, \Pi)$ on a T^{2d} passing through a point $X^I \in T^{2d}$ is parametrized as

$$\mathcal{D}(X^I, \Pi) = \{X^I + t_i \Pi^{iI} | 0 \leq t_i < 1 \text{ and } \Pi \in \mathcal{P}_d\}$$

$\mathcal{P}_d$ is a set of $d \times 2d$ integer matrices of rank d, whose Smith normal form is

$$\Pi = L D_0 V$$

where $L \in PSL(d, \mathbb{Z})$, $V \in PSL(2d, \mathbb{Z})$ and $D_0 = (\mathbb{1}_d \; 0_d)$

- $\mathcal{P}_d$ -> Grassmannian $G(d, 2d)$

the closed d-dimensional plane is defined as a section or cutting plane of T^{2d}, and the Π determines how to slice.

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- A closed d-dimensional *null*-plane is parametrized

$$\mathcal{D}^0(X^I,\Pi)=\{X^I+t_i\Pi^{iI}|0\le t_i<1 \text{ and } \Pi\in\mathcal{P}_d^0\}$$

$\mathcal{P}_d^0$ is a subset of the $\mathcal{P}_d$ such that for an arbitrary element $\Pi\in\mathcal{P}_d^0$, the row vectors Π^i are mutually orthogonal and null

$$\Pi^i{}_I\mathcal{J}^{IJ}(\Pi^t)_J{}^j=0$$

Since the tangent vectors for $\mathcal{D}^0(X^I,\Pi)$ are Π^i, it is a null-plane.

- For $\Pi\in\mathcal{P}_d^0$, the Smith normal form of Π is given by

$$\Pi=LD_0V$$

where $L\in PSL(d,\mathbb{Z})$ and $V\in\mathbf{O}(d,d;\mathbb{Z})$.

- Note that the parametrization of d-plane is not unique, but there is a $PSL(d,\mathbb{Z})$ equivalence relation

$$\Pi^i{}_I \sim a^i{}_j \Pi^j{}_I \,, \qquad a^i{}_j \in PSL(d,\mathbb{Z})$$

- If two slicing matrices Π' and Π are related by $PSL(d,\mathbb{Z})$ rotation, then they parametrize the same d-plane because the $a \in PSL(d,\mathbb{Z})$ can be absorbed into the parameter t^i by redefining $t'_i = t_j a^j{}_i$.

- Radon (X-ray) transform on a torus is an integral transform mapping a continuous function $f(X^I)$ on a T^{2d} to the integrals of this function over the d-dimensional closed planes $\mathcal{D}(X^I, \Pi)$

$$\mathcal{R}f(X^I; \Pi) = \int_0^1 \cdots \int_0^1 \mathrm{d}t_1 \cdots \mathrm{d}t_d f\left(X^I + t_i \Pi^{iI}\right)$$

  where X^I is a point on the T^{2d} and $\Pi^{iI} \in \mathcal{P}_d$.
- X-ray transform for T^{2d} is an injective mapping, and it is possible to define the inverse transformation [Abouelaz, Rouviere, 2011]
- In general, the X-ray transform can be applied to any continuous functions, but we will focus only on weakly constrained fields.

- Let us consider a null plane wave $e_K = e^{2\pi i K_I X^I}$ with an integer momentum K_I satisfying
$$K_I K^I = 0$$
- Then the t integrals in X-ray transform can be done for the e_K trivially
$$\mathcal{R}\, e_K(X^I;\Pi) = \int \mathrm{d}^d t\, e^{2\pi i K_I(X^I + t_i \Pi^{iI})} = e^{2\pi i K_I X^I} \int \mathrm{d}^d t\, e^{2\pi i K_I t_i \Pi^{iI}}$$
$$= e_K\, \delta_{\Pi^{iI} K_I, 0}$$

- Then we have two constraints on K^I for a given Π:

$$(1) \quad \Pi^{iI} K_I = 0\,, \qquad i = 1 \cdots d$$
$$(2) \quad K_I K^I = 0$$

- The first constraint eliminates d degrees of freedom of K^I. Thus K^I is expanded by d-momentum ℓ_i

$$K_I = \ell_i \Psi^i{}_I$$

where $\Psi^i{}_I$ is a $d \times 2d$ integer valued matrix of rank d.

- From the second condition, the row vectors of Ψ^i should be mutually null and orthogonal vectors

$$\Psi^i{}_I \mathcal{J}^{IJ} \Psi^j{}_J = 0$$

and the Ψ^i become a basis of a maximal null subspace N

- Also Ψ and Π are orthogonal by the (1)

- Recall that the orthogonal complement of a maximal null subspace N is identical with itself, $N = N_\perp$.
- Since Π generates $N_\perp$, we can identify Π and Ψ without loss of generality. Then the doubled momentum K_I is represented by

$$K_I = \ell_i \Pi^i{}_I\,, \qquad \text{and} \qquad \Pi^i{}_I \mathcal{J}^{IJ} \Pi^i{}_J = 0$$

  Thus Π defines a null d-dimensional plane $\mathcal{D}^0(X^I, \Pi \in \mathcal{P}^0_d)$.
- The X-ray transform of the e_K can be rewritten by d-dimensional momenta ℓ_i

$$\mathcal{R}e_K(X^I; \Pi^i) = e^{2\pi i \ell_i \Pi^i{}_I X^I} = e^{2\pi i \ell_i z^i}\,, \qquad z^i = \Pi^i{}_I X^I$$

- After X-ray transform, the Fourier basis e_K on T^{2d} reduces to a Fourier basis of d-dimensional null plane defined by $\Pi^i{}_I$.

- To get a X-ray transform for an arbitrary function $f(X^I)$, we carry out Fourier expansion and use the previous result $\mathcal{R}e_K(X^I;\Pi)$

$$\mathcal{R}f(z^i;\Pi^i) = \sum_{K\in\mathbb{Z}^{2d}} \tilde{f}_K e^{2\pi i K_I X^I} \delta_{\Pi^{iI}K_I,0}$$

$$= \sum_{l_i} \tilde{f}'_{l_i} e^{2\pi i l_i z^i} ,$$

where $\tilde{f}'_{l_i} = \tilde{f}_{l_i \Pi^i{}_I}$, and it is reduced to the usual d-dimensional Fourier expansion. This is known as Fourier slice theorem.

- The X-ray transform maps a $2d$-dimensional weakly constrained field to a d-dimensional strongly constrained field on a d-dimensional null plane.

- Inverse X-ray transform : Reconstruction of the original $2d$-dimensional weakly constrained field $f(X^I)$ in terms of d-dimensional strongly constrained fields $\mathcal{R}f(z^i;\Pi)$ [Abouelaz, 2011]

$$f(X^I) = \sum_{\Pi \in \mathcal{P}_d^0} \varphi(\Pi)\hat{f}_\Pi(z^i)$$

where $\varphi(\Pi)$ is a weight factor for convergence of this series

$$\varphi(\Pi^i) = \exp(-\|\Pi\|^2) = \exp(-\sum_{i,I}(\Pi^i{}_I)^2)$$

- The $\hat{f}_\Pi(z^i)$ is defined in terms of $\mathcal{R}f(z^i;\Pi)$

$$\hat{f}_\Pi(z^i;\Pi^i) = \int_{T^{2d}} \mathrm{d}^{2d}Y \sum_K \tfrac{1}{\psi(K)}\mathcal{R}f(\Pi^i{}_I Y^I)e^{2\pi i K_I(X^I - Y^I)}$$

$$= \tfrac{1}{\psi(0)}\mathcal{R}f(z^i;\Pi)$$

- Each $\hat{f}_{\Pi}(z^i)$ is strongly constrained field on a null plane $\mathcal{D}^0(X^I, \Pi)$. Hence,

Weakly constrained fields can be represented as a collection of strongly constrained fields through inverse X-ray transform.

Binary operations for weakly constrained fields

Binary operations for weakly constrained fields

- Weakly constrained fields form the kernel K of the level matching constraint

$$L_0 - \bar{L}_0 = \partial_I \partial^I$$

- The K is not closed by ordinary product. For arbitrary $f, g \in K$,

$$f \cdot g \notin K$$

- Q: How we can define a binary operation which is compatible with level matching constraint?

$$f \circ g \in K$$

- Using the inverse X-ray transform, the $f \cdot g$ is represented as

$$f \cdot g = \sum_{\Pi, \Pi' \in \mathcal{P}_d^0} \varphi(\Pi)\varphi(\Pi')\hat{f}_{\Pi}(z^i)\hat{g}_{\Pi'}(z'^i)$$

- To find an additional condition which makes the ordinary product become compatible with level matching constraint, we act the level matching operator $\partial_I \partial^I$ to the product

$$\partial_I \partial^I (f \cdot g) = 2\partial_I f \partial^I g = 2 \sum_{\Pi, \Pi' \in \mathcal{P}_d^0} \varphi(\Pi)\varphi(\Pi')\Pi^i{}_I \Pi'^{jI} \frac{\partial \hat{f}_{\Pi}}{\partial z^i} \frac{\partial \hat{g}_{\Pi'}}{\partial z'^j} ,$$

- A simple and natural way to vanish the right-hand side is to impose an orthogonality condition on the slicing matrices

$$\Pi^i{}_I \mathcal{J}^{IJ} \Pi'^j{}_J = 0 .$$

- Now we assume that Π and Π' are orthogonal.
- Since the row vectors Π^i define a maximal null subspace, their orthogonal complement is identical with the original maximal null subspace. Thus the Π'^i is represented by a linear combination of Π^i

$$\Pi'^i{}_I = a^i{}_j \Pi^j{}_I \,, \qquad a^i{}_j \in PSL(d;\mathbb{Z})$$

- By the equivalence relation, $\mathcal{D}^0(X^I;\Pi)$ and $\mathcal{D}^0(X^I;a\Pi)$ are identical. Then the X-ray image fields $\hat{f}_\Pi$ and $\hat{g}_{\Pi'}$ live on the same plane.

- Moreover, we can absorb the ${a^i}_j$ into the momenta ℓ_i , which is define by the relation $K_I = \ell_i {\Pi^i}_I$ in the Fourier expansion, by redefining ℓ'_i

$$\ell''_i = \ell'_j {a^j}_i$$

- Without loss of generality, we can always identify Π and Π' if we assume Π and Π' are orthogonal.
- We define a novel binary operation $\circ$ as a product in the space of weakly constrained fields:

$$f(X^I) \circ g(X^I) = \sum_{\Pi \in \mathcal{P}_d^0} \varphi(\Pi) \hat{f}_\Pi(z^i) \cdot \hat{g}_\Pi(z^i) \, .$$

cf. with ordinary product

$$f \cdot g = \sum_{\Pi, \Pi' \in \mathcal{P}_d^0} \varphi(\Pi) \varphi(\Pi') \hat{f}_\Pi(z^i) \hat{g}_{\Pi'}(z'^i)$$

- We can show that the $\circ$-product satisfy the following algebraic properties:
  - Commutativity

$$f \circ g = g \circ f$$

  - Associativity

$$f \circ (g \circ h) = (f \circ g) \circ h$$

  - Distributivity

$$f \circ (g + h) = f \circ g + f \circ h$$

In addition we can define an identity I satisfying $I \circ f = f \circ I = f$

$$I = \sum_{\Pi \in \mathcal{P}_d^0} \varphi(\Pi) \cdot 1$$

- Leibniz rule

$$\partial_I (f \circ g) = \partial_I f \circ g + f \circ \partial_I g$$

- Hull and Zwiebach defined a projector by inserting an operator $\delta_{L_0-\bar{L}_0,0}$ within the Fourier expansion of a function to satisfy level matching constraint. For massless fields, $N=\bar{N}=1$, the $\delta_{L_0-\bar{L}_0,0}$ is represented as

$$\delta_{L_0-\bar{L}_0,0}=\delta_{\partial_I\partial^I,0}$$

  and the projector is defined for an arbitrary field f

$$[\![f]\!]=\sum_{K^I\in\mathbb{Z}^{2d}}\delta_{K_IK^I,0}\tilde{f}_K e^{2\pi i K_I X^I}$$

- It is obvious that $[\![f]\!]$ satisfy

$$\partial_I\partial^I[\![f]\!]=0\,.$$

- The projector for the usual product of two weakly constrained fields f and g is given by

$$[\![f \cdot g]\!] = \sum_{K^I, K'^I} \delta_{K_I K'^I, 0} \tilde{f}_K \tilde{g}_{K'} e^{2\pi i (K+K')_I X^I}$$

where K and K' are null vectors.

- One can show that the strong constraint is automatically satisfied

$$[\![\partial_I f \cdot \partial^I g]\!] = 0$$

and it is commutative

$$[\![fg]\!] = [\![gf]\!]$$

but not associative

$$[\![[\![fg]\!]h]\!] \neq [\![[\![gh]\!]f]\!] \neq [\![[\![hf]\!]g]\!] \neq [\![fgh]\!]$$

- We can rewrite the projector of two weakly constrained fields by using an inverse X-ray transform instead of the Fourier expansion

$$
\begin{aligned}
[\![f \cdot g]\!] &= \sum_{\Pi,\Pi' \in \mathcal{P}_d^0} \varphi(\Pi)\varphi(\Pi')\, \delta_{\partial_I \partial^I,0}\, \hat{f}_\Pi(z^i)\hat{g}_{\Pi'}(z') \\
&= \sum_{\Pi,\Pi' \in \mathcal{P}_d^0} \varphi(\Pi)\varphi(\Pi') \sum_{\ell,\ell'} \delta_{\ell_i \Pi^i{}_I \ell'_j \Pi'^{jI},0}\, \tilde{\hat{f}}_{\Pi,\ell}\, \tilde{\hat{g}}_{\Pi',\ell'}\, e^{2\pi i(\ell_i \Pi^i{}_I + \ell'_j \Pi'^j{}_I) X^I}\,,
\end{aligned}
$$

- In order to make sense the Kronecker-delta we impose a vanishing condition

$$
\ell_i \ell'_j\, \Pi^i{}_I \Pi'^{jI} = 0\,.
$$

- If Π and Π' are orthogonal, this condition is satisfied trivially. It corresponds to $\circ$-product.
- Nevertheless Π and Π' are not orthogonal, it is possible to satisfy due to Fourier zero-modes.

- Example: For $O(2,2)$ case, if we assume that the $\hat{f}_\Pi$ is depend only on z^2, $\hat{f}_\Pi(z^2)$, and $\hat{g}_{\Pi'}$ is depend only on z'^1, $\hat{g}_{\Pi'}(z'^1)$, then the ℓ_2 and ℓ'_1 are remained and $\ell_1 = \ell'_2 = 0$.
- If we denote $t^{ij} = \Pi^i{}_I \Pi'^{jI}$ and assume that

$$t^{21} = 0$$

then the $\ell_2 t^{21} \ell'_1$ vanish. The other elements also vanish due to the zero-modes

$$\ell_1 t^{11} \ell'_1 = \ell_1 t^{12} \ell'_2 = \ell_2 t^{22} \ell'_2 = 0$$

the zero mode contribution is missing in $\circ$-product.

- Therefore, we can separate HZ projector, $[\![f \cdot g]\!]$, into the associative part and the non-associative part as

$$[\![f \cdot g]\!] = f \circ g + f \star g \,,$$

- The $\star$-product represents the non-associative part but satisfies level matching constraint

$$\partial^I f \star \partial_I g = 0$$

- $\circ$-product implies when we consider OPE between two vertex operators, the momenta should be located on a same plane.

$$\ell_i \Pi_i^I = \begin{pmatrix} n_i \\ w^i \end{pmatrix}, \qquad \ell'_i \Pi_i^I = \begin{pmatrix} n'_i \\ w'^i \end{pmatrix}$$

- Then the unwanted factor which arises in two OPEs with different ordering is automatically disappeared

$$\exp[\pi i (n w' + w n')] = \exp[\pi i (\ell_i \ell'_j \Pi^{iI} \Pi^j{}_I)] = 1$$

Thus we don't need any cocycle factor for $\circ$-product.

Associative subsector of Weakly Constrained Double Field Theory

- Q: Under what conditions do we expect the action to give a reasonable description of the massless degrees of freedom of string theory?
- Massive tower of massive string sates: $m_s \simeq 1/\sqrt{\alpha'}$
  Kaluza-Klein momentum modes: $m_{KK} \simeq 1/R$
  string winding modes with $m_w \simeq R/\alpha'$.
- For manifest T-duality, we should treat momentum and winding modes on an equal footing. Thus the compactification scale should be of order of self-dual radius $R \simeq \sqrt{\alpha}$.
- The all the mass scales are of the same order, $m_s \simeq m_{KK} \simeq m_w$.
- There is no mass hierarch! there is no specific limit which truncates the massive string states.
- A: There is no such a condition.

- Recall electroweak subsector of standard model. Even if we cannot ignore strong interaction in general, we can focus only on electroweak subsector as a well-defined independent theory.
- If we turn off $SU(3)$ gauge symmetry and gauge field, we can get a consistent $SU(2) \times U(1)$ subsector.
- Subtheory: A theory forming part of a larger theory. Action is decomposed as

$$S_{\rm tot} = S_{\rm sub} + S_{\rm extra}$$

and

$$\delta_{\rm tot} = \delta_{\rm sub} + \delta_{\rm extra}$$

- As a consistency $S_{\rm sub}$ should be inv. under $\delta_{\rm sub}$ and gauge symmetry form a closed subalgebra.

- Although we cannot decouple string massive excitations, we can focus on massless subsector to study winding mode dynamics in a simple setup.
- Gague symmetry :

δ^{full}(massless fields) $=$ (massless fields only)$+$(massive fields + massless fields)

If we denote the massless field sector as δ^0, then it should form a subalgebra of the full gauge algebra

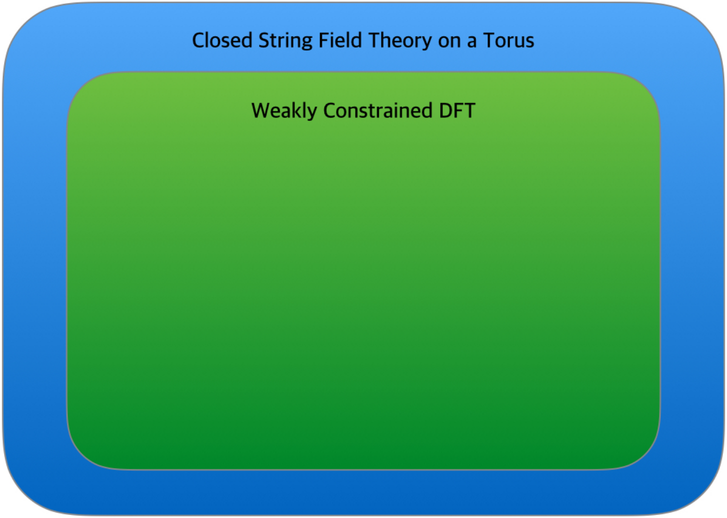
$$[\delta^0_X, \delta^0_Y] \text{ (massless fields)} = \delta^0_Z \text{(massless fields)}$$

- Action It should include a massless subsector in the action

$$\mathcal{L}_{\text{full}} = \mathcal{L}_{\text{massless}} + \mathcal{L}_{\text{massive+massless}}$$

and $\mathcal{L}_{\text{massless}}$ should be invariant under the δ^0.

- Weakly constrained DFT



- It is very difficult to construct any field theory with the HZ projector due to the non-associativity. Even $O(1,1)$ case is hopeless.
- Assume that there exits the full WDFT in terms of HZ-projector

$$S_{\mathrm{WDFT}}[\mathcal{H},d,[\![\cdot\cdot]\!],C(k,\hat{P})]\,,\qquad \delta\{\mathcal{H},d\}=\delta\{\mathcal{H},d\}([\![\cdot\cdot]\!],C(k,\hat{P}))$$

- Using $[\![f\cdot g]\!]=f\circ g+f\star g$, it is always possible to decompose the theory as

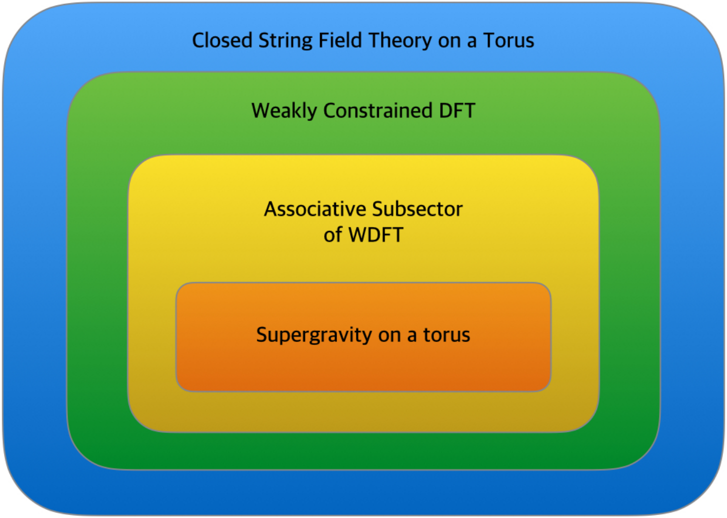
$$S_{\mathrm{WDFT}}=S_{\mathrm{AWDFT}}[\circ]+S_{\mathrm{NA}}[\circ,\star,C[k,\hat{P}]]$$

  as well as the gauge symmetry

$$\delta\{\mathcal{H},d\}=\delta^{\mathrm{AWDFT}}\{\mathcal{H},d\}[\circ]+\delta^{\mathrm{NA}}\{\mathcal{H},d\}[\circ,\star,C[k,\hat{P}]]$$

- The associative subsector of full WDFT is a well defined subtheory.

- Associative subsector of WDFT



- We have to define $\mathbf{O}(d,d;\mathbb{Z})$ group equipped with $\circ$-product. To distinguish with the usual $\mathbf{O}(d,d)$ group, we denote as $\mathbf{O}(d,d;\mathbb{Z})_\circ$.
- Assume that $\mathcal{J}_\circ$ is the $\mathbf{O}(d,d;\mathbb{Z})_\circ$ metric which is defined as

$$\mathcal{J}_\circ = \begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix}$$

where the identity matrix I_d is defined by

$$I_d = \sum_{\Pi} \varphi(\Pi)\, \mathbb{1}_d$$

where $\mathbb{1}_d = \text{diag}(1, \cdots, 1)$. Note that $\mathcal{J}$ is a constant matrix, but it is not the usual $\mathbf{O}(d,d)$ metric

$$\mathcal{J}_{\circ IJ} \neq \mathcal{J}_{IJ} = \begin{pmatrix} 0 & \delta^i{}_j \\ \delta_i{}^j & 0 \end{pmatrix} .$$

- $\mathbf{O}(d,d;\mathbb{Z})_\circ$ is defined by a set of $2d \times 2d$ matrices satisfying

$$\mathcal{O}^t \circ \mathcal{J}_\circ \circ \mathcal{O} = \mathcal{J}_\circ$$

  where $\mathcal{O} \in \mathbf{O}(d,d;\mathbb{Z})_\circ$.

- $\mathcal{J}_\circ$ and $\mathcal{O}$ are expanded by inverse X-ray transform

$$\mathcal{J}_\circ = \sum_\Pi \varphi(\Pi) \hat{\mathcal{J}}_\Pi\,, \qquad \mathcal{O} = \sum_\Pi \varphi(\Pi)\, \hat{\mathcal{O}}_\Pi(z_i)$$

- Each X-ray images $\hat{\mathcal{O}}_\Pi$ are usual $\mathbf{O}(d,d;\mathbb{Z})$ elements

$$\hat{\mathcal{O}}_\Pi^t \cdot \hat{\mathcal{J}}_\Pi \cdot \hat{\mathcal{O}}_\Pi = \hat{\mathcal{J}}_\Pi$$

  Thus $\mathbf{O}(d,d;\mathbb{Z})_\circ$ element is represented by a collection of $\mathbf{O}(d,d;\mathbb{Z})$ elements.

- Then we can show that $\mathbf{O}(d,d;\mathbb{Z})_{\circ}$ defines a group. For arbitrary elements $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3 \in \mathbf{O}(d,d;\mathbb{Z})$, they satisfy the following the properties:
  - Closure
$$\mathcal{O}_1 \circ \mathcal{O}_2 \in \mathbf{O}(d,d)$$
  - Associativity
$$\mathcal{O}_1 \circ (\mathcal{O}_2 \circ \mathcal{O}_3) = (\mathcal{O}_1 \circ \mathcal{O}_2) \circ \mathcal{O}_3$$
  - Identity
$$A \circ I_{2d} = I_{2d} \circ A = A$$
  - Inverse
$$A \circ A^{-1} = A^{-1} \circ A = I_{2d}$$

- $\mathbf{O}(d, d; \mathbb{Z})_{\circ}$ tensor transforms as

$$T'_{I_1 \cdots I_m}{}^{J_1 \cdots J_n}(X') = \mathcal{O}_{I_1}{}^{K_1} \circ \cdots \circ \mathcal{O}_{I_1}{}^{K_m} \circ T_{K_1 \cdots K_m}{}^{L_1 \cdots L_n} \circ \mathcal{O}^{J_1}{}_{L_1} \circ \cdots \circ \mathcal{O}^{J_n}{}_{L_n}$$

- Since we are assuming torus case only, it should be $\mathbf{O}(d, d; \mathbb{Z})_{\circ}$ rather than $\mathbf{O}(d, d, \mathbb{R})_{\circ}$

- Weakly constrained fields are represented by summing the all possible strongly constrained fields. Conversely, we may consider a collection of all possible strongly constrained generalized metric

$$\mathcal{H}_{IJ}(X^I) = \sum_{\Pi \in \mathcal{P}_d^0} \varphi(\Pi) \hat{\mathcal{H}}_{\Pi\, IJ}(z^i)$$

- Weakly constrained generalized metric satisfy following conditions

$$\mathcal{H}_{IJ} = \mathcal{H}_{(IJ)} \qquad \mathcal{H} \circ \mathcal{J}_\circ \circ \mathcal{H}^t = \mathcal{J}_\circ^{-1}$$

- Furthermore, $\mathcal{H}$ is an $\mathbf{O}(d, d; \mathbb{Z})_\circ$ tensor

$$\mathcal{H} \longrightarrow \mathcal{O} \circ \mathcal{H} \circ \mathcal{O}^t$$

- As strongly constrained DFT, we can parametrize $\mathcal{H}$

$$\mathcal{H}_{IJ} = \begin{pmatrix} g^{-1} & g^{-1} \circ B \\ B \circ g^{-1} & g - B \circ g^{-1} \circ B \end{pmatrix}$$

where the g^{-1} is defined by

$$g^{-1} \circ g = g \circ g^{-1} = I_d$$

- Even if we consider weakly constrained DFT, the physical degrees of freedom are same as strongly constrained DFT

$$g(x, \tilde{x}) \,, \qquad B(x, \tilde{x}) \,, \qquad \phi(x, \tilde{x})$$

This is consistent with the result of string field theory.

- Physical degrees of freedom is given by weakly constrained generalized metric.

$$\mathcal{H}_{IJ}(X^I) = \sum_{\Pi \in \mathcal{P}_d^0} \varphi(\Pi) \hat{\mathcal{H}}_{\Pi\, IJ}(z^i)$$

- Gauge transformation of each $\hat{\mathcal{H}}_{\Pi\, IJ}(z^i)$ is given by generalized Lie derivative. The gauge transformation of $\mathcal{H}_{IJ}$ should be a collection of generalized Lie derivatives.
- It is natural to speculate that the form of gauge transformation of the weakly constrained fields : replacing all the usual products to $\circ$-product in the generalized Lie derivative

$$\delta_X \mathcal{H}_{IJ} = X^K \circ \partial_K \mathcal{H}_{IJ} + \left(\partial_I X^K - \partial^K X_I\right) \circ \mathcal{H}_{KJ} + \left(\partial_J X^K - \partial^K X_J\right) \circ \mathcal{H}_{IK}\,,$$

$$\delta_X d = X^P \circ \partial_P d - \tfrac{1}{2} \partial_P X^P\,.$$

- It is straight to show that the gauge transform is closed exactly

$$[\delta_X, \delta_Y]\,\mathcal{H}_{MN} = \delta_{[X,Y]_C}\mathcal{H}_{MN}\,,$$

where the generalized version of C-bracket is defined by

$$[X,Y]_C^M = X^N \circ \partial_N Y^M - \tfrac{1}{2} X^N \circ \partial^M Y_N - \big(X \leftrightarrow Y\big)$$

- Under the $\tilde{\partial}^i$-expansion, the gauge transform is expanded by

$$\delta^{(0)}\mathcal{E}_{ij} = \partial_i \Lambda_j - \partial_j \Lambda_i + \xi^k \circ \partial_k \mathcal{E}_{ij} + \partial_i \xi^k \circ \mathcal{E}_{kj} + \partial_j \xi^k \circ \mathcal{E}_{ik}$$

$$\delta^{(1)}\mathcal{E}_{ij} = -\mathcal{E}_{ik} \circ \big(\tilde{\partial}^k \xi^l - \partial^l \xi^k\big) \circ \mathcal{E}_{lj} + \Lambda_k \circ \tilde{\partial}^k \mathcal{E}_{ij} - \tilde{\partial}^k \Lambda_i \circ \mathcal{E}_{kj} - \tilde{\partial}^k \Lambda_j \circ \mathcal{E}_{ik}$$

where $\mathcal{E} = g + B$. This is exactly same as Hohm, Hull and Zwiebach's tilde derivative expansion except the product.

- Suppose an arbitrary action

$$S = \int \mathrm{d}^{2d} X \, L(x, \tilde{x})$$

where $L(x, \tilde{x})$ is a Lagrangian density.

- Since any $2d$-dimensional functions are expanded by X-ray transform

$$L(x, \tilde{x}) = \sum_{\Pi} \varphi(\Pi) \hat{L}_{\Pi}(z_{\Pi})$$

- We propose an action for an associative subsector of weakly constrained DFT

$$\mathcal{S}_{\text{AWDFT}} = \int \mathrm{d}^{2d}X \, [e^{-2d}]_\circ \circ \mathcal{L}_{\text{AWDFT}}$$

where the Lagrangian $\mathcal{L}_{\text{AWDFT}}$ is given by

$$\begin{aligned}\mathcal{L}_{\text{AWDFT}} = \; & 4\mathcal{H}^{IJ} \circ \partial_I \partial_J d - \partial_I \partial_J \mathcal{H}^{IJ} - 4\mathcal{H}^{IJ} \circ \partial_I d \circ \partial_J d + 4\partial_I \mathcal{H}^{IJ} \circ \partial_J d \\ & + \tfrac{1}{8}\mathcal{H}^{IJ} \circ \partial_I \mathcal{H}^{KL} \circ \partial_J \mathcal{H}_{KL} - \tfrac{1}{2}\mathcal{H}^{IJ} \circ \partial_I \mathcal{H}^{KL} \circ \partial_K \mathcal{H}_{JL}\end{aligned}$$

- The exponentiation of the d, $[e^{-2d}]_\circ$, is defined by

$$\begin{aligned}[e^{-2d}]_\circ &= I - 2d + \tfrac{1}{2}(2d) \circ (2d) - \tfrac{1}{3!}(2d) \circ (2d) \circ (2d) + \cdots \\ &= \sum_{\Pi \in \mathcal{P}_d^0} \sum_{m \geq 0} \varphi(\Pi) \tfrac{1}{m!}\big(-2\hat{d}_\Pi(z^i)\big)^m \\ &= \sum_{\Pi \in \mathcal{P}_d^0} \varphi(\Pi) e^{-2\hat{d}_\Pi} \,.\end{aligned}$$

- Using the definition of $\circ$-product, the action is expanded as

$$\mathcal{L}_{\mathsf{AWDFT}} = \sum_{\Pi \in \mathcal{P}_d^0} \varphi(\Pi) \hat{L}_\Pi(z^i)$$

- Each $\hat{S}_\Pi$ is a strongly constrained DFT action on a d-dimensional null plane $\mathcal{D}^0(X^I, \Pi)$

$$\hat{\mathcal{L}}_\Pi = e^{2\hat{d}_\Pi}\big(4\hat{\mathcal{H}}_\Pi{}^{IJ}\partial_I\partial_J\hat{d}_\Pi - \partial_I\partial_J\hat{\mathcal{H}}_\Pi{}^{IJ} - 4\hat{\mathcal{H}}_\Pi^{IJ}\partial_I\hat{d}\partial_J\hat{d}_\Pi$$
$$+ 4\partial_I\hat{\mathcal{H}}_\Pi{}^{IJ}\partial_J\hat{d} + \tfrac{1}{8}\hat{\mathcal{H}}_\Pi{}^{IJ}\partial_I\hat{\mathcal{H}}_\Pi{}^{KL}\partial_J\hat{\mathcal{H}}_{\Pi KL} - \tfrac{1}{2}\hat{\mathcal{H}}_\Pi{}^{IJ}\partial_I\hat{\mathcal{H}}_\Pi{}^{KL}\partial_K\hat{\mathcal{H}}_{\Pi JL}\big)$$

- Three different concepts: Local coordinate, section condition and polarization
- Section condition defines a d-dimensional plane(or torus) where a strongly constrained DFT lives within double torus T^{2d}.
- Polarization Θ provides a consistent way to separate the T^d and $\tilde{T}^d$ within the double torus T^{2d} . [Hull, 2004]
- For simplicity we identify local coordinate with polarization. Identify x as a usual coordinate, and $\tilde{x}$ as a winding coordinate. Also section condition is identified null-planes, Π.

- In general, there is no reason that section condition is identical with polarization. However we can always identify these using $O(d, d)$ rotation.
- In AWDFT case, we cannot identify all of them. There is a single global plarization, but there are infinite number of section conditions.
- Since we cannot identify section condtion and polarization, except one, each strongly constrained DFT has non-trivial winding dependences and interaction between momentum and winding modes.

tilde-derivative expansion

- zeroth-order

$$\mathcal{L}^{(0)} = [e^{2d}]_\circ \circ \Big[-\tfrac{1}{4} g^{ik} \circ g^{jl} \circ g^{pq} \circ \big(\partial_p \mathcal{E}_{kl} \circ \partial_q \mathcal{E}_{ij} - \partial_i \mathcal{E}_{lp} \circ \partial_j \mathcal{E}_{kq} - \partial_i \mathcal{E}_{pl} \circ \partial_j \mathcal{E}_{qk} \big) + 2\partial^i d \circ \partial^j g_{ij} + 4\partial^i d \circ \partial_i d \Big] ,$$

where $\partial^i = g^{ij} \circ \partial_j$.

- The next order takes the form

$$\mathcal{L}^{(1)} = [e^{2d}]_\circ \circ \Big[\tfrac{1}{2} g^{ik} \circ g^{jl} \circ g^{pq} \circ \big(\mathcal{E}_{pr} \circ \tilde{\partial}^r \mathcal{E}_{kl} \circ \partial_q \mathcal{E}_{ij} - \mathcal{E}_{ir} \circ \tilde{\partial}^r \mathcal{E}_{ip} \circ \partial_k \mathcal{E}_{jq} + \mathcal{E}_{rl} \circ \tilde{\partial}^r \mathcal{E}_{pi} \circ \partial_k \mathcal{E}_{qj} \big) + g^{ip} \circ g^{jq} \circ \big(\mathcal{E}_{rq} \circ \partial_p d \circ \tilde{\partial}^r \mathcal{E}_{ij} - \mathcal{E}_{pr} \circ \tilde{\partial}^r d \circ$$

$$+ \mathcal{E}_{rp} \circ \tilde{\partial}^r d \circ \partial_q \mathcal{E}_{ij} - \mathcal{E}_{qr} \circ \partial_p d \circ \tilde{\partial}^r \mathcal{E}_{ji} \big) - 8 g^{ij} \circ \mathcal{E}_{ik} \circ \tilde{\partial}^k d \circ \partial_j d \Big] ,$$

- We constructed an associative subsector of WDFT: Gauge symmetry, $\mathbf{O}(d, d; \mathbb{Z})_\circ$ and gauge invariant action.
- This is the string effective theory beyond supergravity limit. AWDFT is not rewriting supergravity at all!
- From the X-ray transform and $\circ$-product, AWDFT is defined in a very simple and straightforward way.
- Is it a unique associative subsector besides supergravity?
- Is it possible to construct the full WDFT?
- Is it possible to construct a non-associative subsector of WDFT which is interpolating full WDFT and associative WDFT?

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Thank you for attention