Foliations, non-commutative geometry and exceptional generalized geometry

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Outline

1. Introduction
   - M-theory flux compactifications to AdS_3
   - The chiral and nonchiral loci
   - A topological no-go theorem

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   - M-theory flux compactifications to AdS_3
   - The chiral and nonchiral loci
   - A topological no-go theorem
2. Kähler-Atiyah formulation for the $\mathcal{N} = 1$ case
   - The non-chiral locus
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Outline

Motivation

A litmus case for exceptional generalized geometry/exceptional field theory is that of eight-dimensional compactifications, since this is the first dimension for which the "problem of dual gravitons" arises.

Any physically meaningful and mathematically consistent formulation of "$\mathrm{E}_{8(8)}$ geometry" should be able to account for the most general geometry (including fluxes) of supersymmetric backgrounds of eleven-dimensional supergravity given by warped products of N with M, where N, M are spin manifolds of dimensions $\dim N = 3$ and $\dim M = 4$, with N pseudo-Riemannian and M Riemannian and compact.

When the corresponding compactification preserves the minimal amount of supersymmetry ($\mathcal{N}=1$ in 3-dimensional language), the case $N = \mathbb{R}^{2,1}$ leads to the condition that M is conformally of $\mathrm{Spin}(7)$ holonomy. However, the geometry is much more subtle when $N = \mathrm{AdS}_{2,1}$. To have a good testing ground for $\mathrm{E}_{8(8)}$ generalized geometry, we must first:

Give a *useful* geometric description of the most general Riemannian spin 8-manifolds M with fluxes which preserve $\mathcal{N} = 1_3$ supersymmetry when compactifying 11-dim SUGRA down to $\mathrm{AdS}_{2,1}$ manifolds (N, g_3).

A *useful description* is one which satisfies the following conditions:

(a) Is *geometric* in some conceptual sense (i.e. it is not simply a bunch of equations which one makes no attempt to understand conceptually).

(b) Gives a *mathematically equivalent* characterization of the supersymmetry conditions, without loosing information and without making unspecified supplementary assumptions. In particular, such an equivalent description should be *globally* correct.

Previous work on such compactifications:

- Martelli and Sparks (2001): proposes a set of equations which fail to satisfy both conditions (a) and (b) above.
- Tsimpis (2007): proposes equations which satisfy condition (b) but not condition (a).

A useful geometric description involves the theory of cosmooth singular foliations (singular Haefliger structures) endowed with longitudinal G2 structures. In general, such "supersymmetric" foliations can have both minimal and non-minimal components, so some of their leaves may be dense in certain subsets of M. This provides an interesting connection to non-commutative geometry and suggests that "$E_{8(8)}$ geometries" should admit similar singular foliations.

Such a description was given in:

- [1]. E.M. Babalic, C.I. Lazaroiu, *Singular foliations for M-theory compactifications*, JHEP03(2015)116.
- [2]. E. M. Babalic, C. I. Lazaroiu, *Foliated 8-manifolds for M-theory compactifications*, JHEP01(2015)140.

All results of these references are mathematically rigorous and globally valid.

$$\mathbf{M} = N \times M \ , \ \mathbf{G} = \nu_3 \wedge \mathbf{f} + \mathbf{F} \ , \ \mathbf{f} \in \Omega^1(M) \ , \ \mathbf{F} \in \Omega^4(M) \ .$$

When $\mathcal{N} = 1_{d=3}$ supersymmetry is preserved, we have an internal part ξ of the susy generator, which is a *real pinor* field on M.

- If ξ is everywhere chiral $\Longrightarrow$ no fluxes at the classical level, M has $\mathrm{Spin}(7)$ holonomy.
- If ξ is everywhere non-chiral $\Longrightarrow$ regular foliation with leafwise G_2 structure.
- If ξ is chiral somewhere but not everywhere $\Longrightarrow$ cosmooth singular foliation (Haefliger structure) with leafwise G_2 structure.

Compactifications down to AdS_3

- SUGRA action in 11 dimensions (involving the SUGRA fields **g**, **C**, **Ψ**):

$$S_{11} = \frac{1}{2G_{11}^2}\int d^{11}y\left[\mathbf{R}\nu - \frac{1}{2}\mathbf{G}\wedge\star\mathbf{G} - \frac{1}{6}\mathbf{G}\wedge\mathbf{G}\wedge\mathbf{C}\right] + \text{terms involving } \boldsymbol{\Psi}$$

- The metric on $\mathbf{M} = N \times M$ is a warped product:

$$\mathrm{d}\mathbf{s}_{11}^2 = e^{2\Delta}\mathrm{d}s_{11}^2 \quad , \quad \mathrm{d}s_{11}^2 = \mathrm{d}s_3^2 + g_{mn}\mathrm{d}x^m\mathrm{d}x^n \quad , \quad \Delta \in C^\infty(M,\mathbb{R}) \quad .$$

- $\mathbf{G} = e^{3\Delta}G \quad , \quad G = \mathrm{vol}_3\wedge f + F \ , \quad f\in\Omega^1(M), \quad F\in\Omega^4(M)$

- **Susy conditions**: $\delta_\eta\boldsymbol{\Psi}_A = \mathbf{D}_A\eta = 0 \quad , \quad A = 0, \dots, 10$

$$\eta = e^{\frac{\Delta}{2}}\eta \quad \text{with} \quad \eta = \zeta\otimes\xi \quad , \quad \zeta\in\Gamma(N,S_3) \ , \quad \xi\in\Gamma(M,S) \ ,$$

The chiral and nonchiral loci

$$\xi = \xi^+ + \xi^- \quad , \quad \xi^\pm \stackrel{\text{def.}}{=} \frac{1}{2}(1 \pm \gamma(\nu))\xi \; \in \Gamma(M, S^\pm)$$

$$||\xi||^2 = ||\xi^+||^2 + ||\xi^-||^2 = 1 \quad , \quad b = ||\xi^+||^2 - ||\xi^-||^2 \quad \Longleftrightarrow \quad \boxed{||\xi^\pm||^2 = \frac{1}{2}(1 \pm b)}$$

- **The purely non-chiral locus** $\mathcal{U}$ (ξ is Majorana, but not Weyl, $b \neq \pm 1$):

$$\mathcal{U} \stackrel{\text{def.}}{=} \{p \in M | \xi \not\in S_p^+ \cup S_p^-\} = \{p \in M | \xi_p^+ \neq 0 \text{ and } \; \xi_p^- \neq 0\} = \{p \in M | |b(p)| < 1\}$$

- **The chiral locus** $\mathcal{W} = \mathcal{W}^+ \sqcup \mathcal{W}^-$:

$$\mathcal{W}^\pm \stackrel{\text{def.}}{=} \{p \in M | \xi_p \in S_p^\pm\} = \{p \in M | b(p) = \pm 1\} = \{p \in M | \xi_p^\mp = 0\} \quad .$$

- **The loci** $\mathcal{U}^\pm$:

$$\mathcal{U}^\pm \stackrel{\text{def.}}{=} \mathcal{U} \cup \mathcal{W}^\pm = \{p \in M | \xi_p^\pm \neq 0\} \quad .$$

A topological no-go theorem

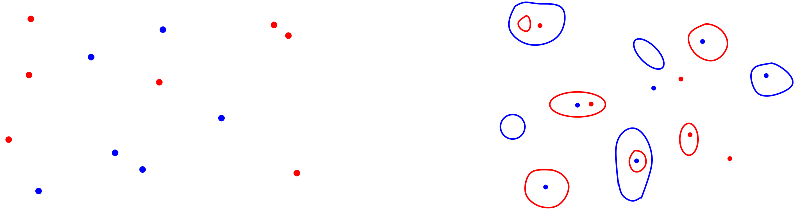
Bianchi identity and e.o.m. for **G**:

$$\mathrm{d}\mathbf{F} = \mathrm{d}\mathbf{f} = 0 \ , \ \mathrm{d}\star\mathbf{G} + \frac{1}{2}\mathbf{G}\wedge\mathbf{G} = 0 \ . \tag{1}$$

Theorem. Assume that the supersymmetry conditions, the Bianchi identity and equations of motion for **G** as well as the Einstein equations are satisfied. There exist only the following four possibilities:

1. The set $\mathcal{W}^+$ coincides with M and hence $\mathcal{W}^-$ and $\mathcal{U}$ are empty. In this case, ξ is a chiral spinor of positive chirality which is covariantly constant on M and we have $\kappa = f = F = 0$ while Δ is constant on M.
2. The set $\mathcal{W}^-$ coincides with M and hence $\mathcal{W}^+$ and $\mathcal{U}$ are empty. In this case, ξ is a chiral spinor of negative chirality which is covariantly constant on M and we have $\kappa = f = F = 0$ while Δ is constant on M.
3. The set $\mathcal{U}$ coincides with M and hence $\mathcal{W}^+$ and $\mathcal{W}^-$ are empty.
4. At least one of the sets $\mathcal{W}^+$ or $\mathcal{W}^-$ is non-empty but both of these sets have empty interior. In this case, $\mathcal{U}$ is dense in M and the union $\mathcal{W} = \mathcal{W}^+ \cup \mathcal{W}^-$ coincides with the topological frontier of $\mathcal{U}$.

Character of chiral and nonchiral loci



Sketch of chiral loci of two subcases of Case 4: when $\mathcal{W}^+$ and $\mathcal{W}^-$ are finite sets of points and when they are submanifolds of various dimensions. The non-chiral locus $\mathcal{U}$ is the complement of $\mathcal{W}$ in M and is indicated by white space.

Reduction of SUSY to a CGKS problem

For ζ a Killing spinor on N, susy conditions $\Longrightarrow$ CGKS equations on M:

$$\boxed{\mathbb{D}\xi = 0 \quad , \quad Q\xi = 0}$$

$$\mathbb{D}_X = \nabla^S_X + \frac{1}{4}\gamma(X\lrcorner F) + \frac{1}{4}\gamma((X_\sharp \wedge f)\nu) + \kappa\gamma(X\lrcorner\nu) \quad , \quad X \in \Gamma(M, TM)$$

$$Q = \frac{1}{2}\gamma(\mathrm{d}\Delta) - \frac{1}{6}\gamma(\iota_f \nu) - \frac{1}{12}\gamma(F) - \kappa\gamma(\nu)$$

Kähler-Atiyah formalism – rigorous formulation of the "method of bilinears" using Kähler-Atiyah bundles: eliminate pinors using the differential forms:

$$\mathscr{B}(\xi, \gamma_{a_1\ldots a_k}\xi')e^{a_1\cdots a_k} \in \Omega^k(M) \quad , \quad \xi, \xi' \in \Gamma(S, M)$$

and perform operations using the **geometric product**:

$$\boxed{\omega \diamond \eta = \sum_{k=0}^{d} (-1)^{\left[\frac{k+1}{2}\right]} \pi^k(\omega) \triangle_k \eta} \quad , \quad \triangle_k = \frac{1}{k!}\wedge_k \quad , \quad \omega, \eta \in \Omega(M)$$

Geometric algebra analysis for the $\mathcal{N}=1$ case

Theorem: Giving a globally-defined smooth real pinor $\xi \in \Gamma(M,S)$ satisfying the CGKS equations is equivalent to giving a globally-defined inhomogeneous form:

$$\boxed{\check{E} = \frac{1}{16}\sum_{k=1}^{8}\frac{1}{k!}\mathscr{B}(\xi,\gamma_{a_1\dots a_k}\xi)e^{a_1\cdots a_k} = \frac{1}{16}(1+V+Y+Z+b\nu)} \in \Omega(M)$$

such that:

$$\boxed{\nabla_m \check{E} = -[\check{A}_m,\check{E}]_{-,\diamond} \quad , \quad \check{Q}\diamond\check{E}=0} \qquad (2)$$

where

$$||\xi||^2 = 1\ ,\ b\in\mathcal{C}^\infty(M,\mathbb{R})(\mathbb{R},M)\ ,\ V\in\Omega^1(M)\ ,\ Y\in\Omega^4(M)\ ,\ Z\in\Omega^5(M)$$
$$\check{A}_m = \frac{1}{4}e_m \lrcorner F + \frac{1}{4}(e_{m_\sharp}\wedge f)\diamond\nu + \kappa e_{m_\sharp}\diamond\nu \quad , \quad \check{Q} = \frac{1}{2}\mathrm{d}\Delta - \frac{1}{6}f\diamond\nu - \frac{1}{12}F - \kappa\nu$$

Geometric algebra analysis for the $\mathcal{N} = 1$ case

The **Fierz identities** are encoded by the relations:

$$\boxed{\check{E}^2 = \check{E} \ \ , \ \ \mathcal{S}(\check{E}) = 1 \ \ , \ \ \tau(\check{E}) = \check{E} \ \ , \ \ |\mathcal{S}(\nu\check{E})| < 1} \tag{3}$$

and are equivalent with:

$$\begin{aligned}
&||V||^2 = 1 - b^2 > 0 \ \ , \\
&\iota_V * Z = 0 \\
&\iota_V Z = Y - b * Y \\
&(\iota_u(*Z)) \wedge (\iota_v(*Z)) \wedge (*Z) = -6 < u \wedge V, v \wedge V > \iota_V \nu \ \ , \ \ \forall u, v \in \Omega^1(M)
\end{aligned}$$

The non-chiral locus

One has the decomposition $\boxed{\check{E}|_{\mathcal{U}} = \frac{1}{16}(1+V+b\nu)(1+\psi) = P|_{\mathcal{U}}\Pi}$ where

$$P = \frac{1}{2}(1+V+b\nu) \quad , \quad \Pi = \frac{1}{8}(1+\psi)$$

(3) is equivalent with:

$$\boxed{V^2 = ||V||^2 = 1-b^2 \quad , \quad Y = (1+b\nu)\psi \quad , \quad Z = V\psi}$$

$$\text{where}: \quad \psi = \frac{1}{1-b^2}\iota_V Z \qquad (\perp V)$$

Let: $\varphi = *(\hat{V}\wedge\psi) = *_\perp\psi \quad (\perp V)$, where $\hat{V} = \frac{V}{||V||}$

The self-dual and anti-selfual parts of ψ:

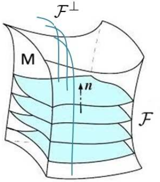
$$\psi^{\pm} = \frac{1}{2}(\psi \pm *\psi) = \frac{1}{2}(\psi \pm \hat{V}\wedge\varphi) \in \Omega^4(\mathcal{U})$$

The supersymmetry conditions imply:

$$\mathrm{d}\boldsymbol{\omega} = 0 \quad , \quad \boldsymbol{\omega} = \mathbf{f} - \mathrm{d}\mathbf{b} \tag{4}$$

where $\boldsymbol{\omega} = 4\kappa e^{3\mathrm{d}\Delta} V \quad , \quad \mathbf{b} = e^{3\mathrm{d}\Delta} b \quad , \quad \mathbf{f} = e^{3\mathrm{d}\Delta} f$

The Frobenius integrability theorem implies that the (generalized) distribution $\mathcal{D} = \ker V$ (or $\boldsymbol{\omega}$) integrates to a (singular) codimension one foliation $\mathcal{F}$, which has a longitudinal G_2 structure given by the coassociative 4-form ψ (or equivalently by the associative 3-form $\varphi = *_\perp \psi$). The complementary distribution determine a foliation $\mathcal{F}^\perp$ whose leaves are integral curves of $n = \hat{V}^\sharp = \frac{V^\sharp}{||V||}$.



Parameterization of the 4-form fluxes

Since any form can be decomposed into parallel and orthogonal parts to any one-form, we have:

$$F = F_\perp + \hat{V} \wedge F_\top \quad , \quad f = f_\perp + f_\top \hat{V}$$

with components $F_\perp, F_\top, f_\perp, f_\top \in \Omega_7(M, \mathcal{D})$ living on the 7-dim. distribution.

The G_2 structure gives decompositions:

$$F_\perp = F_\perp^{(1)} + F_\perp^{(7)} + F_\perp^{(27)} \equiv F_\perp^{(7)} + F_\perp^{(\mathcal{S})} \quad \in \Omega^4(M, \mathcal{D})$$
$$F_\top = F_\top^{(1)} + F_\top^{(7)} + F_\top^{(27)} \equiv F_\top^{(7)} + F_\top^{(\mathcal{S})} \quad \in \Omega^3(M, \mathcal{D}) \quad , \quad \mathcal{D} = T\mathcal{F}$$

with the parameterization:

$$\boxed{F_\perp^{(7)} = \alpha_1 \wedge \varphi \quad , \quad F_\perp^{(S)} = -\hat{h}_{kl} e^k \wedge \iota_{e^l} \psi = -\frac{4}{7}\mathrm{tr}_g(\hat{h})\psi - h_{kl}^{(0)} e^k \wedge \iota_{e^l}\psi}$$

$$\boxed{F_\top^{(7)} = -\iota_{\alpha_2}\psi \quad , \quad F_\top^{(S)} = \chi_{kl} e^k \wedge \iota_{e^l}\varphi = \frac{3}{7}\mathrm{tr}_g(\chi)\varphi - \chi_{kl}^{(0)} e^k \wedge \iota_{e^l}\varphi}$$

$\alpha_1, \alpha_2 \in \Omega^1(M, \mathcal{D})$ and $\hat{h}, \chi$ are symmetric tensors.

Solving the $\check{Q}$-constraints.

Theorem 1. Let $||V|| = \sqrt{1-b^2}$. Then the $\check{Q}$-constraints are *equivalent* with the following relations, which determine (in terms of $\Delta, b, \hat{V}, \psi$ and f) the components of $F_\top^{(1)}$, $F_\perp^{(1)}$ and $F_\top^{(7)}$, $F_\perp^{(7)}$:

$$\boxed{\begin{aligned}
\alpha_1 &= \frac{1}{2||V||}(f - 3b\mathrm{d}\Delta)_\perp \ , \\
\alpha_2 &= -\frac{1}{2||V||}(bf - 3\mathrm{d}\Delta)_\perp \ , \\
\mathrm{tr}_g(\hat{h}) &= -\frac{3}{4}\mathrm{tr}_g(h) = \frac{1}{2||V||}(bf - 3\mathrm{d}\Delta)_\top \ , \\
\mathrm{tr}_g(\hat{\chi}) &= -\frac{3}{4}\mathrm{tr}_g(\chi) = 3\kappa - \frac{1}{2||V||}(f - 3b\mathrm{d}\Delta)_\top \ .
\end{aligned}} \qquad (5)$$

Notice that the $\check{Q}$-constraints do *not* determine the components $F_\top^{(27)}$ and $F_\perp^{(27)}$.

Intrinsic and extrinsic geometry of the foliation

The **fundamental equations** of the foliation:

$$\begin{aligned}
&\nabla_n n = H \quad (\perp n)\ , \quad (\textit{Gauss eq. for } \mathcal{F}^\perp)\\
&\nabla_{X_\perp} n = -AX_\perp \quad (\perp n)\ , \quad (\textit{Weingarten eq. for } \mathcal{F})\\
&\nabla_n(X_\perp) = -g(H, X_\perp)n + D_n(X_\perp)\ , \quad (\textit{Weingarten eq. for } \mathcal{F}^\perp)\\
&\nabla_{X_\perp}(Y_\perp) = \nabla^\perp_{X_\perp}(Y_\perp) + g(AX_\perp, Y_\perp)n \quad (\textit{Gauss eq. for } \mathcal{F})
\end{aligned}$$

Have:

$$D_n\varphi = 3\iota_\vartheta\psi \quad , \quad D_n\psi = -3\vartheta\wedge\varphi \ , \tag{6}$$

where $\vartheta \in \Omega^1(\mathcal{D})$ parameterizes the adapted part of the normal connection. The **torsion forms** $\tau_k \in \Omega^k(M, \mathcal{D})$ of the longitudinal G_2 structure are uniquely determined by:

$$\boxed{\mathrm{d}_\perp\psi = 4\tau_1\wedge\psi + *_\perp\tau_2\ , \quad \mathrm{d}_\perp\varphi = \tau_0\psi + 3\tau_1\wedge\varphi + *_\perp\tau_3}$$

Theorem 2. Let $||V|| = \sqrt{1-b^2}$ and suppose that (F, f) is consistent with the quadruple $(\Delta, b, \hat{V}, \psi)$, i.e. that the $\check{Q}$-constraints are satisfied. Then the supersymmetry constraints are *equivalent* with the following conditions:

1. The function $b \in \mathcal{C}^\infty(M, (-1, 1))$ satisfies:

$$\boxed{e^{-3\Delta}\mathrm{d}(e^{3\Delta}b) = f - 4\kappa\sqrt{1-b^2}\hat{V}} \tag{7}$$

2. The fundamental tensors H and A of $\mathcal{F}$ and $\mathcal{F}^\perp$ are given by the following expressions in terms of b, ψ and f, F:

$$\begin{aligned} H_\sharp &= \frac{2}{||V||}\alpha_2 = -\frac{1}{||V||^2}(bf - 3\mathrm{d}\Delta)_\perp \,, \\ AX_\perp &= \frac{1}{||V||}\Big[(b\chi^{(0)}_{ij} - h^{(0)}_{ij})X^j_\perp e^i + \frac{1}{7}(14\kappa b - 8\mathrm{tr}_g(\hat{h}) - 6b\,\mathrm{tr}_g(\hat{\chi}))X_\perp\Big] = \\ &= \frac{1}{||V||}\Big[(b\chi^{(0)}_{ij} - h^{(0)}_{ij})X^j_\perp e^i + \frac{1}{7}\Big(-4\kappa b + 9||V||(\mathrm{d}\Delta)_\top - \frac{1}{||V||}(bf - 3\mathrm{d}\Delta)_\top\Big)X_\perp\Big] \,, \end{aligned} \tag{8}$$

3. The one-form $\vartheta \in \Omega(\mathcal{D})$ is given by the following relation in terms of Δ, b and f:

$$\boxed{\vartheta = \frac{b\alpha_2 - \alpha_1}{3||V||} = \frac{1}{6||V||^2}\big[-(1+b^2)f + 6b\mathrm{d}\Delta\big]_\perp} \tag{9}$$

4. The torsion classes of the leafwise G_2 structure (in the conventions of are given by the following expressions in terms of Δ, b and f, F:

$$\begin{aligned} \boldsymbol{\tau}_0 &= \frac{4}{7||V||}(b\,\mathrm{tr}_g(\hat{h}) - \mathrm{tr}_g(\hat{\chi}) + 7\kappa) = \frac{4}{7||V||}\Big[4\kappa + \frac{(1+b^2)f_\top - 6b(\mathrm{d}\Delta)_\top}{2||V||}\Big] \,, \\ \boldsymbol{\tau}_1 &= -\frac{3}{2}(\mathrm{d}\Delta)_\perp \,, \\ \boldsymbol{\tau}_2 &= 0 \,, \\ \boldsymbol{\tau}_3 &= \frac{1}{||V||}(\chi^{(0)}_{ij} - bh^{(0)}_{ij})e^i \wedge \iota_{e_j}\varphi = \frac{1}{||V||}(F^{(27)}_\top - b *_\perp F^{(27)}_\perp) \,. \end{aligned} \tag{10}$$

Eliminating the fluxes

Theorem 3. The following statements are equivalent:

(A) There exist $f \in \Omega^1(M)$ and $F \in \Omega^4(M)$ such that the susy equations admit at least one non-trivial solution ξ which is everywhere non-chiral (and which we can take to be everywhere of norm one).

(B) There exist $\Delta \in \mathcal{C}^\infty(M,\mathbb{R})$, $b \in \mathcal{C}^\infty(M,(-1,1))$, $\hat{V} \in \Omega^1(M)$ and $\varphi \in \Omega^3(M)$ such that:

1. Δ, b, $\hat{V}$ and φ satisfy the conditions:

$$||\hat{V}|| = 1 \ \ , \ \ \iota_{\hat{V}}\varphi = 0 \ \ . \tag{11}$$

   Furthermore, the Frobenius distribution $\mathcal{D} \stackrel{\mathrm{def.}}{=} \ker \hat{V}$ is integrable and we let $\mathcal{F}$ be the foliation which integrates it.

2. The quantities H, $\mathrm{tr}A$ and ϑ of the foliation $\mathcal{F}$ are given by:

$$\boxed{\begin{aligned} H_\sharp &= \frac{2}{||V||}\alpha_2 = -\frac{b}{||V||^2}(\mathrm{d}b)_\perp + 3(\mathrm{d}\Delta)_\perp = \frac{\mathrm{d}_\perp ||V||}{||V||} + 3(\mathrm{d}\Delta)_\perp \ , \\ \mathrm{tr}A &= 12(\mathrm{d}\Delta)_\top - \frac{b(\mathrm{d}b)_\top}{||V||^2} - 8\kappa\frac{b}{||V||} = 12\partial_n\Delta + \frac{\partial_n ||V|| - 8\kappa b}{||V||} \ , \\ \vartheta &= -\frac{1+b^2}{6||V||^2}(\mathrm{d}b)_\perp + \frac{b}{2}(\mathrm{d}\Delta)_\perp \ \ . \end{aligned}} \tag{12}$$

3. φ induces a leafwise G_2 structure on $\mathcal{F}$ whose torsion classes satisfy:

$$\boxed{\begin{aligned} \boldsymbol{\tau}_0 &= \frac{4}{7||V||}\left[2\kappa(3+b^2) - \frac{3b}{2}||V||(\mathrm{d}\Delta)_\top + \frac{1+b^2}{2||V||}(\mathrm{d}b)_\top\right] \ , \\ \boldsymbol{\tau}_1 &= -\frac{3}{2}(\mathrm{d}\Delta)_\perp \ , \\ \boldsymbol{\tau}_2 &= 0 \ \ . \end{aligned}} \tag{13}$$

Eliminating the fluxes

In this case, the forms f and F are uniquely determined by b, Δ, V and φ. Namely, the one-form f is given by:

$$\boxed{f = 4\kappa V + e^{-3\Delta}\mathrm{d}(e^{3\Delta}b)}~, \qquad (14)$$

while F is given as follows:

(a) We have $F^{(1)}_{\top} = \frac{3}{7}\mathrm{tr}_g(\chi)\varphi = -\frac{4}{7}\mathrm{tr}_g(\hat{\chi})\varphi$ and $F^{(1)}_{\perp} = -\frac{4}{7}\mathrm{tr}_g(\hat{h})\psi$, with:

$$\boxed{\mathrm{tr}_g(\hat{h}) = -\frac{3||V||}{2}(\mathrm{d}\Delta)_\top + 2\kappa b + \frac{b}{2||V||}(\mathrm{d}b)_\top ~,~ \mathrm{tr}_g(\hat{\chi}) = \kappa - \frac{1}{2||V||}(\mathrm{d}b)_\top} \qquad (15)$$

(b) We have $F^{(7)}_{\top} = -\iota_{\alpha_2}\psi$ and $F^{(7)}_{\perp} = \alpha_1 \wedge \varphi$, with:

$$\boxed{\alpha_1 = \frac{1}{2||V||}(\mathrm{d}b)_\perp ~,~ \alpha_2 = -\frac{b}{2||V||}(\mathrm{d}b)_\perp + \frac{3||V||}{2}(\mathrm{d}\Delta)_\perp = \frac{\mathrm{d}_\perp ||V||}{||V||} + \frac{3||V||}{2}(\mathrm{d}\Delta)_\perp} \qquad (16)$$

(c) We have:

$$\boxed{\begin{aligned} h^{(0)}_{ij} &= -\frac{b}{4||V||}[\langle e_i \lrcorner \varphi, e_j \lrcorner \boldsymbol{\tau}_3\rangle + (i \leftrightarrow j)] - \frac{1}{||V||}A^{(0)}_{ij} = \frac{b}{||V||}t_{ij} - \frac{1}{||V||}A^{(0)}_{ij} ~,\\ \chi^{(0)}_{ij} &= -\frac{1}{4||V||}[\langle e_i \lrcorner \varphi, e_j \lrcorner \boldsymbol{\tau}_3\rangle + (i \leftrightarrow j)] - \frac{b}{||V||}A^{(0)}_{ij} = \frac{1}{||V||}t_{ij} - \frac{b}{||V||}A^{(0)}_{ij} ~, \end{aligned}}$$

where $A^{(0)}$ is the traceless part of the Weingarten tensor of $\mathcal{F}$ while $\boldsymbol{\tau}_3$ is the rank 3 torsion class of the leafwise G_2 structure.

Remark. The differential and codifferential relations for V do not depend on the fluxes:

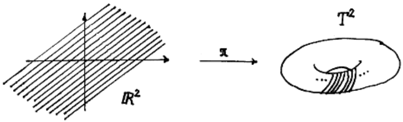
$$\begin{aligned} \mathrm{d}V &= 3V \wedge (\mathrm{d}\Delta)_\perp \ , \\ \delta V &= -8\kappa b + 12||V||(\mathrm{d}\Delta)_\top \ . \end{aligned}$$

Topology of $\mathcal{F}$ in the everywhere non-chiral case

Assuming that M is compact and connected and V nowhere vanishing, the foliation $\mathcal{F}$ is defined by the closed nowhere vanishing one-form $\boldsymbol{\omega} = 4\kappa e^{3\Delta} V$.

$$per_{\boldsymbol{\omega}}([\gamma]) = \oint_{\gamma} \boldsymbol{\omega} = \oint_{\gamma} \mathbf{f} = per_{\mathbf{f}}([\gamma])$$

The foliation $\mathcal{F}$ is a fibration (namely a fibration over S^1) iff. all periods of $\boldsymbol{\omega}$ can be commonly rescaled to integers ($\boldsymbol{\omega}$ is projectively rational).



When $\boldsymbol{\omega}$ is projectively irrational, each leaf of $\mathcal{F}$ is non-compact and dense in M, hence $\mathcal{F}$ is not a fibration. The periods of **G** on *noncompact* 4-cycles of **M** are not quantized, thus we cannot conclude $per_{\mathbf{f}}([\gamma])$ need to be commonly rescalable to integers in M-theory. Hence the case when $\mathcal{F}$ is not a fibration might arise as a consistent background in M-theory. Morita-equivalent models for NC geometry of leaf space, Latour obstruction etc: see refs.

The singular distribution

G structure	$\mathrm{Spin}(7)_+$	$\mathrm{Spin}(7)_-$	G_2 (on $\mathcal{D}\|_\mathcal{U}$)	$\mathrm{SO}(7)$ ($\mathcal{D}\|_\mathcal{U}$)
spinor	η^+	η^-	$\eta_0=\frac{1}{\sqrt{2}}(\eta^+ + \eta^-)$	—
idempotent	$\Pi^+=\frac{1}{16}(1+\Phi^+ + \nu)$	$\Pi^-=\frac{1}{16}(1+\Phi^- - \nu)$	$\Pi=\Pi^+ + \Pi^- = \frac{1}{8}(1+\psi)$	$P=\frac{1}{2}(1+V+b\nu)$
forms	$\Phi^+ = 2\psi^+$	$\Phi^- = 2\psi^-$	φ and $\psi=*_\perp\varphi$	b and V
extends to	$\mathcal{U}^+$	$\mathcal{U}^-$	$\mathcal{U}$	$\mathcal{U}$

Summary of various G structures and of their reflections in the Kähler-Atiyah algebra.

We have:

$$M=\bar{\mathcal{U}}=\mathcal{U}\cup\mathcal{W}\quad,\quad \mathcal{W}=\mathrm{Fr}\,\mathcal{U}$$

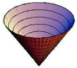
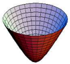
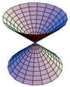
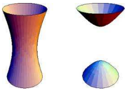
The 4-form $\psi^\pm$ has a continuous extension to the locus $\mathcal{U}^\pm \stackrel{\mathrm{def.}}{=} \mathcal{U}\cup\mathcal{W}^\pm$:

$$\bar{\psi}^\pm=\frac{1}{1\pm b}Y^\pm|_{\mathcal{U}^\pm}$$

and describes a $\mathrm{Spin}(7)_\pm$ structure. On the locus $\mathcal{W}^\pm$ we have $b=\pm1,, Y^\pm=2\bar{\psi}^\pm$ and $V=Z=Y^\mp=0$. The generalized distribution $\mathcal{D}=\ker V=\ker\boldsymbol{\omega}$ determines a cosmooth singular foliation $\bar{\mathcal{F}}$ of M, which degenerates along the chiral locus $\mathcal{W}$.

Since $\mathcal{D}$ is cosmooth rather than smooth, the notion of singular foliation which is appropriate in our case is that of Haefliger structure. Singular foliations can be very complicated and little is known about their topology and geometry. The description of $\bar{\mathcal{F}}$ simplifies when $\boldsymbol{\omega}$ is of Morse or Bott-Morse type.

The singular foliation in the Morse case

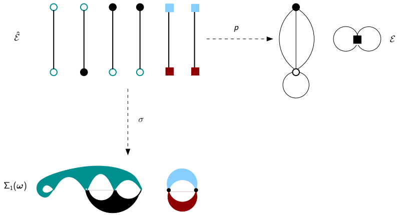
Name	Morse index	Local form of $\mathcal{L}_p$	Local form of regular leaves
Center	0 or n	$\bullet = \{p\}$	
Weak saddle	between 2 and $n-2$		
Strong saddle	1 or $n-1$		

Types of singular points p. The first and third figure on the right depict the case $d = 3$ for centers and strong saddles, while the second figure attempts to depict the case $d > 3$ for a weak saddle (notice that weak saddles do not exist unless $d > 3$). In that case, the topology of the leaves does not change locally when they "pass through" the weak saddle point.

Classification of leaves

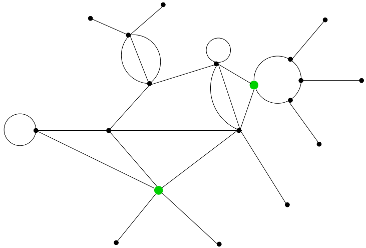
type of L	compactifiable		non-compactifiable
	compact	non-compact	
ordinary	Y	—	Y
special	—	Y	Y
$\mathrm{Card}(\mathrm{Fr}L)$	finite		infinite

Classification of the leaves of $\mathcal{F}_{\boldsymbol{\omega}}$, where the allowed combinations are indicated by the letter "Y". A compactifiable leaf is ordinary iff. it is compact and it is special iff. it is non-compact. A non-compactifiable leaf may be either ordinary or special. Non-compactifiable leaves coincide with those leaves whose frontier is an infinite set, while compactifiable leaves are those leaves whose frontier is finite.



Example of the graphs $\hat{\mathcal{E}}$ and $\mathcal{E}$ for a Morse form foliation $\bar{\mathcal{F}}_{\boldsymbol{\omega}}$ with two compact strong singular leaves. The regular foliation $\mathcal{F}_{\boldsymbol{\omega}}$ of M^* has four special leaves, each of which is compactifiable; they are depicted using four different colors. At the bottom of the picture, we depict $\Sigma_1(\boldsymbol{\omega})$ as well as the schematic the shape of the special leaves in the case $d = 3$. The strong singular leaves of $\bar{\mathcal{F}}_{\boldsymbol{\omega}}$ correspond to the left and right parts of the figure at the bottom; each of them is a union of two special leaves of $\mathcal{F}_{\boldsymbol{\omega}}$ and of singular points. Each special leaf corresponds to a vertex of $\mathcal{E}$.

The foliation graph



An example of foliation graph. Regular (a.k.a type I) vertices are represented by black dots, while exceptional (a.k.a. type II) vertices are represented by green blobs. All terminal vertices are regular vertices and correspond to center singularities. Notice that the graph can have multiple edges as well as loops.

(a) Foliation graph when $\mathcal{W} = \emptyset$ and $\rho(\boldsymbol{\omega}) = 1$.

(b) Foliation graph when $\mathcal{W} = \emptyset$ and $\rho(\boldsymbol{\omega}) > 1$.

Degenerate foliation graphs in the everywhere non-chiral case.

Open problems:

- Moduli spaces
- Characterize the noncommutative geometry of the leaf space for general cosmooth singular foliations (this requires adapting the work of Androulidakis and Zambon from the smooth to the cosmooth case).
- Give a conceptual explanation of why Haefliger structures appear when studying higher-dimensional supersymmetric backgrounds (a phenomenon which turns out to be quite general).
- Develop the theory of "G-structured" Haefliger structures (can be done by enriching the notion of Haefliger groupoid).
- Connections with the theory of stratifications and G-spaces/G-manifolds. The latter turn out to vastly generalize the extremely special classes of "membrane solutions" which were considered until now in the physics literature (such as the Bena-Warner/LLM solutions).
- Connection with homotopy theory.
- Connections with non-commutative field theory.
- Connections with real algebraic geometry (Nash-Tognolli theorem).

Now that we have a useful description of such backgrounds, understand how it fits in with a good theory of $E_{8(8)}$ exceptional generalized geometry/exceptional double field theory.