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Generalised T-duality and Integrable Deformations

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Duality and Novel Geometry in M-theory
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Motivation and Outline

Explore the landscape of dualities; DFT/EFT beyond the section; Applications!!! e.g. Holography

TODAY:

1. Recap/Fresh Angle on world-sheet Doubled Formalism
2. Generalised T-dualities: Non-Abelian, Poisson-Lie
3. New integrable 2d (C)QFTs & Generalised Duality

1. World-sheet Doubled Formalism

The Tseytlin Model Revisited

Amazing progress in D/EFT mostly from a space-time perspective, but lets not ignore the world sheet. Important if we want to get to grips with the non-LEEA nature

Consider a non-linear σ-model

$$\mathcal{L}[X] = \partial_+ X^I E_{IJ} \partial_- X^J \equiv \partial_+ X^I (G_{IJ} + B_{IJ}) \partial_- X^J$$

Adapted coordinates for $U(1)^d$ isometry $X^I = \{x^i, Y^\alpha\}$.
Buscher part I: gauge $\oplus$ flat connection with Lag. multipliers $\tilde{x}_i$.

$$\mathcal{L}_g = \partial_+ x g \partial_- x + \mathcal{J}_+ A_- + \mathcal{J}_- A_+ + A_- g A_+$$
$$\mathcal{J}_+ = E^T \partial_+ x - \partial_- \tilde{x} \ , \quad \mathcal{J}_- = E \partial_+ x + \partial_- \tilde{x}$$

Buscher part II: gauge fix on x and integrate out connection $\Rightarrow$ Dual model

- Instead keep both x and $\tilde{x}$, partially fix on A and partial integrate out A $\Rightarrow$ Double model
- Axial Gauge Choice [Rocek Tsyetlin '98]

$$A_+ = A_- = a$$

- Integrate out a and get a theory for $\mathbb{X} = \{x, \tilde{x}\}$

$$\mathcal{L}_{Tseytlin} = -\mathcal{H}_{IJ}\partial_\sigma \mathbb{X}^I \partial_\sigma \mathbb{X}^J + \eta_{IJ}\partial_\sigma \mathbb{X}^I \partial_\tau \mathbb{X}^J + \Omega_{IJ}\partial_\sigma \mathbb{X}^I \partial_\tau \mathbb{X}^J + \dots$$
$$\mathcal{H} = \begin{pmatrix} g - bg^{-1}b & -bg^{-1} \\ g^{-1}b & g^{-1} \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- Note 1: Topological term inevitable!
- Note 2: Residual gauge invariance $\delta\mathbb{X} = \xi(\tau) \Rightarrow 1^{st}$ order eqm for chiral Bosons

$$d\mathbb{X}^I = \star(\eta\mathcal{H})^I{}_J d\mathbb{X}^J$$

PST Model via Gauging

- Cuter fixing condition with some auxiliary function $f(\sigma^+, \sigma^-)$

$$A_+\partial_- f = A_-\partial_+ f \Leftrightarrow A_\pm = a\,\partial_\pm f \ ,$$

- 'Covariant' Doubled Action *à la* PST [Pasti Sorokin Tonin]

$$\mathcal{L}_{PST} = \mathcal{H}\partial_+\mathbb{X}\partial_-\mathbb{X} - \frac{\partial_+ f}{\partial_- f}(P_+\partial_-\mathbb{X})^2 - \frac{\partial_- f}{\partial_+ f}(P_-\partial_+\mathbb{X})^2 - \Omega\partial_+\mathbb{X}\partial_-\mathbb{X} + \dots$$

- Projectors $2P_\pm = 1 \pm \eta\mathcal{H}$
- Residual gauge invariance $\delta\mathbb{X} = \xi(f) \Rightarrow 1^{st}$ order eqm for chiral Bosons
- Duality symmetric dilaton emerges from gaussian integral over *single* component a

$$a^i g_{ij} a^j \rightsquigarrow \Phi = \phi - \frac{1}{4}\log\det g$$

- Given f this defines a theory

$$Z[f] = \int [d\mathbb{X}][db][dc]e^{-i\int \mathcal{L}_{PST}+\mathcal{L}_{gh}}$$
$$\mathcal{L}_{gh} = \partial_+ f\, b\partial_- c - \partial_- f\, b\partial_+ c$$

- For full covariance we should allow f to be a field i.e. average over gauge fixing choices

$$Z = \frac{1}{vol}\int [df]Z[f]$$

- Physics shouldn't (hopefully) depend on $f \Rightarrow$ 'PST' symmetry for all fields:

$$\delta f = \epsilon\ , \quad \delta\mathbb{X} = \epsilon\left(\frac{P_+\partial_-\mathbb{X}}{\partial_- f} + \frac{P_-\partial_+\mathbb{X}}{\partial_+ f}\right)$$

- Fixing PST symmetry $f(\sigma^+,\sigma^-) = \tau$ (no more ghosts!) gives back Tseytlin action
- Cautions: measure on of f; fixing need be only locally defined
- Work in progress: extend gauging approach to superspace, open sector etc.

2. Generalised T-dualities

Non-Abelian T-duality

Abelian dualities have a natural analogue when space time Killing vectors forming a group: Non-Abelian T-duality [de la Ossa & Quevedo '93]

Simplest example: 'Principal Chiral Model' Lagrangian

$$\boxed{\mathcal{L} = L_+^a E_{ab} L_-^b}$$

A theory of maps $g : \Sigma \to G$ with one-forms

$$L = g^{-1} dg = L_i^a dx^i T_a \ , \quad dL^a = \frac{1}{2} f_{ab}{}^c L^b \wedge L^c$$

Buscher the G_L global symmetry $g \to hg$. Gauge with minimal coupling

$$\partial g \to Dg = \partial g - Ag \ , \quad A = iA^a T_a$$

Lagrange multiplier term

$$\tilde{x}_a F_{+-}^a = \tilde{x}_a \left(\partial_+ A_- - \partial_- A_+ - [A_+, A_-] \right)^a$$

Integration out of the gauge fields and fixing $g = 1 \Rightarrow$ T-dual theory

$$\mathcal{L}_{T-dual} = \partial_+ \tilde{x}^T (E + f)^{-1} \partial_- \tilde{x} \ , \quad f = f_{ab}{}^c \tilde{x}_c$$

- Unlike Abelian T-duality not expected to be exact CFT duality
- Isometries (and e.g. super symmetries) of the target space that don't commute with duality are destroyed.
- Realisation as a canonical transformation of phase space variables
- Principal Chiral Models on groups or cosets are important $AdS_n \times S^n$
- Non-abelian T-duality can lift to the RR-sector and given life in SUGRA **[Sfetsos & Thompson '10]**
- Remarkable utility in context of holography as a solution generating tool **[Sfetsos, Thompson, Lozano, O Colgain, Nunez, Itsios, Macpherson ......]**

A Double model for non-Abelian T-duality

- Instead, choose Axial gauge to get a theory for $\mathbb{X}^M = \{x, \tilde{x}\}$

$$\mathbb{L}^A = \begin{pmatrix} L^a \\ \tilde{L}_a \end{pmatrix} \,, \quad \tilde{L}_a = (ad_g)_a{}^b d\tilde{x}_b$$

$$\mathcal{L}_{Tseytlin} = -\mathcal{H}_{AB}\mathbb{L}^A_\sigma \mathbb{L}^B_\sigma + \eta_{AB}\mathbb{L}^A_\sigma \mathbb{L}^B_\tau + \Omega_{AB}\mathbb{L}^A_\sigma \mathbb{L}^B_\tau + \dots$$

- In coordinate basis $\mathcal{H}_{MN}(\mathbb{X}) = \mathbb{L}^A_M \mathcal{H}_{AB}\mathbb{L}^B_N$ depends in general on all the x (but not the $\tilde{x}$)
- 'Topological' term contributes a potential for a three form flux

$$H = d\Omega = f_{ab}{}^c L^a \wedge L^b \wedge \tilde{L}_c$$

Suggests chiral-WZW [Klimcik & Severa, Sfetsos, Hull & Reid-Edwards]

$$\boxed{S = \int_\Sigma d^2\sigma - \mathcal{H}_{AB}\mathbb{L}^A_\sigma \mathbb{L}^B_\sigma + \eta_{AB}\mathbb{L}^A_\sigma \mathbb{L}^B_\tau + \int_{\mathcal{M}_3} f_{AB}{}^D \eta_{DC}\mathbb{L}^A \wedge \mathbb{L}^B \wedge \mathbb{L}^C}$$

Mathematica Detour: The Drinfeld Double

- The Drinfeld Double is a Lie Algebra $\mathcal{D}$ which can be decomposed as the sum $\mathcal{D} = \mathcal{G} + \tilde{\mathcal{G}}$ for two maximally isotropic sub-algebras. If $T_A = \{T_a, \tilde{T}^a\}$ then

$$\langle T_a | T_b \rangle = \langle \tilde{T}^a | \tilde{T}^b \rangle = 0 \Rightarrow \langle T_A | T_B \rangle = \eta_{AB}$$

- Mixed Jacobi identity restrict choices of $\mathcal{G}$ and $\tilde{\mathcal{G}}$. Some examples:
  - Abelian Double $\mathcal{D} = u(1)^d + u(1)^d$
  - Semi-Abelian Double $\mathcal{D} = \mathcal{G} + u(1)^d$
  - Non-Abelian Double $\mathcal{D} = \mathcal{G} + \tilde{\mathcal{G}}$ (e.g. $so(3,1) = su(2) + e_3$)
- The Doubled σ-model lives on the Drinfeld Double [Klimcik & Severa] (also [Sfetsos, Hull & Reid-Edwards])!
- Structure constants & WZW structure $\Rightarrow$ torsion $= \pm$ spin connection

Drinfeld Double & The Section Condition

Regular T-dual pairs of σ-models extracted by parameterising $h \in \mathcal{D}$ as $h = g\tilde{g}$ and integrating out $\tilde{g}$ or as $h = \tilde{g}g$ and integrating out g

- $\mathcal{D} = u(1)^d + u(1)^d \Rightarrow$ Abelian T-duality $\Rightarrow \mathcal{H}_{MN}$ constant
- $\mathcal{D} = \mathcal{G} + u(1)^d \Rightarrow$ non-Abelian T-duality $\Rightarrow \mathcal{H}_{MN}(x)$ on-section
- $\mathcal{D} = \mathcal{G} + \tilde{\mathcal{G}} \Rightarrow$ Poisson-Lie T-duality $\Rightarrow \mathcal{H}_{MN}(x, \tilde{x})$ beyond-section

β-function of $\mathcal{H}_{AB}$ implies scalar potential of gauged supergravity!!

[Dall'agata,Prezas; Sfetsos-Siampos-DT]

$$\frac{d\mathcal{H}_{AB}}{dt} = \frac{1}{4}\left(\mathcal{H}_{AC}\mathcal{H}_{BF} - \eta_{AC}\eta_{BF}\right)\left(\mathcal{H}^{KD}\mathcal{H}^{HE} - \eta^{KD}\eta^{HE}\right) f_{KH}^{C} f_{DE}^{F}$$

Compare with DFT Scherk-Schwarz!

PLT details

PL T-duality which is an equivalence between two σ-models

$$S[g] = \frac{1}{2\pi t}\int d^2\sigma L_+^T (E-\Pi)^{-1} L_- \,, \quad g \in \mathcal{G}\,,$$
$$\tilde{S}[\tilde{g}] = \frac{1}{2\pi t}\int d^2\sigma \tilde{L}_+^T (E^{-1}-\tilde{\Pi})^{-1} \tilde{L}_- \,, \quad \tilde{g} \in \tilde{\mathcal{G}}\,.$$

The group theoretic matrix Π

$$a_a{}^b = \langle g^{-1} T_a g, \tilde{T}^b\rangle \,, \quad b^{ab} = \langle g^{-1}\tilde{T}^a g, \tilde{T}^b\rangle \,, \quad \Pi = b^T a$$

Typically these backgrounds have *no isometries* so don't expect conserved currents however non-commutative conservation with respect to dual group

$$d\star\mathcal{J}_a = \tilde{f}^{bc}{}_a \star\mathcal{J}_b \wedge \star\mathcal{J}_b \qquad (1)$$

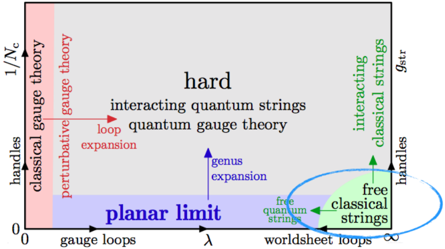
Thus $\star\mathcal{J}$ should be pure gauge in a dual algebra (Field Equations $\Leftrightarrow$ Bianchi identity)

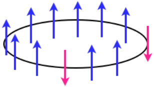
3. New Integrable Models

Integrability in AdS-CFT

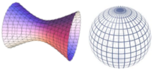
Integrability has been a game changer for $\mathcal{N} = 4$ SYM

$$\lambda = g_{YM}^2 N_c \qquad \left(\frac{R}{\sqrt{\alpha'}}\right)^4 = 4\pi N_c g_s \qquad 4\pi g_s = g_{YM}^2$$





$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$



Source: Main image from 1012.3982 **Beisert et al. Others from Wikipedia/DCT**

Integrability in AdS-CFT

Is integrability just a feature of maximal supersymmetry?

- QCD in high energy limits [Lipatov; Faddeev, Korchemsky]
- Marginal (real)-β deformations that break SUSY to $\mathcal{N} = 1$

$$\int d^4x\, d^4\theta\, \mathrm{Tr}\left(e^{i\beta}XYZ - e^{-i\beta}ZXY\right)$$

- String holographic spacetime given by doing T-dualities (TsT) [Lunin,Maldacena] is integrable [Frolov]
- Other generalisations e.g. null TsT transformations and Schrodinger deformations [Bobev, Kundu] , non-comm YM etc. recently classified in terms of Yang-Baxter equation [Matsumoto,Yoshida; Van Tongeren]
- New η-deformations [Delduc,Magro, Vicedo] based on Yang-Baxter σ-models [Klimcik] preserve no SUSY and only some $U(1)$

- Integrability: a game-changer for AdS/CFT
- Can we deform AdS/CFT *and* keep integrability?
- Two new ideas: "η" and "λ" integrable deformations
- Closely connected via generalised T-dualities

Recap: the PCM

- Toy model: *Principal Chiral Model* [PCM] on S^3

$$S = \frac{\kappa^2}{4\pi}\int_\Sigma d^2\sigma \, Tr\left(g^{-1}\partial_+ g g^{-1}\partial_- g\right) \ , \quad g : \Sigma \to SU(2)$$

- Non-conformal proto-QCD model
- $SU(2)_L \times SU(2)_R$ symmetry
- **Integrable**: Lax formulation and ∞ conserved charges

$$A(z) = \frac{1}{1-z^2}L + \frac{z}{1-z^2}\star L \ , \quad dA - A\wedge A = 0$$

$$T(z) = P\exp\int d\sigma A_\sigma$$

A toy model for η

- Deform [Cherednik '81]:

$$S = \frac{\kappa^2}{4\pi}\int_\Sigma d^2\sigma\, Tr\left(g^{-1}\partial_+ g g^{-1}\partial_- g\right) + C J_+^3 J_-^3$$

- Integrable but $SU(2)_L \times SU(2)_R \rightarrow SU(2)_L \times U(1)_R$
- Non-local charges recover semi-classical version of $\mathcal{U}_q(\mathfrak{sl}_2)$ [Kawaguchi, Matsumoto, Yoshida '11, '12]

$$\{Q_R^+, Q_R^-\}_{P.B.} = \frac{q^{Q_R^3} - q^{-Q_R^3}}{q - q^{-1}}\ , \quad q = \exp\left(\frac{\sqrt{C}}{1+C}\right)$$

Yang-Baxter and η

- Arbitrary groups [Klimcik '02] based on modified Yang-Baxter eq:

$$[\mathcal{R}A, \mathcal{R}B] - \mathcal{R}([\mathcal{R}A, B] + [A, \mathcal{R}B]) = [A, B] \ , \quad \forall A, B \in \mathfrak{g}$$

$$S_\eta = \frac{1}{2\pi t} \int_\Sigma d^2\sigma Tr \left(g^{-1}\partial_+ g, \frac{1}{1-\eta\mathcal{R}} g^{-1}\partial_- g \right)$$

- Poisson algebra of G_R gives quantum group [Delduc, Magro, Vicedo 1308]
- Cosets and super-cosets i.e. $AdS_5 \times S^5$ superstring [Delduc, Magro, Vicedo 1309]
- Perturbative matching to q-deformed S-matrix [Artyunov, Borsato, Frolov 1312]
  [Beisert, Krooteev '08; Hoare, Hollowood, Miramontes '11; de Leeuw, Regelskis, Matsumoto '11]

λ-deformations

A simple recipe for integrable λ deformations [Sfetsos 1312]

1. **Double** the d.o.f.: $\kappa^2 S_{PCM}[\tilde{g}] + kS_{WZW}[g]$
2. **Gauge** G_L in PCM and G_{diag} in WZW
3. **Gauge Fix** $\tilde{g} = \mathbb{1}$
4. **Integrate out** non-propagating gauge fields

$$S_\lambda = kS_{WZW} + \frac{k}{2\pi}\int Tr(g^{-1}\partial_+ g \frac{1}{\lambda^{-1} + Ad_g}\partial_- g g^{-1})$$

$$\lambda = \frac{k}{\kappa^2 + k}$$

λ-limits

Nice behaviour in limits of small and large deformations:

- $\lambda \to 0$: current bilinear perturbation

$$S_\lambda|_{\lambda\to 0} \approx k S_{WZW} + \frac{k}{\pi}\int \lambda J_+^a J_-^a + \mathcal{O}(\lambda^2)$$

- $\lambda \to 1$: non-Abelian T-dual of PCM

$$S_\lambda|_{\lambda\to 1} \approx \frac{1}{\pi}\int \partial_+ X^a (\delta_{ab} + f_{ab}{}^c X_c)^{-1} \partial_- X^b + \mathcal{O}(k^{-1})$$

λ-space time

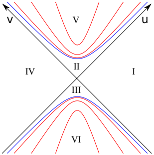
Gets pretty messy, so lets explicitly give S_2:

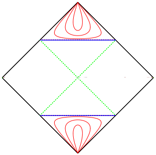
$$ds^2 = k\left(\frac{1-\lambda}{1+\lambda}(d\omega^2 + cot^2\omega d\phi^2) + \frac{4\lambda}{1-\lambda^2}(cos\phi d\omega + sin\phi cot\omega d\phi)^2\right)$$

After wick rotation $\omega \to i\rho, \phi \to it$, $k \to -k$ this gives a $2d$-geometry that is a deformation of the Witten black hole. In Kruskal coordinates:

$$u = \cosh\rho(e^{-t} + \lambda e^{t}), v = \cosh\rho(e^{t} + \lambda e^{-t})$$

$$ds^2 = k(1-\lambda^2)\frac{dudv}{f(u,v)}\ , \quad f(u,v) = (u-\lambda v)(v-\lambda u) - (1-\lambda^2)^2\ .$$





λ-developments

- Applies to (super)-cosets [Hollowood, Miramontes, Schmidtt 1408, 1409]
- Conjectured to be quantum group deformation with q root unity [HMS 1409,1506; Itsios et al. 1409]
- β-functions [Itsios, Sfetsos, Siampos 1405] vanish for $PSU(2,2|4)/SO(4,1) \times SO(5)$ [Appadu, Hollowood 1507]
- Type II supergravity embeddings of bosonic λ-deformations [Sfetsos,DT 1410; Demulder, Sfetsos, DT 1504]
- Multi-parameter integrable λ-deformations [Siampos, Sfetsos, DT 1506]

η, λ and Poisson-Lie

η and λ connected by generalised Poisson Lie T-duality

[Vicedo 1504; Hoare & Tseytlin 1504; Siampos Sfetsos DT 1506; Klimcik 1508]

- Modified conservation law for currents of broken G_R in η-model:

$$d \star J_a = \tilde{f}^{bc}{}_a J_b \wedge J_c$$

- $\tilde{f}^{bc}{}_a$ structure constants for $\mathfrak{g}_{\mathcal{R}}$

$$[A, B]_{\mathcal{R}} = [\mathcal{R}A, B] - [A, \mathcal{R}B]$$

- Mathematically $\mathfrak{g} \oplus \mathfrak{g}_{\mathcal{R}} \simeq \mathfrak{g}^{\mathbb{C}}$ defines a Drinfel'd Double

η, λ and Poisson-Lie

- Can still T-dualise these currents
- Drinfel'd Double provides the Doubled Formalism!
- Analytic continue certain Euler angles and deformation parameters

$$\eta \to i\frac{1-\lambda}{1+\lambda}\ , \quad t \to \frac{\pi(1+\lambda)}{(1-\lambda)}$$

- Acting on the parameter q we have

$$q = e^{\eta t} \leftrightarrow q = e^{\frac{i\pi}{k}}$$

Conclusions and Outlook

η and λ open a new window onto integrable deformations in the AdS/CFT conjecture and generalised T-dualities

What's next?

1. Implications of Poisson-Lie duality for DFT?
2. Consistent CFTs?
3. New scenario's e.g. $AdS_4 \times CP^3$?
4. Demonstrate the Mass gap in η-deformed $2-d$ QFTs
5. *Implication on gauge theory side?*

Appendix

Currents for squashed S^3

With $J = g^{-1}dg$ define $U(1)_R$ current

$$j^{R,3} = 2(1+C)J^3$$

and non-local currents

$$j^{R,\pm}_{\mu} = 2e^{\gamma\chi}\left(\eta_{\mu\mu} + i\sqrt{C}\epsilon_{\mu\nu}\right) Tr(T^{\pm}J^{\nu})$$

with non-local contributions

$$\chi(x) \sim \int dy \epsilon(x-y) j^{R,3}_t(y)$$

Conserved charges

$$Q^{\pm} = \int j^{R,\pm}_t = Q^{\pm}_0 \pm i\sqrt{C}Q^{\pm}_1 + \dots$$

Q_0 and Q_1 generate Yangian

Yangian

Schematic picture. Generators $Q_0 = J$ and $Q_1 = Q$ obey

$$[J, J] = FJ \ , \quad [J, Q] = [Q, J] = FQ$$

Co-product

$$\Delta J = J \otimes 1 + 1 \otimes J \ , \quad \Delta Q = Q \otimes 1 + 1 \otimes Q + \alpha/2FJ \otimes J$$

Serre relations (since Δ acts as structure preserving map)

$$[Q, [Q, J]] - [J, [Q, Q]] \sim F^4 J^3$$

(extra relation needed for $SU(2)$

$\mathcal{U}_q(\mathfrak{sl}_2)$

Classical Lie algebra

$$[H, X_\pm] = \pm 2 X_\pm \ , \quad [X_+, X_-] = H$$

Quantum group: algebra generated by $1, X_\pm$ and $q^{\pm H/2}$

$$[X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}} \ , \quad Ad_{q^{H/2}} X_\pm = q^{\pm 1} X_\pm$$

Co-product:

$$\Delta q^{\pm H/2} = q^{\pm H/2} \otimes q^{\pm H/2} \ , \quad \Delta X_\pm = X_\pm \otimes q^{H/2} + q^{-H/2} \otimes X_\pm$$

η-space time

For AdS_5 the η deformed space time is given by

$$ds_5^2 = -\frac{1+\rho^2}{1-\varkappa^2\rho^2}dt^2 + \frac{d\rho^2}{(1-\varkappa^2\rho^2)(1+\rho^2)} + \frac{\rho^2\cos^2\zeta}{1+\varkappa^2\rho^4\sin^2\eta}d\psi_1^2$$
$$\frac{1}{1+\varkappa^2\rho^4\sin^2\eta}d\zeta^2 + \rho^2\sin^2\zeta\, d\psi_2^2$$