LIE SUPERALGEBRAS in (GAUGED) SUPERGRAVITY and EXCEPTIONAL GEOMETRY

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D	7	6	5	4	3
$\mathfrak{e}_n$	$\mathfrak{e}_4 = \mathfrak{sl}(5,\mathbb{R})$	$\mathfrak{e}_5 = \mathfrak{so}(5,5)$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$
R_1	$\overline{\mathbf{10}}$	$\mathbf{16}_c$	$\overline{\mathbf{27}}$	$\mathbf{56}$	$\mathbf{248}$
R_2	$\mathbf{5}$	$\mathbf{10}$	$\mathbf{27}$	$\mathbf{133}$	
R_3	$\overline{\mathbf{5}}$	$\mathbf{16}_s$	$\mathbf{78}$		
R_4	$\mathbf{10}$	$\mathbf{45}$			
R_5	$\mathbf{24}$				

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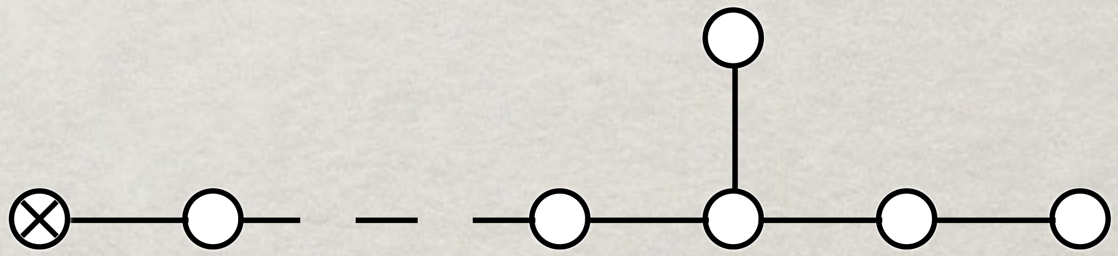
These representations can be derived by extending $\mathfrak{e}_n$ to an infinite-dimensional Borcherds superalgebra $\mathscr{B}$.

Extending $\mathfrak{e}_n$ to $\mathscr{B}$ corresponds to adding a gray node to the Dynkin diagram.



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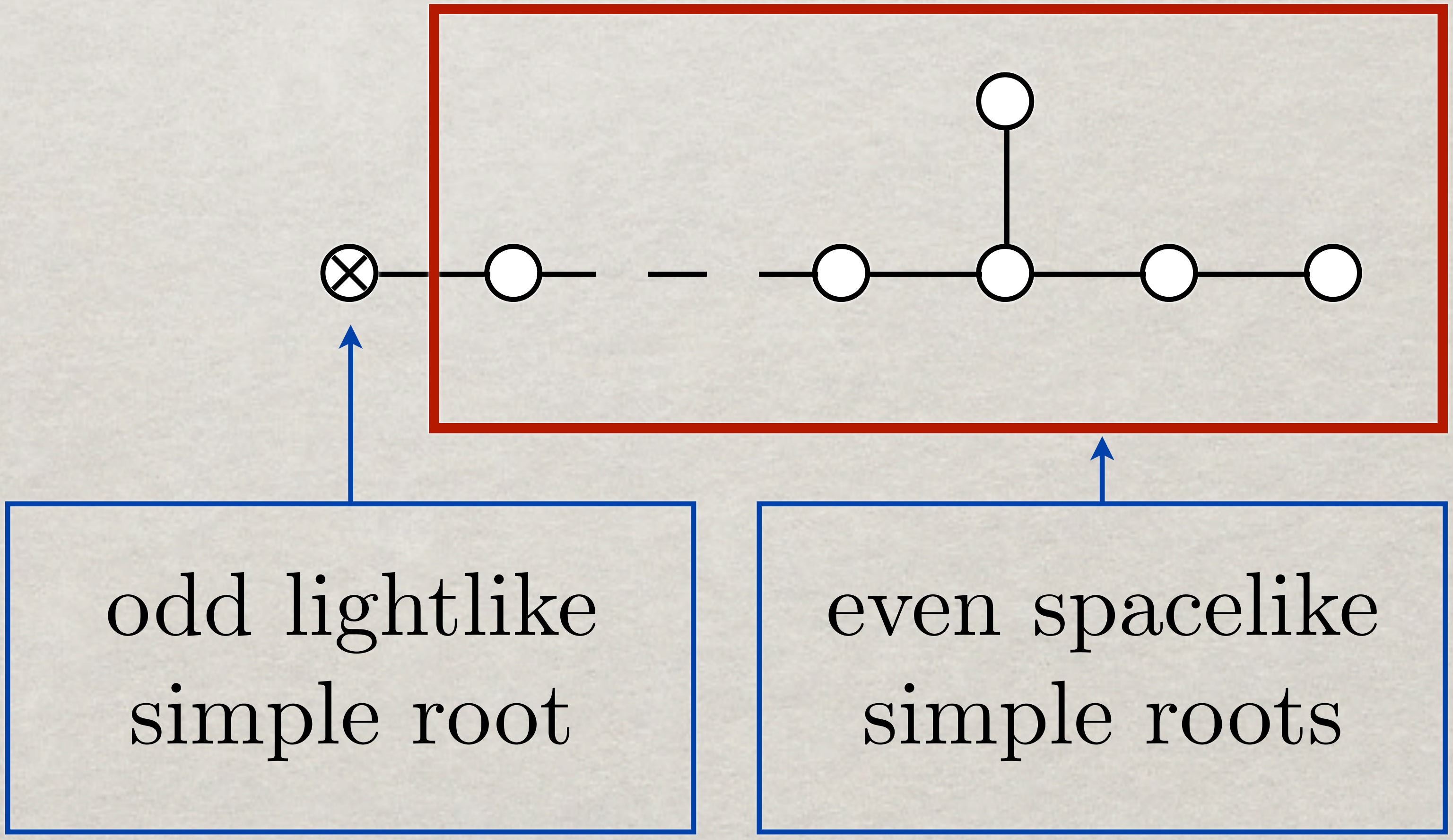

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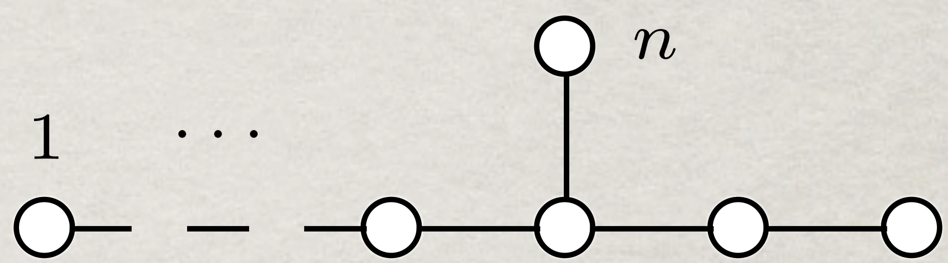
$$a_{IJ} = \begin{pmatrix} 0 & -1 & 0 & \cdots \\ -1 & 2 & -1 & \cdots \\ 0 & -1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The Cartan matrix a_{ij} of $\mathfrak{e}_n \subset \mathscr{B}$

The Kac-Moody algebra $\mathfrak{e}_n$ is constructed from its Dynkin diagram



as the Lie algebra generated by $3n$ elements e_i, f_i, h_i modulo the Chevalley-Serre relations

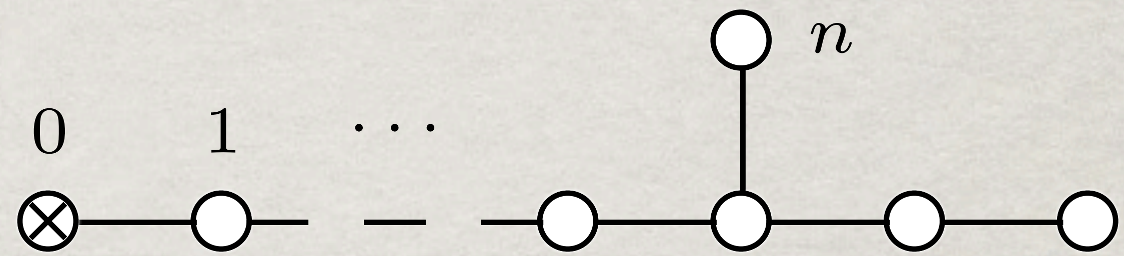
$$[h_i, e_j] = a_{ij} e_j \qquad [e_i, f_j] = \delta_{ij} h_j$$

$$[h_i, f_j] = -a_{ij} f_j \qquad [h_i, h_j] = 0$$

$$(\text{ad } e_i)^{1-a_{ij}}(e_j) = (\text{ad } f_i)^{1-a_{ij}}(f_j) = 0$$

$$i = \quad 1, \ldots, n$$

The Borcherds superalgebra $\mathscr{B}_n$ is constructed from its Dynkin diagram



as the Lie superalgebra generated by $3(n+1)$ elements e_I, f_I, h_I modulo the Chevalley-Serre relations

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odd: e_0, f_0 $\qquad I = 0, 1, \ldots, n$

even: e_i, f_i, h_I $\qquad i = 1, \ldots, n$

Level decomposition of $\mathscr{B}$ with respect to $\mathfrak{e}_n$:

$$\begin{array}{ccccccccccc} & & & & f_0 & & e_i, f_i, h_I & & e_0 & & \\ & & & & \cap & & \cap & & \cap & & \\ \mathscr{B} & = & \cdots & \oplus & \mathscr{U}_{-1} & \oplus & (\mathfrak{e}_n \oplus \mathbb{R}) & \oplus & \mathscr{U}_1 & \oplus & \cdots \end{array}$$

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Basis of the subspace $\mathscr{U}_p$ of $\mathscr{B}$ at level $p \geqslant 1$:

$$E_{\mathcal{M}_1 \cdots \mathcal{M}_p} = [E_{\mathcal{M}_1}, [E_{\mathcal{M}_2}, \ldots [E_{\mathcal{M}_{p-1}}, E_{\mathcal{M}_p}] \cdots]]$$

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The p-form potentials can be combined into an element

$$A = A_m{}^{\mathcal{M}}(\mathrm{d}x^m \otimes E_{\mathcal{M}}) + A_{mn}{}^{\mathcal{M}\mathcal{N}}(\mathrm{d}x^m \wedge \mathrm{d}x^n \otimes E_{\mathcal{M}\mathcal{N}}) + \cdots$$

in the tensor product of the differential algebra and $\mathscr{B}$.

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This can be generalized to gauged supergravity.

[Greitz, Howe, Palmkvist: 1308.4972]

For $D < 11$ the Borcherds superalgebras $\mathscr{B}$ are infinite-dimensional and suggest that $(D-1)$- and D-form potentials corresponding to R_{D-1} and R_D can be added to the theory.

These additional $(D-1)$- and D-form potentials are precisely those allowed by supersymmetry.

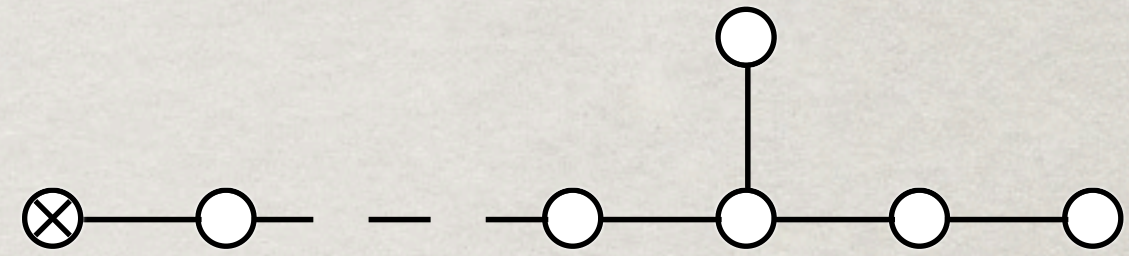
Although non-dynamical, they play an important role in gauged supergravity.

[Bergshoeff, de Roo, Kerstan, Riccioni: hep-th/0506013]

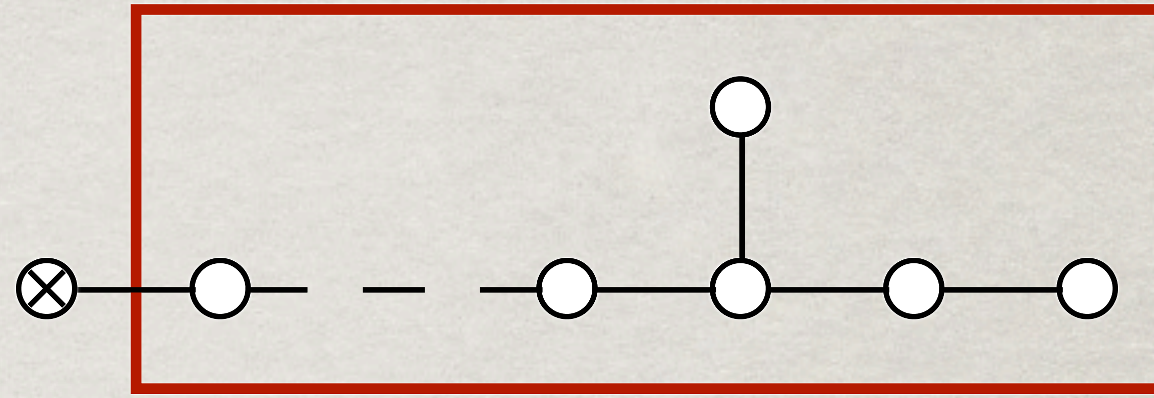
[Bergshoeff, Hartong, Howe, Ortin, Riccioni: 1004.1348]

[Greitz, Howe: 1103.2730, 1103.5053, 1203.5585]

[Howe, Palmkvist: 1503.00015]



$\vdots$						$\vdots$	$\iddots$
$E_{\mathcal{MN}}$					E^{m}	E^{mnpq}	$\cdots$
$E_{\mathcal{M}}$				E_{m}	E^{mn}	E^{mnpqr}	$\cdots$
t_{α}	$\cdots$	F_{mnpqrs}	F_{mnp}	$K^{m}{}_{n}$	E^{mnp}	E^{mnpqrs}	$\cdots$
$F^{\mathcal{M}}$	$\cdots$	F_{mnpqr}	F_{mn}	F^{m}			
$F^{\mathcal{MN}}$	$\cdots$	F_{mnpq}	F_{m}				
$\vdots$	$\iddots$	$\vdots$					



$\vdots$						$\vdots$	$\iddots$
$E_{\mathcal{M}\mathcal{N}}$					E^{m}	E^{mnpq}	$\ldots$
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$F^{\mathcal{M}}$	$\ldots$	F_{mnpqr}	F_{mn}	F^{m}			
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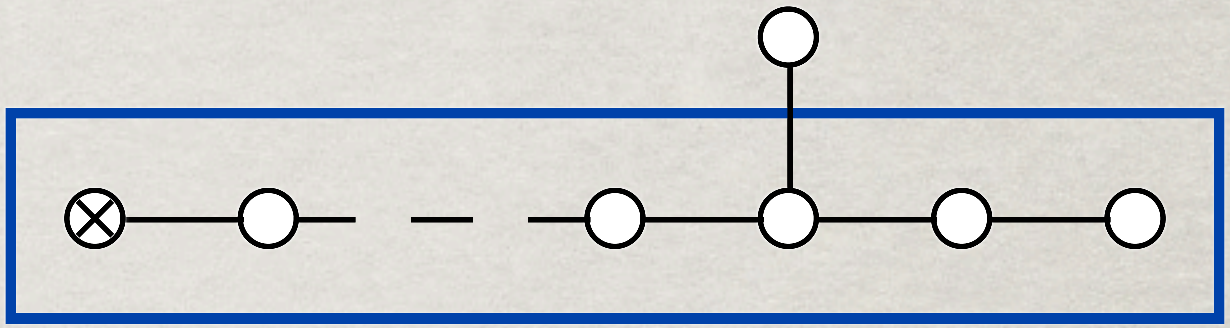

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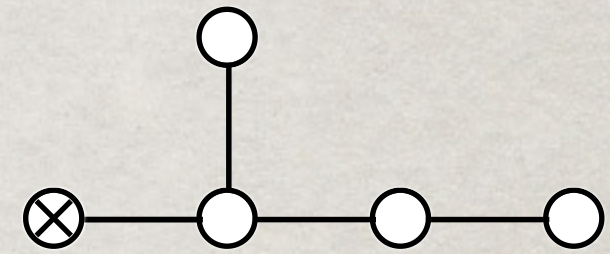
$\vdots$	$\iddots$	$\vdots$					

Example: $n = 4$

$\mathscr{B} \supset \mathfrak{sl}(5) \supset \mathfrak{sl}(4)$



$\vdots$						$\vdots$	$\iddots$
$\mathbf{5}$					$\mathbf{4}$	$\mathbf{1}$	
$\overline{\mathbf{10}}$				$\overline{\mathbf{4}}$	$\mathbf{6}$		
$\mathbf{24}$			$\mathbf{4}$	$(\mathbf{15}+\mathbf{1})$	$\overline{\mathbf{4}}$		
$\mathbf{10}$			$\mathbf{6}$	$\mathbf{4}$			
$\overline{\mathbf{5}}$		$\mathbf{1}$	$\overline{\mathbf{4}}$				
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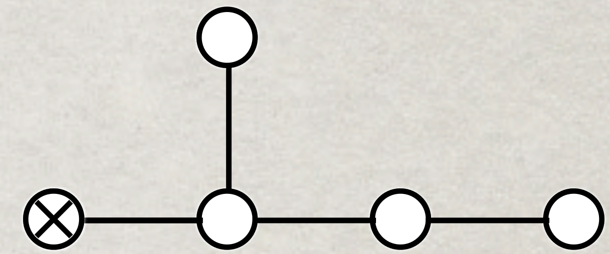

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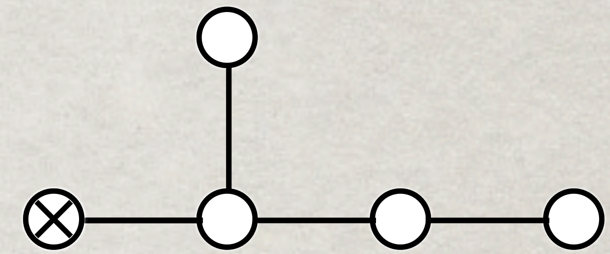

0 1 0 0 0			
0 1 1 0 0	1 1 1 0 0		
0 1 1 1 0	1 1 1 1 0	1 1 2 1 0	
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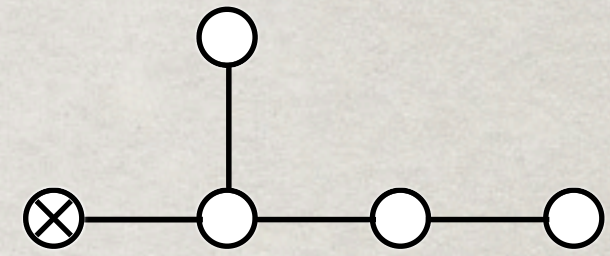

$$\mathscr{U}_1 = (\mathscr{U}_1)^0 \oplus (\mathscr{U}_1)^1 \oplus \cdots$$

$(\mathscr{U}_1)^0$	$(\mathscr{U}_1)^1$		
0 1 0 0 0			
0 1 1 0 0	1 1 1 0 0		
0 1 1 1 0	1 1 1 1 0	1 1 2 1 0	
0 1 1 1 1	1 1 1 1 1	1 1 2 1 1	1 1 2 2 1

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Thus we demand that for any pair of fields A, B, and any pair of elements $U, V \in \mathscr{U}_1$ such that $U^{\mathcal{M}} V^{\mathcal{N}} \partial_{\mathcal{M}} A \, \partial_{\mathcal{N}} B \neq 0$, there is an $x \in \mathfrak{e}_n$ such that $e^{-x} U e^{x}$, $e^{-x} V e^{x} \in (\mathscr{U}_1)^0$.

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Since $[(\mathscr{U}_1)^0, (\mathscr{U}_1)^0] = 0$, this implies $[U, V] = 0$, so that $U^{\mathcal{M}} V^{\mathcal{N}} (\mathbb{P}_2)_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}\mathcal{Q}} = 0$ and then $(\mathbb{P}_2)_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}\mathcal{Q}} \partial_{\mathcal{P}} A \, \partial_{\mathcal{Q}} B = 0$.

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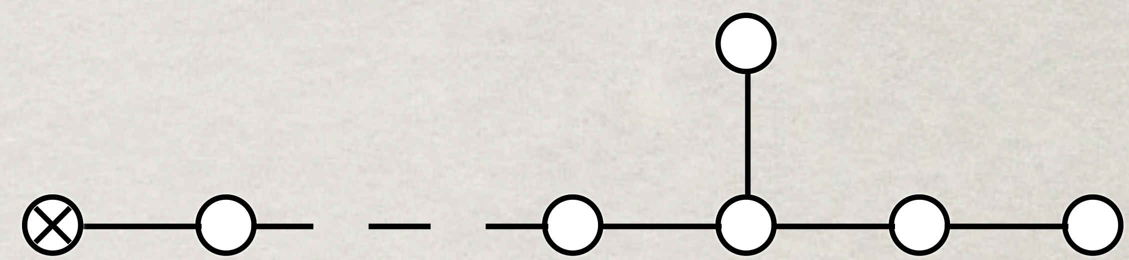
This, together with $(\mathbb{P}_2)_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}\mathcal{Q}} \partial_{\mathcal{P}} \partial_{\mathcal{Q}} A = 0$, is the section condition, which also has solutions corresponding to ten-dimensional type IIB supergravity.



$\vdots$						$\vdots$	$\iddots$
$E_{\mathcal{MN}}$					E^{m}	E^{mnpq}	$\cdots$
$E_{\mathcal{M}}$				E_{m}	E^{mn}	E^{mnpqr}	$\cdots$
t_{α}	$\cdots$	F_{mnpqrs}	F_{mnp}	$K^{m}{}_{n}$	E^{mnp}	E^{mnpqrs}	$\cdots$
$F^{\mathcal{M}}$	$\cdots$	F_{mnpqr}	F_{mn}	F^{m}			
$F^{\mathcal{MN}}$	$\cdots$	F_{mnpq}	F_{m}				
$\vdots$	$\iddots$	$\vdots$					

The transformation of a vector field V under a generalized diffeomorphism generated by a vector field U is given by the generalized Lie derivative

$$\begin{aligned}\mathscr{L}_U V^{\mathcal{M}} &= U^{\mathcal{N}}\partial_{\mathcal{N}}V^{\mathcal{M}} - V^{\mathcal{N}}\partial_{\mathcal{N}}U^{\mathcal{M}} + Y^{\mathcal{M}\mathcal{N}}{}_{\mathcal{P}\mathcal{Q}}\partial_{\mathcal{N}}U^{\mathcal{P}}V^{\mathcal{Q}} \\ &= U^{\mathcal{N}}\partial_{\mathcal{N}}V^{\mathcal{M}} + Z^{\mathcal{M}\mathcal{N}}{}_{\mathcal{P}\mathcal{Q}}\partial_{\mathcal{N}}U^{\mathcal{P}}V^{\mathcal{Q}}\end{aligned}$$

where $Y^{\mathcal{M}\mathcal{N}}{}_{\mathcal{P}\mathcal{Q}}$ and $Z^{\mathcal{M}\mathcal{N}}{}_{\mathcal{P}\mathcal{Q}} = Y^{\mathcal{M}\mathcal{N}}{}_{\mathcal{P}\mathcal{Q}} - \delta_{\mathcal{P}}{}^{\mathcal{M}}\delta_{\mathcal{Q}}{}^{\mathcal{N}}$ are $\mathfrak{e}_n$-invariant tensors, uniquely determined by the requirement that the transformations close under the commutator, $[\mathscr{L}_U, \mathscr{L}_V] = \mathscr{L}_{\frac{1}{2}(\mathscr{L}_U V - \mathscr{L}_V U)}$.

[Coimbra, Strickland-Constable, Waldram: 1112.3989]

[Berman, Cederwall, Kleinschmidt, Thompson: 1208.5884]

The transformations close if the tensor Y satisfies the following identities, up to the section condition,

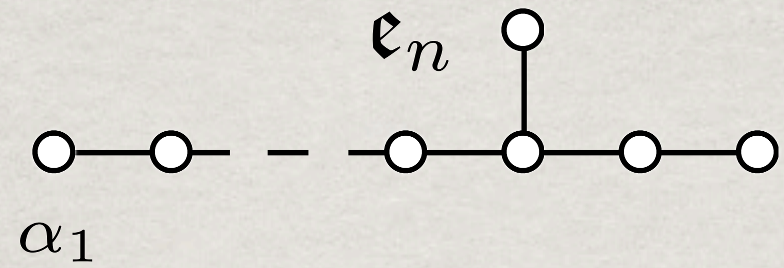
$$Y^{\mathcal{M}\mathcal{N}}{}_{\mathcal{P}\mathcal{Q}}\,\partial_{\mathcal{M}}\otimes\partial_{\mathcal{N}}=0,$$

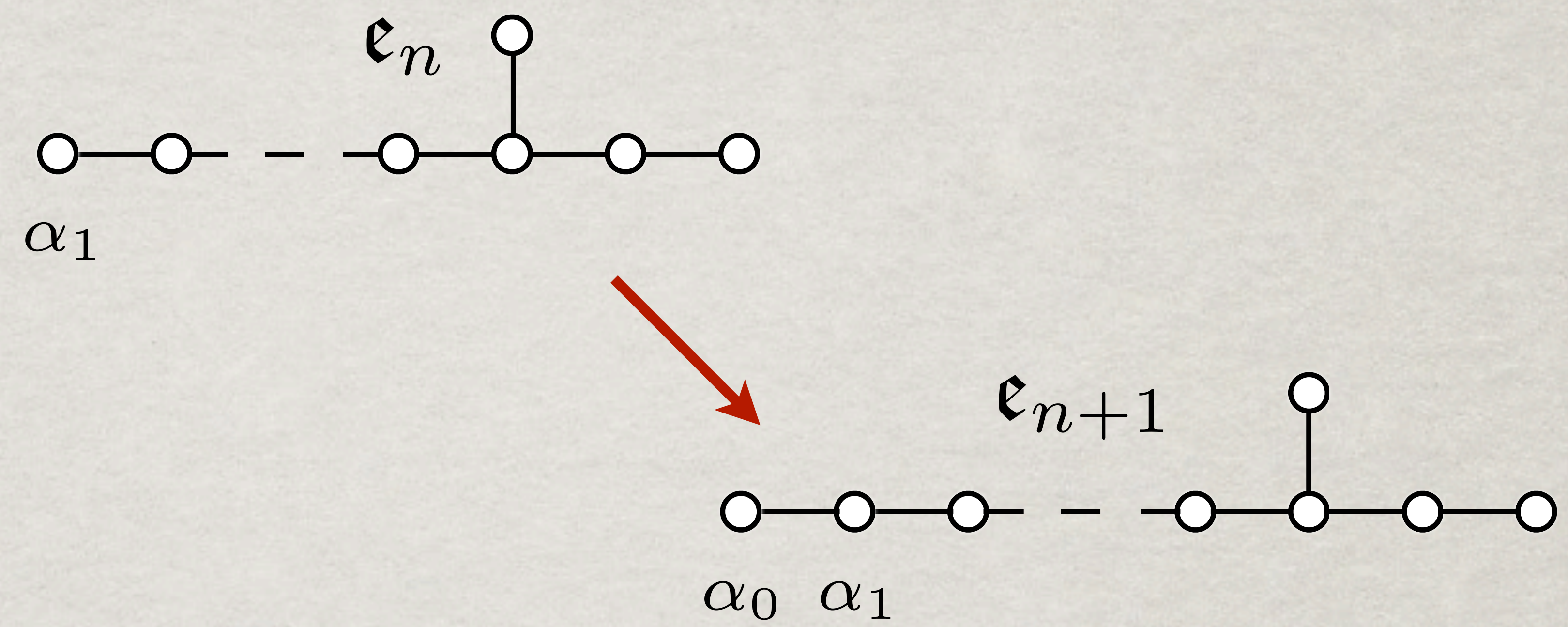
$$(Y^{\mathcal{M}\mathcal{N}}{}_{\mathcal{T}\mathcal{Q}}Y^{\mathcal{T}\mathcal{P}}{}_{\mathcal{R}\mathcal{S}}-Y^{\mathcal{M}\mathcal{N}}{}_{\mathcal{R}\mathcal{S}}\delta^{\mathcal{P}}{}_{\mathcal{Q}})\partial_{(\mathcal{N}}\otimes\partial_{\mathcal{P})}=0,$$

$$\begin{aligned}(Y^{\mathcal{M}\mathcal{N}}{}_{\mathcal{T}\mathcal{Q}}Y^{\mathcal{T}\mathcal{P}}{}_{[\mathcal{S}\mathcal{R}]}+2Y^{\mathcal{M}\mathcal{N}}{}_{[\mathcal{R}|\mathcal{T}|}Y^{\mathcal{T}\mathcal{P}}{}_{\mathcal{S}]\mathcal{Q}}\\ -Y^{\mathcal{M}\mathcal{N}}{}_{[\mathcal{R}\mathcal{S}]}\delta^{\mathcal{P}}{}_{\mathcal{Q}}-2Y^{\mathcal{M}\mathcal{N}}{}_{[\mathcal{S}|\mathcal{Q}|}\delta^{\mathcal{P}}{}_{\mathcal{R}]})\partial_{(\mathcal{N}}\otimes\partial_{\mathcal{P})}=0,\end{aligned}$$

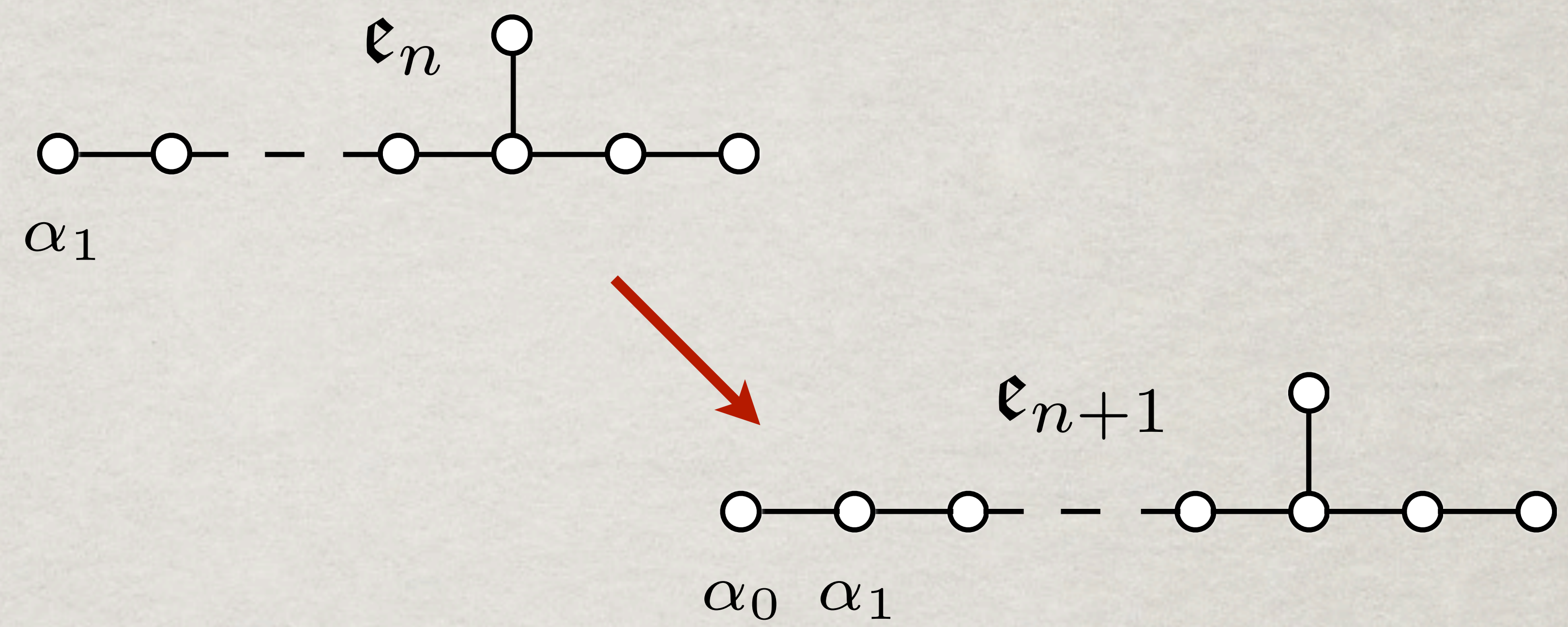
$$\begin{aligned}(Y^{\mathcal{M}\mathcal{N}}{}_{\mathcal{T}\mathcal{Q}}Y^{\mathcal{T}\mathcal{P}}{}_{(\mathcal{S}\mathcal{R})}+2Y^{\mathcal{M}\mathcal{N}}{}_{(\mathcal{R}|\mathcal{T}|}Y^{\mathcal{T}\mathcal{P}}{}_{\mathcal{S})\mathcal{Q}}\\ -Y^{\mathcal{M}\mathcal{N}}{}_{(\mathcal{R}\mathcal{S})}\delta^{\mathcal{P}}{}_{\mathcal{Q}}-2Y^{\mathcal{M}\mathcal{N}}{}_{(\mathcal{S}|\mathcal{Q}|}\delta^{\mathcal{P}}{}_{\mathcal{R})})\partial_{[\mathcal{N}}\otimes\partial_{\mathcal{P}]}=0.\end{aligned}$$

[Berman, Cederwall, Kleinschmidt, Thompson: 1208.5884]


$\mathfrak{e}_n$
α_1

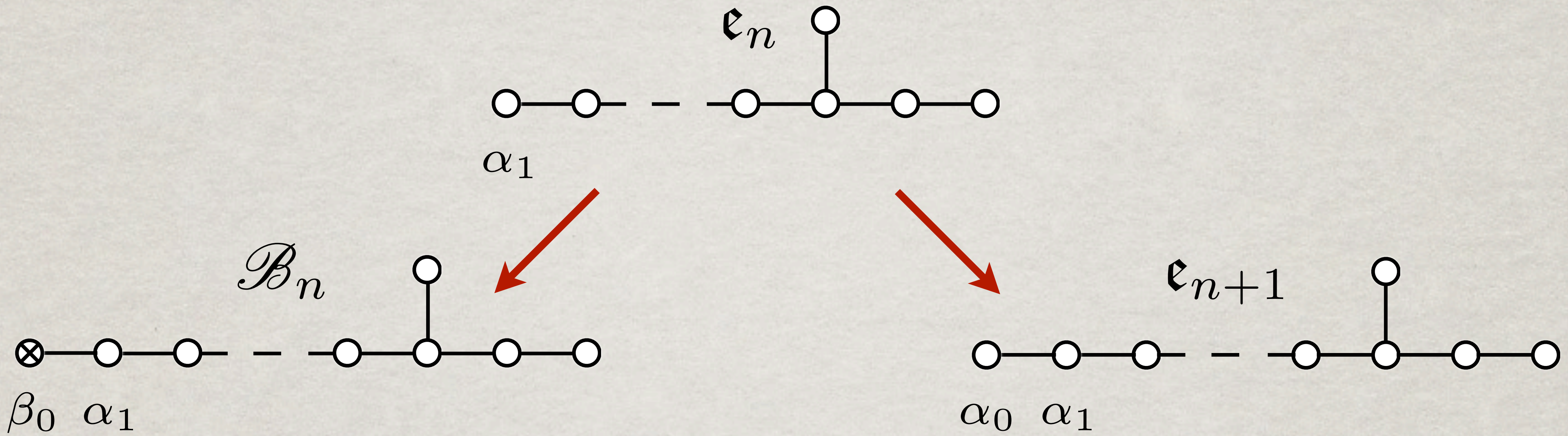


$\mathfrak{e}_n$
α_1
$\mathfrak{e}_{n+1}$
α_0
α_1



$$(\alpha_0, \alpha_1) = -1$$

$$(\alpha_0, \alpha_0) = 2$$



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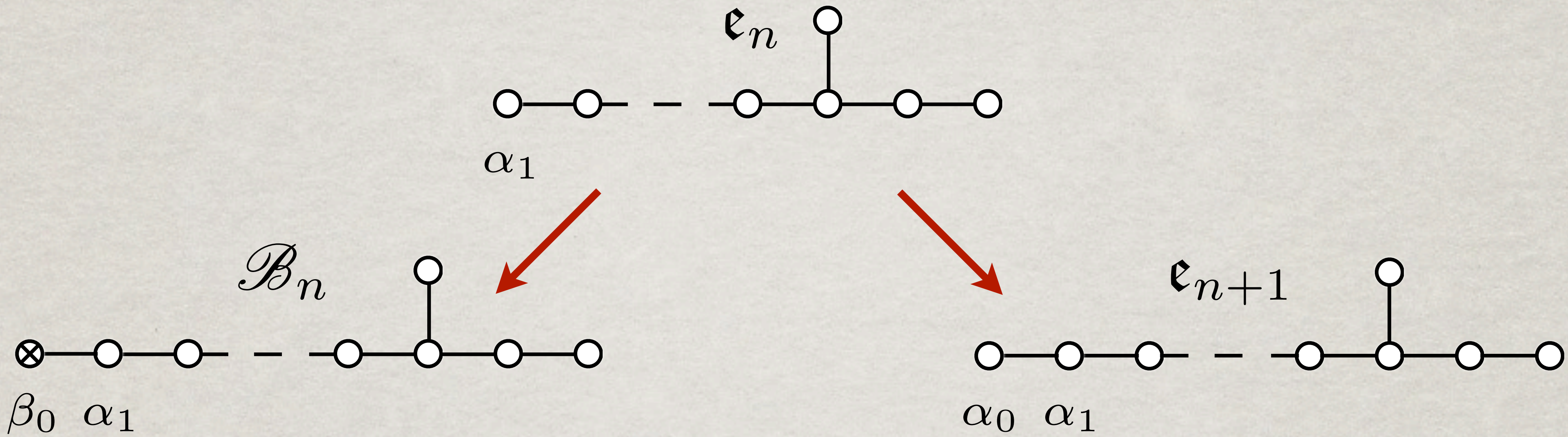

$$(\beta_0, \alpha_1) = -1$$

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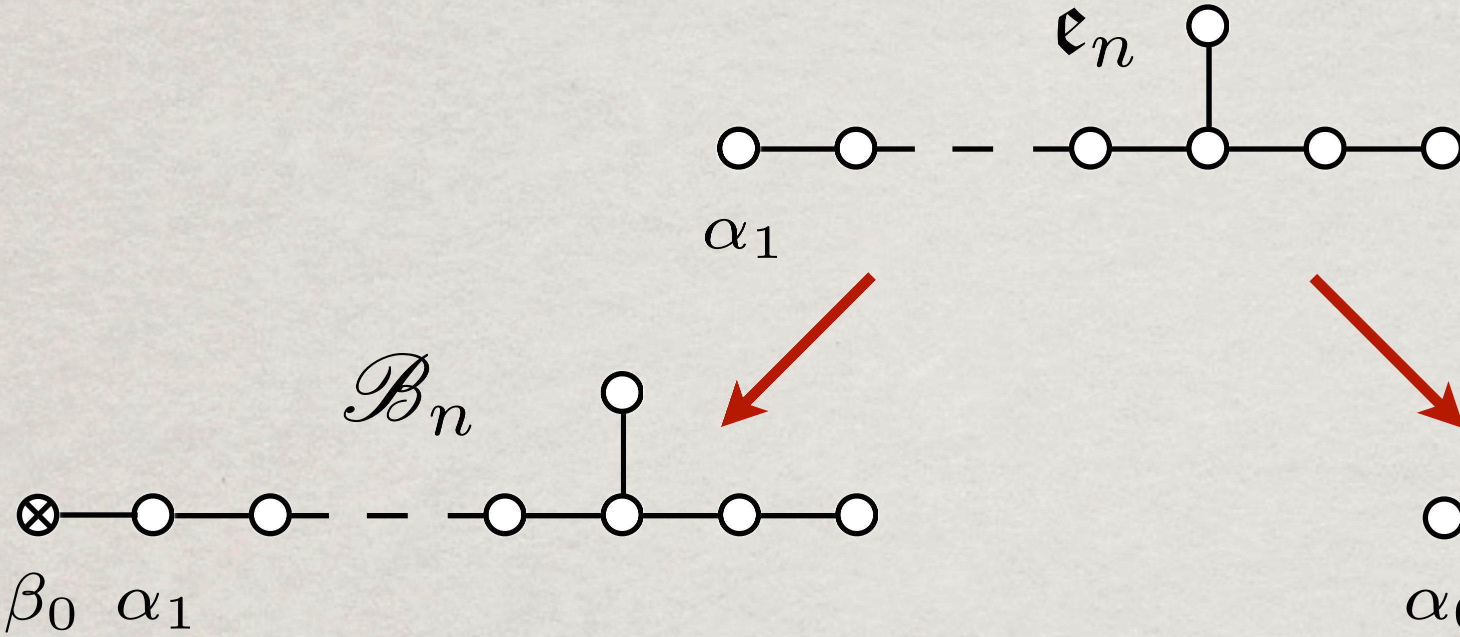

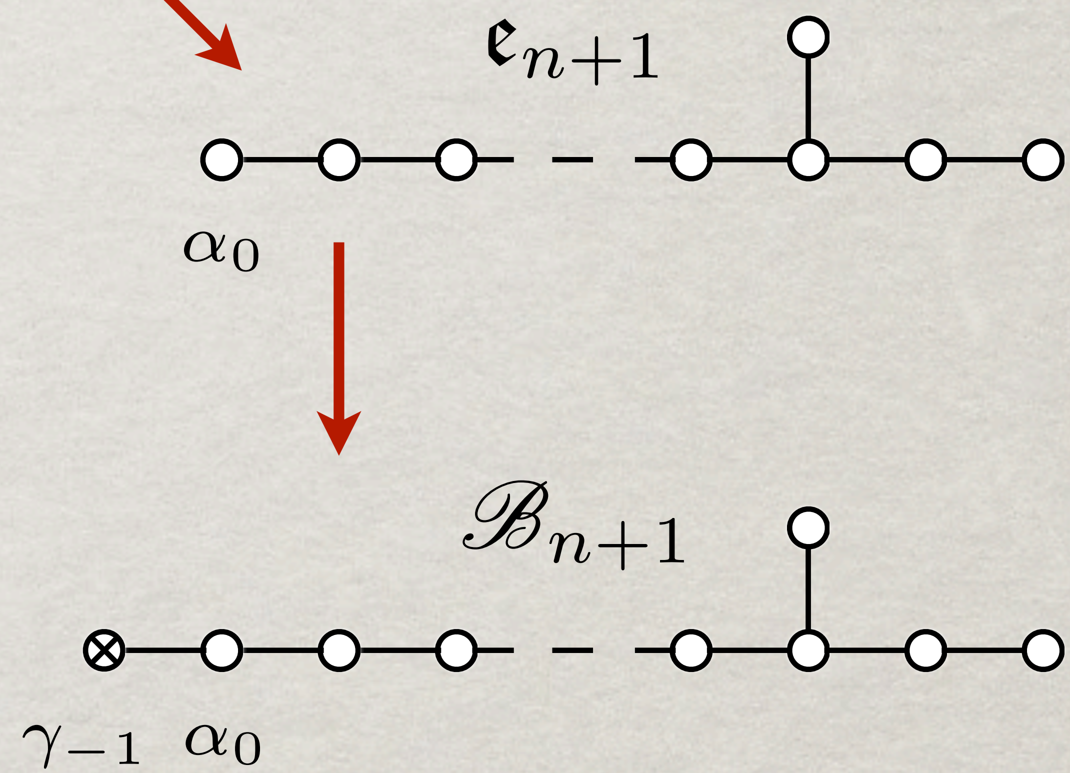

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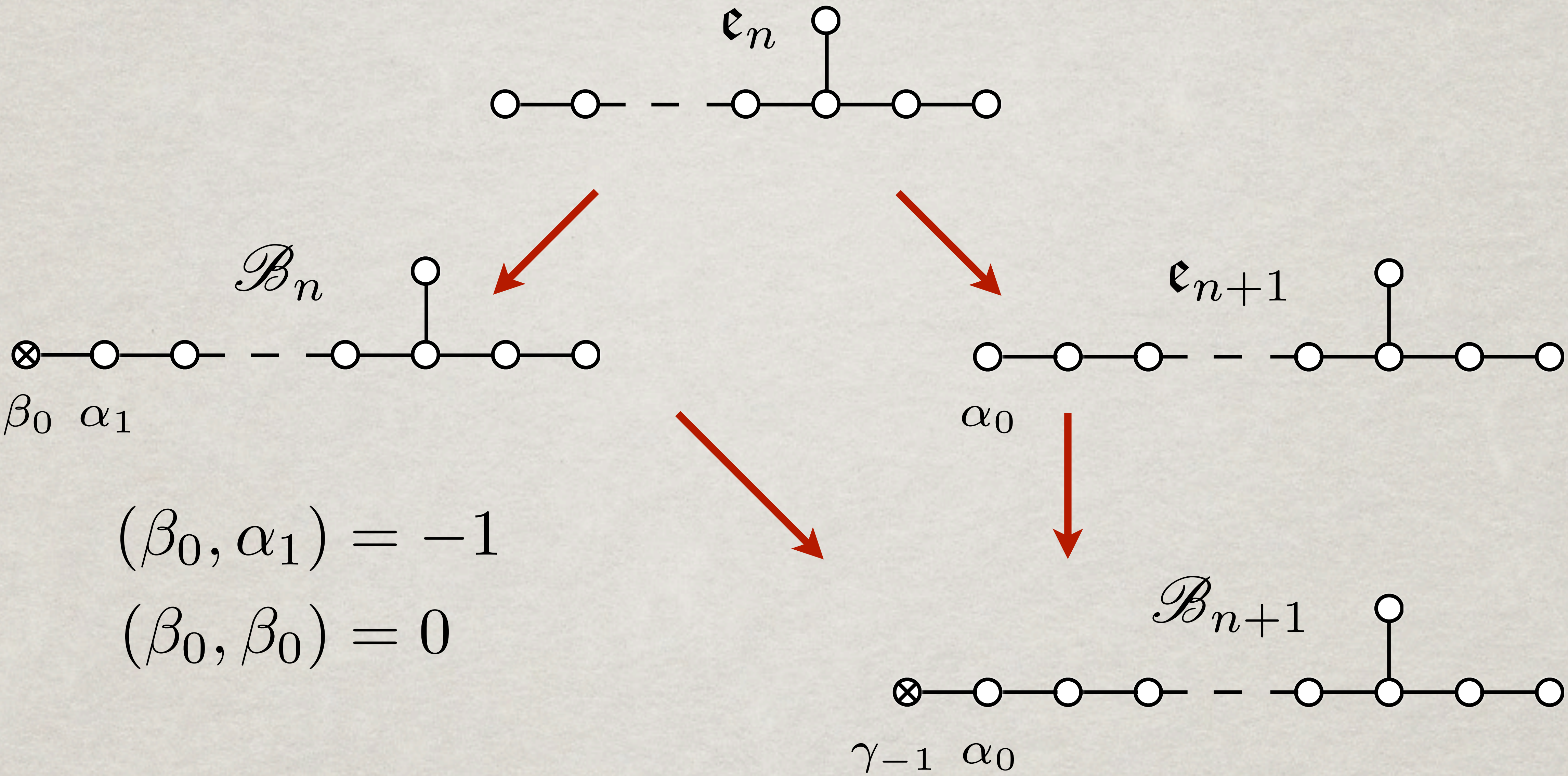
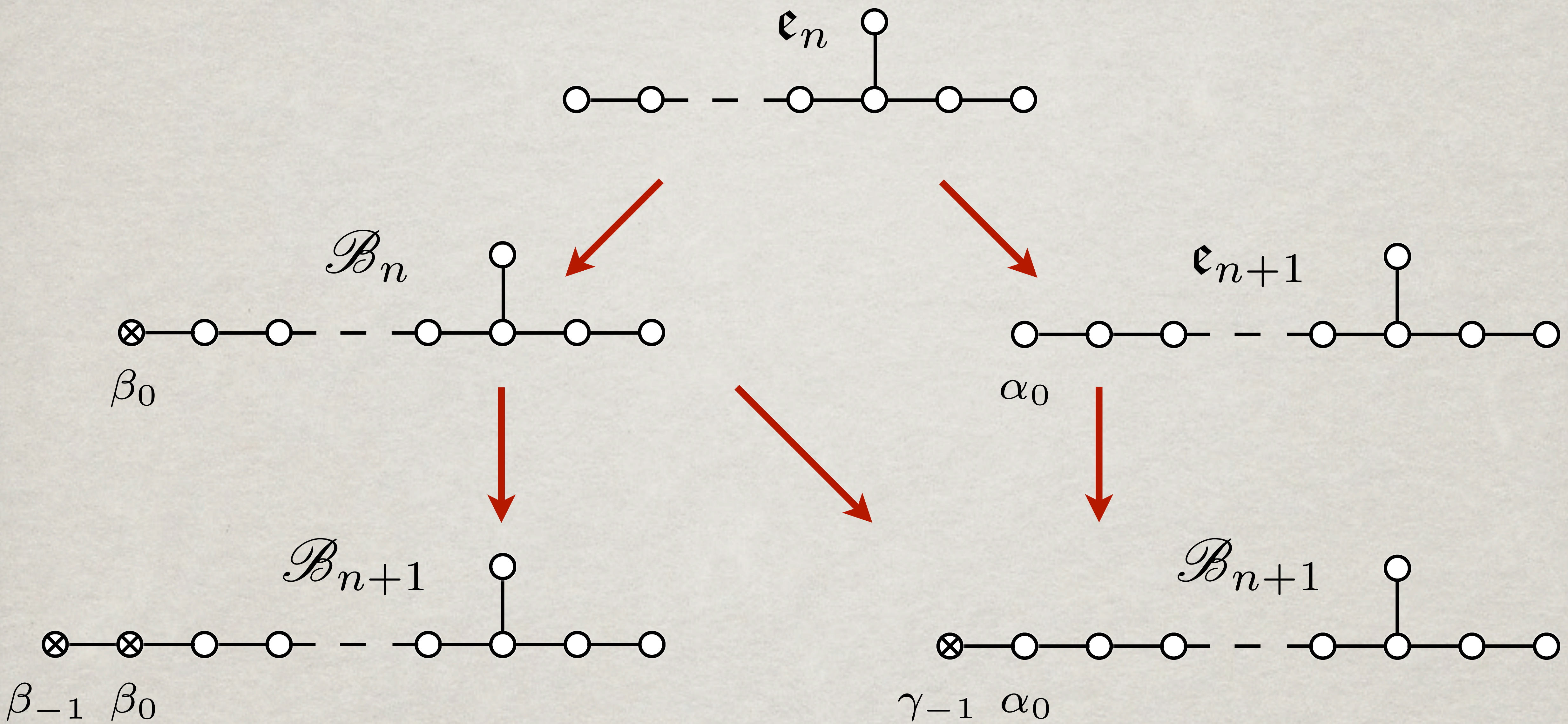


$$(\beta_0, \alpha_1) = -1$$

$$(\beta_0, \beta_0) = 0$$

$$\beta_0 = \gamma_{-1} + \alpha_0$$

[Kleinschmidt, Palmkvist: 1301.1346]



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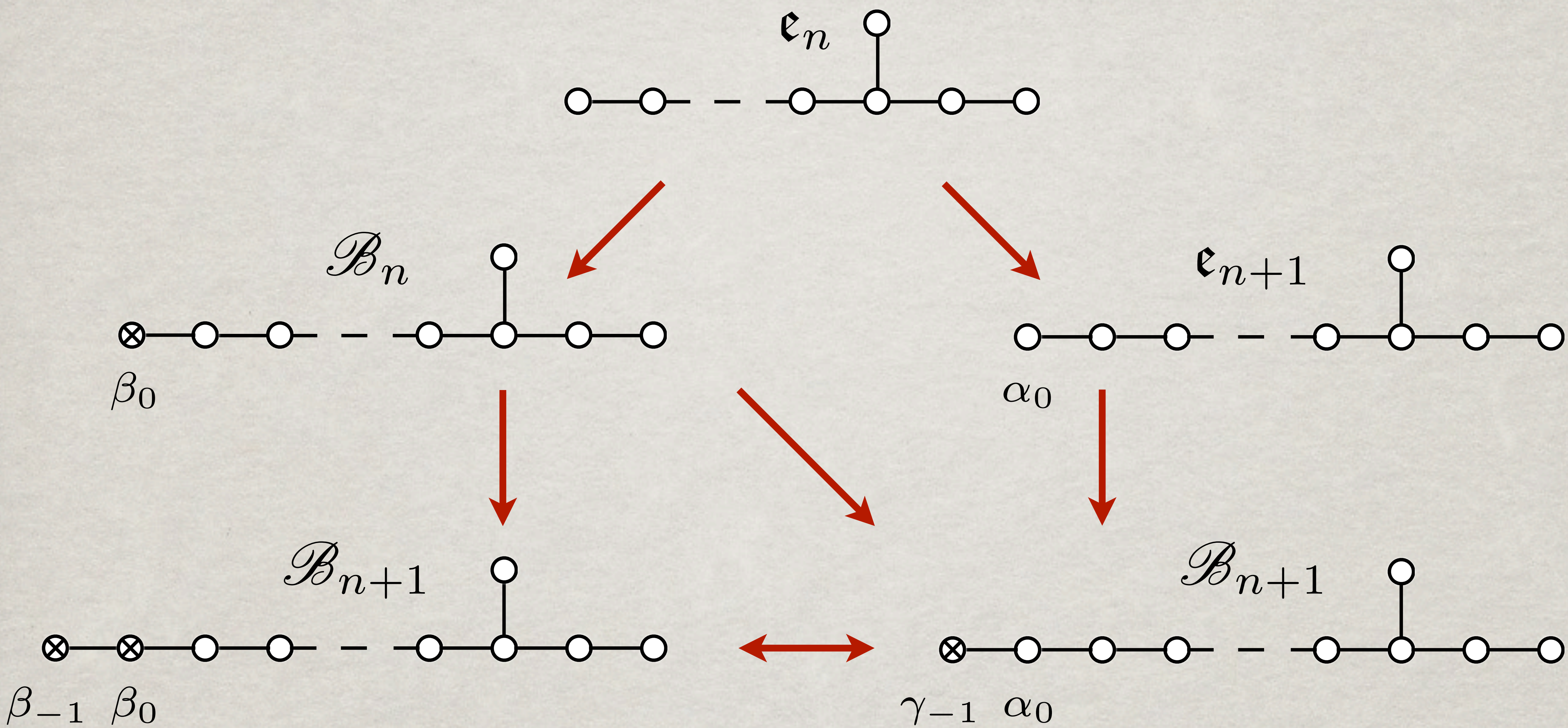
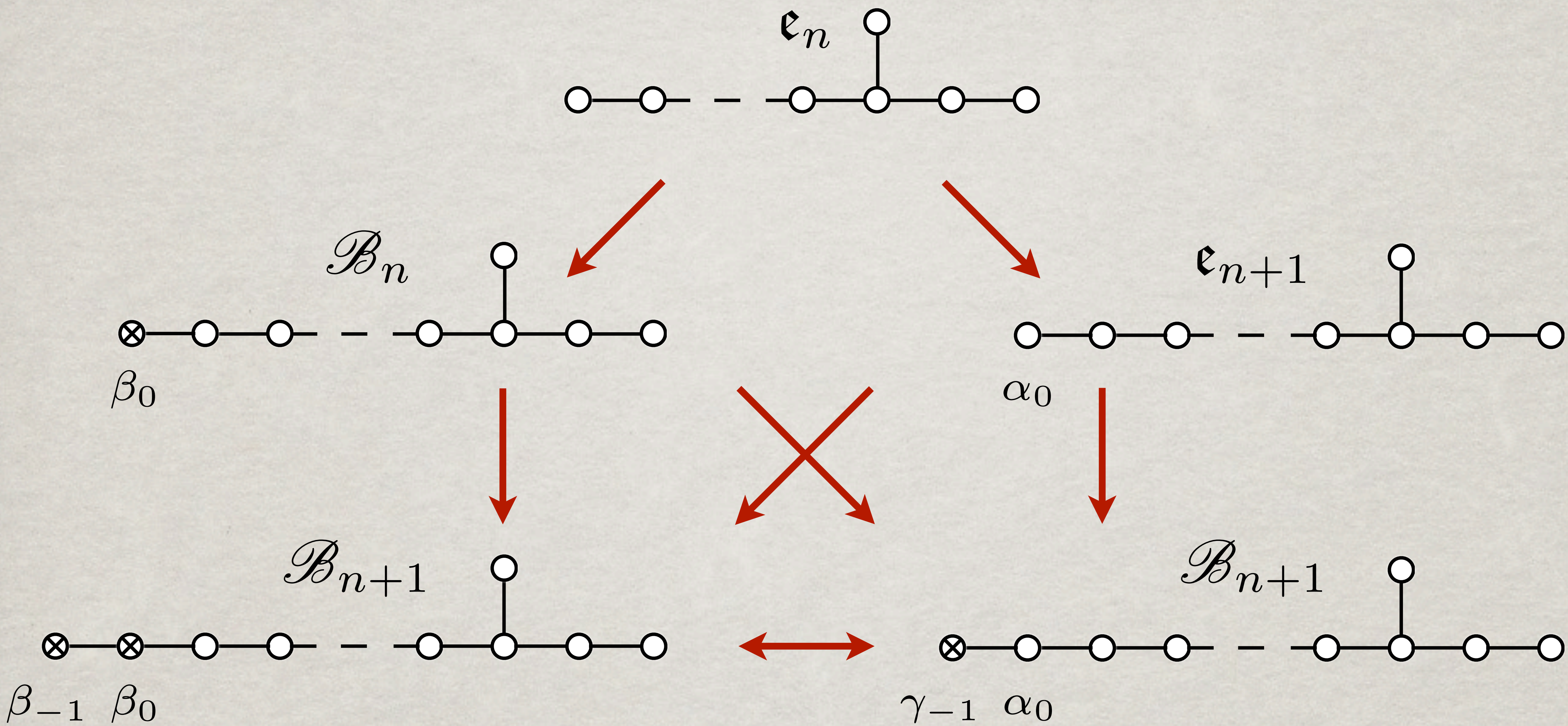
[Kleinschmidt, Palmkvist: 1301.1346]



$$\beta_0 = \gamma_{-1} + \alpha_0 \qquad \beta_{-1} = -\gamma_{-1}$$

[Kleinschmidt, Palmkvist: 1301.1346]



$\beta_0 = \gamma_{-1} + \alpha_0$ $\quad$ $\beta_{-1} = -\gamma_{-1}$ $\quad$ $\alpha_0 = \beta_{-1} + \beta_0$

[Kleinschmidt, Palmkvist: 1301.1346]

Consider a vector field V as an odd element

$$V = V^{\mathscr{M}} E_{\mathscr{M}} \in \mathscr{U}_1 \subset \mathscr{B}_n \subset \mathscr{B}_{n+1}$$

with a corresponding even element

$$\tilde{V} = [e_{-1}, V] = V^{\mathscr{M}} \tilde{E}_{\mathscr{M}} \in \tilde{\mathscr{U}}_1 \subset \mathfrak{e}_{n+1} \subset \mathscr{B}_{n+1}.$$

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Then the generalized Lie derivative can be written

$$\mathscr{L}_U V = [[U, \tilde{F}^{\mathcal{N}}], \partial_{\mathcal{N}} \tilde{V}] - [[\partial_{\mathcal{N}} \tilde{U}, \tilde{F}^{\mathcal{N}}], V]$$

or equivalently

$$\mathscr{L}_U \tilde{V} = -[[\tilde{U}, F^{\mathcal{N}}], \partial_{\mathcal{N}} V] - [[\partial_{\mathcal{N}} U, F^{\mathcal{N}}], \tilde{V}].$$

In the subspace $\mathscr{U}_1 \oplus \tilde{\mathscr{U}_1}$ of $\mathscr{B}_{n+1}$, the triple product

$$(\hat{E}_{\mathcal{M}}\hat{E}_{\mathcal{N}}\hat{E}_{\mathcal{P}}) = [[\hat{E}_{\mathcal{M}}, \hat{F}^{\mathcal{N}}], \hat{E}_{\mathcal{P}}],$$

where $\hat{E}_{\mathcal{M}} = E_{\mathcal{M}} + \tilde{E}_{\mathcal{M}}$ and $\hat{F}_{\mathcal{M}} = F_{\mathcal{M}} + \tilde{F}_{\mathcal{M}}$, satisfies

$$(uv(xyz)) - (xy(uvz)) = ((uvx)yz) - (x(vuy)z)$$

as a consequence of the Jacobi identity in $\mathscr{B}_{n+1}$.

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$$(\hat{E}_{\mathcal{M}} \hat{E}_{\mathcal{N}} \hat{E}_{\mathcal{P}}) = [[\hat{E}_{\mathcal{M}}, \hat{F}^{\mathcal{N}}], \hat{E}_{\mathcal{P}}] = -2\, Z^{\mathcal{N}\mathcal{Q}}{}_{\mathcal{P}\mathcal{M}}\, \hat{E}_{\mathcal{Q}},$$

where $\hat{E}_{\mathcal{M}} = E_{\mathcal{M}} + \tilde{E}_{\mathcal{M}}$ and $\hat{F}_{\mathcal{M}} = F_{\mathcal{M}} + \tilde{F}_{\mathcal{M}}$, satisfies

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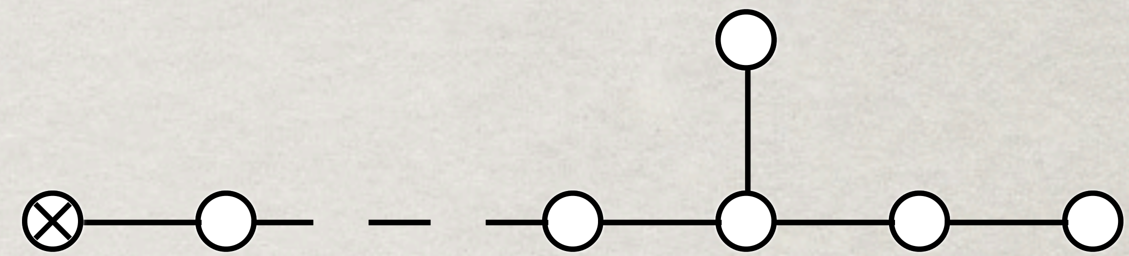
as a consequence of the Jacobi identity in $\mathscr{B}_{n+1}$.

From this identity, and the $\mathbb{Z}$-grading of $\mathfrak{e}_{n+1}$ with respect to $\mathfrak{e}_n$, we can derive the closure identities for the $\mathfrak{e}_n$ invariant tensor $Y^{\mathcal{M}\mathcal{N}}{}_{\mathcal{P}\mathcal{Q}} = Z^{\mathcal{M}\mathcal{N}}{}_{\mathcal{P}\mathcal{Q}} + \delta_{\mathcal{P}}{}^{\mathcal{M}} \delta_{\mathcal{Q}}{}^{\mathcal{N}}$.



$\vdots$						$\vdots$	$\iddots$
$E_{\mathcal{MN}}$					E^{m}	E^{mnpq}	$\cdots$
$E_{\mathcal{M}}$				E_{m}	E^{mn}	E^{mnpqr}	$\cdots$
t_{α}	$\cdots$	F_{mnpqrs}	F_{mnp}	$K^{m}{}_{n}$	E^{mnp}	E^{mnpqrs}	$\cdots$
$F^{\mathcal{M}}$	$\cdots$	F_{mnpqr}	F_{mn}	F^{m}			
$F^{\mathcal{MN}}$	$\cdots$	F_{mnpq}	F_{m}				
$\vdots$	$\iddots$	$\vdots$					

The same expression

$$\mathscr{L}_U \tilde{V} = -[[\tilde{U}, F^{\mathcal{N}}], \partial_{\mathcal{N}} V] - [[\partial_{\mathcal{N}} U, F^{\mathcal{N}}], \tilde{V}]$$

is also valid for ordinary geometry and doubled geometry by restricting U and V to subalgebras of $\mathscr{B}_n$

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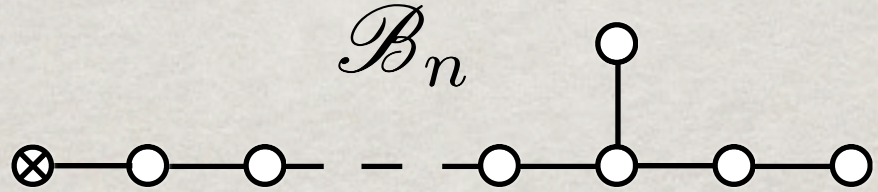

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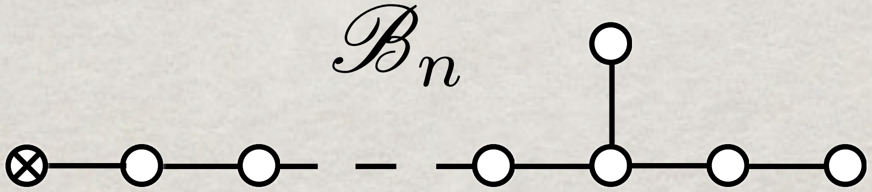
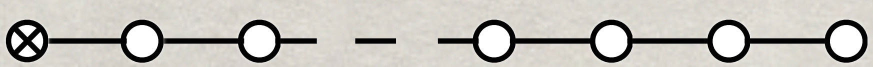


ordinary geometry



$A(n-1,0) = \mathfrak{sl}(n|1)$

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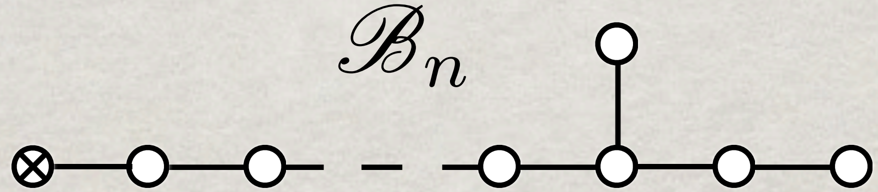
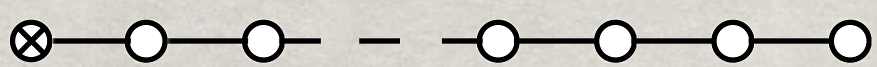


ordinary geometry

doubled geometry



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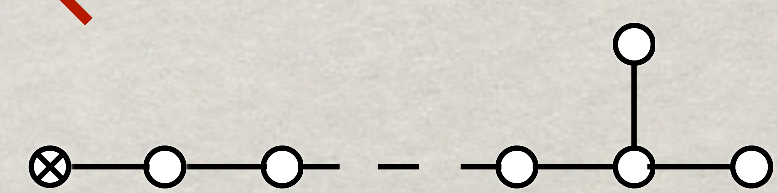

$D(n-1,1) = \mathfrak{osp}(2n-2|2)$

It follows from the expression

$$\mathscr{L}_U \tilde{V} = -[[\tilde{U}, F^{\mathcal{N}}], \partial_{\mathcal{N}} V] - [[\partial_{\mathcal{N}} U, F^{\mathcal{N}}], \tilde{V}],$$

that the infinite sequence of $\mathfrak{e}_n$-representations $R_1, R_2, \ldots$ describes the infinite reducibility of the transformations.

[Berman, Cederwall, Kleinschmidt, Thompson: 1208.5884]
[Cederwall, Palmkvist: 1503.06215]

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[Berman, Cederwall, Kleinschmidt, Thompson: 1208.5884]
[Cederwall, Palmkvist: 1503.06215]

The same sequence appears in tensor hierarchies considered in exceptional field theory, related to those in gauged supergravity.

[Hohm, Samtleben: 1312.0614, 1312.4542, 1406.3348, 1410.8145]
[Aldazabal, Graña, Marqués, Rosabal: 1302.5419, 1312.4549]
[de Wit, Samtleben: 0501243] [de Wit, Nicolai, Samtleben: 0801.1294]

To do:

- In the applications to (gauged) supergravity:

  Include the gravitational degrees of freedom

- In the applications to exceptional geometry:

  Continue to $\mathfrak{e}_9$, $\mathfrak{e}_{10}$, $\mathfrak{e}_{11}$ (infinity-dimensional!)