Double Sigma Model for Strings in Constant and Non-Constant Backgrounds

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Duality and Novel Geometry in M-Theory - POSTECH
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Based on:

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Plan of the talk

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1. Introduction and Motivation
2. Hodge-Dual Symmetric Free Scalar Fields in 2D

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T-Duality

- T-duality is an old subject in string theory. It is a discrete symmetry implying that in many cases two different geometries for the extra-dimensions are physically equivalent: string physics at a very small scale cannot be distinguished from the one at a large scale. It is also a clear indication that ordinary geometric concepts can break down in string theory at the string scale.

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- In the simplest case of circular compactification, T-duality is encoded, for bosonic closed strings, in the simultaneous transformations $R \leftrightarrow \alpha'/R$ and $p_a \leftrightarrow w^a/\alpha'$ under which $X^a = X_L^a + X_R^a \leftrightarrow \tilde{X}_a \equiv X_L^a - X_R^a$, with w^a playing the role of momentum mode for $\tilde{X}_a$. These transformations leave the mass spectrum invariant.

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- In toroidal compactifications (with constant backgrounds $G_{\mu\nu}$ and $B_{\mu\nu}$) T-duality is described by $O(D, D; \mathbb{Z})$ transformations.

O(D,D) Duality in String Theory

- Already at the classical level the indefinite orthogonal group $O(D, D; \mathbb{R})$ appears naturally in the Hamiltonian description of the usual bosonic string model.
- With $*$ the Hodge operator with respect to $h = \mathrm{diag}(-1, 1)$, the action is:

$$S[X; G, B] = \frac{T}{2} \int \left[G_{ab}(X) dX^a \wedge *dX^b + B_{ab}(X) dX^a \wedge dX^b \right]$$

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- Varying S with respect to X^a yields the equation of motion:

$$d*dX^a + \Gamma^a{}_{bc}dX^b \wedge *dX^c = \frac{1}{2}G^{am}H_{mbc}dX^b \wedge dX^c$$

with $H = dB$ and $\Gamma^a{}_{bc} = \frac{1}{2}G^{am}(\partial_b G_{mc} + \partial_c G_{mb} - \partial_m G_{bc})$ the coefficients of the Levi Civita connection.

O(D,D) Duality in String Theory

- The dynamics of the theory is determined by the equations of motion for the coordinates X^a accompanied with the constraints (in the conformal gauge):

$$G_{ab}(\dot{X}^a\dot{X}^b + X'^a X'^b) = 0 \qquad G_{ab}\dot{X}^a X'^b = 0.$$

These come from the vanishing of the energy-momentum tensor $T_{\alpha\beta} = 0$, i.e. from the equation of motion for a general world-sheet metric h.

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- The Hamiltonian density $\mathcal{H} = \frac{T}{2} G_{ab}(\dot{X}^a \dot{X}^b + X'^a X'^b)$ is computed from the Lagrangian density by performing a Legendre transformation with respect to the canonical momentum $P_a = \frac{\partial L}{\partial \dot{X}^a} = \frac{1}{2\pi\alpha'}\left(G_{ab}\dot{X}^b + B_{ab}X'^b\right)$ and $\dot{X}^a = 2\pi\alpha' G^{ab}P_b - G^{ab}B_{bc}X'^c$ but *also* from a Legendre transformation with respect to the canonical winding $W_a = \frac{\partial L}{\partial X'^a} = -\frac{1}{2\pi\alpha'}\left(G_{ab}X'^b + B_{ab}\dot{X}^b\right)$ and X'^a.

O(D,D) Invariance of the String Hamiltonian Density

- The Hamiltonian density can be written equivalently as:

$$\mathcal{H} = \frac{1}{4\pi\alpha'}\begin{pmatrix} \partial_\sigma X \\ 2\pi\alpha' P \end{pmatrix}^t \mathcal{M}(G,B)\begin{pmatrix} \partial_\sigma X \\ 2\pi\alpha' P \end{pmatrix}$$
$$= \frac{1}{4\pi\alpha'}\begin{pmatrix} \partial_\tau X \\ -2\pi\alpha' W \end{pmatrix}^t \mathcal{M}(G,B)\begin{pmatrix} \partial_\tau X \\ -2\pi\alpha' W \end{pmatrix}$$

where the *generalized metric* is introduced:

$$\mathcal{M}(G,B) = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}$$

- Defining the 2D-dimensional $O(D,D)$ generalized vectors in $TM \bigoplus T^*M$:

$$A_P(X) \equiv \partial_\sigma X^a \partial_a + 2\pi\alpha' P_a dX^a$$
$$A_W(X) \equiv \partial_\tau X^a \partial_a - 2\pi\alpha' W_a dX^a$$

one can see that the Hamiltonian density is proportional to the squared length of A_P and A_W as measured by the generalized metric $\mathcal{M}$.

Constraints and Generalized Vectors

- In terms of the generalized vector A_P the constraints, i.e. the components of the energy-momentum tensor can be rewritten as:

$$A_P^t \mathcal{M} A_P = 0 \qquad A_P^t \Omega A_P = 0.$$

The first constraint sets the Hamiltonian density to zero, hence the second constraint completely determines the dynamics and it is rewritten in terms of the matrix $\Omega = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$, i.e. the invariant metric of the group $O(D, D)$ defined by the $D \times D$ matrices T satisfying the condition $T^t \Omega T = \Omega$. In particular the generalized metric is an element of $O(D, D)$.

- All the admissible generalized vectors satisfying the second constraints are related by $O(D, D)$ transformations via $A'_P = \mathcal{T} A_P$. For A'_P to solve the first constraint as well, the generalized metric has to be transformed according to a compensating $O(D, D)$ transformation $\mathcal{T}^{-1}$

$O(D,D;R)$ in the presence of constant backgrounds

- In the presence of constant backgrounds (G,B), the equations of motion for the string coordinates are a set of conservation laws on the world-sheet:

$$\partial_\alpha J^\alpha_a = 0 \rightarrow J^\alpha_a = \eta^{\alpha\beta} G_{ab}\partial_\beta X^b + \epsilon^{\alpha\beta} B_{ab}\partial_\beta X^b$$

- Locally, one can express such currents as:

$$\eta^{\alpha\beta} G_{ab}\partial_\beta X^b + \epsilon^{\alpha\beta} B_{ab}\partial_\beta X^b \equiv \epsilon^{\alpha\beta}\partial_\beta \tilde{X}_a \quad \rightarrow \text{ dual coordinates}$$

in terms of which the action S can be rewritten as:

$$S[X;G,B] = \frac{T}{2}\int \left[\tilde{G}_{ab} d\tilde{X}^a \wedge * d\tilde{X}^b + \tilde{B}_{ab}(X) d\tilde{X}^a \wedge d\tilde{X}^b\right]$$

with $\tilde{G} = (G - BG^{-1}B)^{-1}$ and $\tilde{B} = -G^{-1}B\tilde{G}$.

- The equations of motion for the coordinates $\chi = (X, \tilde{X})$ can be combined into a single equation $O(D,D)$-invariant:

$$M\partial_\alpha \chi = \Omega \epsilon_{\alpha\beta}\partial^\beta \chi$$

- For $B=0$, the equations of motion become the usual Hodge-duality condition for $X^a, \tilde{X}_a$.

$O(D, D; \mathbb{R}) \to O(D, D; \mathbb{Z})$

- If the closed string coordinates are defined on a compact target manifold, the dual coordinates will satisfy the same periodicity conditions and then T-duality maps two theories of the same type into one another $\to$ exact symmetry.
- For closed strings, toroidal compactification means:

$$X^a(\sigma, \tau) \equiv X^a(\sigma + \pi, \tau) + 2\pi L^a \quad L^a = \sum_{i=1}^{d} w_i R_i e_i^a$$

with w_i being the winding numbers and e_i^a vector basis on T^d.

- In the compact space $O(D, D; \mathbb{R}) \to O(D, D; \mathbb{Z})$. The latter becomes the T-duality group of the toroidal compactification. For closed strings on compactified dimensions, this group becomes a *symmetry* not only of the mass spectrum and the vacuum partition function but also of the scattering amplitudes.

T-dual invariant bosonic string formulation

- The presence of the $O(D, D)$ symmetry suggests to extend the standard formulation of String Theory, based on the Polyakov action, by introducing this symmetry at the level of the world-sheet sigma-model. It would be interesting, therefore, looking for a manifestly $O(D, D)$-dual invariant formulation of the string theory.

- The introduction of *both* the coordinates X^a *and* the dual ones $\tilde{X}_a$ is required. Such formulation is based on a doubling of the string coordinates in the target space.

Doubling Coordinates: Motivation

- The main goal of this new action would be to explore more closely aspects of stringy geometry and, hence, of string gravity. In fact, if interested in writing down the complete effective field theory of such generalized sigma-model, one should consider, correspondingly to the introduction of X^a and $\tilde{X}_a$, a dependence of the fields associated with string states on such coordinates. In this way, double string effective field theory becomes a double field theory.

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- What the well-known effective gravitational action of a closed string

$$S = \int dX \sqrt{G} e^{-2\phi} \left[R + 4(\partial\phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right]$$

becomes when G, B and ϕ are dependent on X^a and $\tilde{X}_a$? Which symmetries and what properties would it have? This could shed light on aspects of string gravity unexplored thus far.

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- How the string theory would look like when the T-duality is manifested in the sigma-model Lagrangian density?

Hodge-Duality Symmetry for 2D Scalar Fields

- The usual Lagrangian of a 2D scalar field ϕ

$$\mathcal{L} = -\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi = \frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi'^2$$

can be rewritten in a manifestly invariant form under $\phi \leftrightarrow \tilde{\phi}$, its Hodge dual defined by $\partial_\alpha\tilde{\phi} = -\epsilon_{\alpha\beta}\partial^\beta\phi$ ($\epsilon^{01} = 1$).

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- Two steps are necessary.
- The first consists in rewriting $\mathcal{L}$ in a first order form, after introducing an auxiliary field p whose equation of motion reproduces $p = \dot{\phi}$.
- The second consists in trading p for the new field $\tilde{\phi}$ defined through $p \equiv \tilde{\phi}'$. It is easy to see that this procedure leads to the following symmetric Lagrangian:

$$\mathcal{L}_{sym} = \frac{1}{2}\left[\dot{\phi}\tilde{\phi}' + \phi'\dot{\tilde{\phi}} - \phi'^2 - \tilde{\phi}'^2\right]$$

- The manifest Lorentz invariance has disappeared, but it holds on-shell.

Free scalars fields in 2D - Equations of motion

- The equations of motion for ϕ and $\tilde{\phi}$ result to be respectively:

$$\partial_\sigma \left[\partial_\sigma \phi - \partial_\tau \tilde{\phi} \right] = 0 \quad ; \quad \partial_\sigma \left[\partial_\sigma \tilde{\phi} - \partial_\tau \phi \right] = 0$$

$$\partial_\sigma \phi - \partial_\tau \tilde{\phi} = f(\tau) \quad ; \quad \partial_\sigma \tilde{\phi} - \partial_\tau \phi = \tilde{f}(\tau)$$

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- Hence, they can be rewritten as first-order equations:

$$\partial_\sigma \phi - \partial_\tau \tilde{\phi} = 0$$
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by invoking another symmetry of $\mathcal{L}_{sym}$, i.e. the one under a *shift*:

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- The equations of motion reproduce on-shell the duality conditions, after gauging away $f(\tau)$ and $\tilde{f}(\tau)$.

Floreanini-Jackiw Lagrangians for chiral fields

- The symmetric Lagrangian $\mathcal{L}_{sym}$ can be diagonalized by introducing a pair of scalar fields ϕ_+ and ϕ_- defined by:

$$\phi \equiv \frac{1}{\sqrt{2}}(\phi_+ + \phi_-) \quad ; \quad \tilde{\phi} \equiv \frac{1}{\sqrt{2}}(\phi_+ - \phi_-)$$

in terms of which it becomes the sum of two Floreanini-Jackiw Lagrangians, the one associate with ϕ_+ and the other with ϕ_-:

$$\mathcal{L}_{sym} = \mathcal{L}_+(\phi_+) + \mathcal{L}_-(\phi_-)$$

with

$$\mathcal{L}_\pm(\phi_\pm) = \pm\frac{1}{2}\dot{\phi}_\pm\phi'_\pm - \frac{1}{2}\phi'^2_\pm$$

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- It is only on-shell that $\phi_\pm$ become functions of $\sigma \pm \tau$:

$$\dot{\phi}_+ = \phi'_+ \qquad \dot{\phi}_- = -\phi'_-$$

Symmetries

- $\mathcal{L}_{sym}$ is invariant under space-time translations acting as (the constant parameters of the transformations are omitted):

$$\delta_\tau \phi = \dot{\phi} \quad ; \quad \delta_\sigma \phi = \phi^{'}$$

and under *modified global Lorentz transformations*:

$$\delta_L \phi = \tau \phi' + \sigma \tilde{\phi}' \quad ; \quad \delta_L \tilde{\phi} = \tau \tilde{\phi}' + \sigma \phi'$$

that on-shell become the usual two-dimensional Lorentz rotations:

$$\delta_L \phi = \tau \phi' + \sigma \dot{\phi} \quad ; \quad \delta_L \tilde{\phi} = \tau \tilde{\phi}' + \sigma \dot{\tilde{\phi}}$$

- The Lorentz invariance is recovered on-shell.

Chiral and Non-chiral Basis

- The free Lagrangians considered here can be rewritten, in both cases, as:

$$\mathcal{L}_0 = \frac{1}{2}\left(C_{ij}\partial_0\Phi^i\partial_1\Phi^j + M_{ij}\partial_1\Phi^i\partial_1\Phi^j\right) \ .$$

In the chiral basis $\Phi^i = (\phi_+, \phi_-)$ $(i = 1, 2)$

$$C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} ;$$

in the non-chiral basis $\Phi^i = (\phi, \tilde{\phi})$

$$C \equiv \Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad .$$

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- In the string case, C and M will become, respectively, the $O(D, D)$ invariant metric and the *generalized metric*.

Two-dimensional scalar fields on curved space

- In order to couple $\mathcal{L}_{sym}$ (or the two FJ Lagrangians for chiral scalar fields) to an external 2-bein $e^{a}{}_{\alpha}$ one has to replace $\partial_a \rightarrow e_a^{\alpha}\partial_\alpha$ and to multiply by $e \equiv \det e^{a}{}_{\alpha}$:

$$\begin{aligned}\mathcal{L}_{sym} &= \frac{1}{2} e \left[e_0^{\alpha} e_1^{\beta} \partial_n \phi \partial_m \tilde{\phi} + e_1^{\alpha} e_0^{\beta} \partial_\alpha \phi \partial_\beta \tilde{\phi} \right. \\ &\left. - e_1^{\alpha} e_1^{\beta} \partial_\alpha \phi \partial_\beta \phi - e_1^{\alpha} e_1^{\beta} \partial_\alpha \tilde{\phi} \partial_\beta \tilde{\phi} \right]\end{aligned}$$

- After eliminating $\tilde{\phi}$ through its equation of motion, one returns to the usual scalar Lagrangian:

$$\mathcal{L} = \frac{1}{2} e \eta^{ab} e_a{}^{\alpha} e_b^{\beta} \partial_\alpha \phi \partial_\beta \phi$$

General String Sigma-Model

- General string "sigma model":

$$S = -\frac{T}{2} \int d^2\sigma e \left[C_{ij} \nabla_0 \chi^i \nabla_1 \chi^j + M_{ij} \nabla_1 \chi^i \nabla_1 \chi^j \right]$$

- ${e^a}_\alpha \rightarrow$ zweibein defined on the string world-sheet.

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- $C_{ij} = C_{ji}$ and $M_{ij} = M_{ji}$; $\nabla_a\chi^i = e^\alpha_a\partial_\alpha\chi^i$, the functions χ^i the string coordinates in an N-dimensional target space.

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Symmetries

1. Invariance under diffeomorphisms:
$$\sigma^\alpha \rightarrow \sigma'^\alpha(\sigma)$$
2. Invariance under Weyl transformations:
$$e^a{}_\alpha \rightarrow \lambda(\sigma)e^a{}_\alpha$$
3. Request of invariance under local Lorentz transformations: ${e^a}_\alpha \rightarrow {e'^a}_\alpha = {\Lambda^a}_b(\sigma){e^b}_\alpha$ where ${\Lambda^a}_b$ is an arbitrary Lorentz matrix $SO(1,1)$.

Requiring local Lorentz invariance

- The action S is not manifestly invariant under the group $SO(1,1)$ of local Lorentz transformations:

$$\delta e^a{}_\alpha = \alpha(\sigma)\epsilon^a{}_b(\sigma)e^b{}_\alpha$$

but such invariance has to hold since physical observables are independent on the choice of the vielbein. Hence, the theory is required to be locally Lorentz invariant on shell.

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- Since the variation of S under an infinitesimal local Lorentz transformation results to be:

$$\frac{\delta S}{\delta e^a{}_\alpha}\delta e^a{}_\alpha = \alpha(\sigma)\frac{e}{2}\epsilon^a{}_b t_a{}^b$$

the above requirement implies:

$$\epsilon^{ab}t_{ab} = 0 \qquad t_a{}^b \equiv -\frac{2}{T}\frac{1}{e}\frac{\delta S}{\delta e^a{}_\alpha}e^b{}_\alpha$$

Flat gauge

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- The equations of motion for ${e^a}_\alpha$ give $t_{ab} = 0$ providing constraints that have to imposed at classical and quantum levels, analogously to what happens in the ordinary formulation with $T_{\alpha\beta} = -\frac{2}{T\sqrt{g}} \frac{\delta S}{\delta g^{\alpha\beta}} = 0$. Hence, on the solutions of these equations the local Lorentz invariance holds.

Flat gauge

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- Local symmetries (Reparametrization + Weyl + Modified Lorentz inv.) allow to fix the flat gauge

$$e_\alpha^{\ a} = \delta_\alpha^{\ a} .$$

Constraints and Equations of Motion

- The constraint $\epsilon^{ab} t_{ab} = 0$ can be rewritten in the following way:

$$\left[C_{ij}\partial_0\chi^j + M_{ij}\partial_1\chi^j\right](C^{-1})^{ik}\left[C_{kl}\partial_0\chi^l + M_{kl}\partial_1\chi^l)\right] + \left[C - MC^{-1}M\right]_{ij}\partial_1\chi^i\,\partial_1\chi^j = 0.$$

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- Equations of motion for χ^i:

$$\partial_1\left[C_{ij}\partial_0\chi^j + M_{ij}\partial_1\chi^j\right] - \Gamma^l{}_{ik}C_{lj}\partial_0\chi^j\partial_1\chi^k - \frac{1}{2}(\partial_i M_{jk})\partial_1\chi^j\partial_1\chi^k = 0$$

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- Boundary conditions:

$$\left[\left(\frac{1}{2}C_{ij}\partial_0\chi^j + M_{ij}\partial_1\chi^j\right)\right]_{\sigma=0}^{\sigma=\pi} = 0$$

Constant Backgrounds

- When C and M are constant, the equations of motion for χ^i drastically simplifies into:

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- The further local gauge invariance of the action under shifts as:

$$\delta \chi^i = f^i(\tau, \sigma) \quad \text{with} \quad \nabla_1 f^i = 0$$

allows to rewrite the equation of motion for χ^i as:

$$C_{ij} \partial_0 \chi^j + M_{ij} \partial_1 \chi^j = 0$$

with boundary conditions dictated by the vanishing of the surface integral:

$$\frac{1}{2} \int d\tau C_{ij} \left[\partial_0 \chi^j \delta \chi^i \right] |_{\sigma=0}^{\sigma=\tau} = 0$$

describing both open strings with Dirichlet boundary conditions and closed strings.

Emerging out of $O(D, D)$

- This causes the constraint on the ϵ-trace to become:
$$\left[C - MC^{-1}M\right]_{ij} \partial_1\chi^i \, \partial_1\chi^j = 0$$
implying the restriction on C and M: $C = MCM$.
- After rotating and rescaling χ^i, C can always be put in the diagonal form:
$$C = (1, \cdots, 1, -1, \cdots, -1)$$
with N_+ eigenvalues 1 and N_- eigenvalues -1 and $N = N_+ + N_-$. So the action can be interpreted as describing N_+ chiral and N_- antichiral scalars interacting via the bilinear term $(M_{ij} + \delta_{ij})\nabla_1\chi^i\nabla_1\chi^j$ and the absence of a quantum Lorentz anomaly requires $N_+ = N_- = D = \frac{N}{2}$. Hence, $N = 2D$.
- C becomes the $O(D, D)$ invariant metric while $C = MCM$ implies that M is an $O(D, D)$ element.

Non-chiral coordinates

- It is possible to make a change of coordinates in the $2D$-dimensional target space according to the definition:

$$X^{\mu} \equiv \frac{1}{\sqrt{2}}\left(X_{+}^{\mu} + X_{-}^{\mu}\right) \quad ; \quad \tilde{X}_{\mu} \equiv \delta_{\mu\nu}\frac{1}{\sqrt{2}}\left(X_{+}^{\nu} - X_{-}^{\nu}\right)$$

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- It makes the matrix C become off-diagonal:

$$C_{ij} = -\Omega_{ij} \;\; ; \;\; \Omega_{ij} = \begin{pmatrix} 0_{\mu\nu} & \mathbb{I}_\mu^{\ \nu} \\ \mathbb{I}^\mu_{\ \nu} & 0^{\mu\nu} \end{pmatrix}$$

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with $(\Omega)_{ij} = (\Omega^{-1})^{ij}$.

- The expression for M results to be:

$$M_{ij} = \begin{pmatrix} (G - B\,G^{-1}B)_{\mu\nu} & (B\,G^{-1})_{\mu}^{\ \nu} \\ (-G^{-1}\,B)^{\mu}_{\ \nu} & (G^{-1})^{\mu\nu} \end{pmatrix}$$

being M parametrized by D^2.

$O(D, D)$ INVARIANCE

- The sigma-model action can be expressed, in the non-chiral basis, as:

$$S = -\frac{T}{2}\int d^2\sigma \left[\Omega_{ij}\partial_0\chi^i\partial_1\chi^j - M_{ij}\partial_1\chi^i\partial_1\chi^j\right].$$

- It is invariant under the combined $O(D, D)$ transformations of χ^i *and* the matrix of the couplings parameters in M:

$$\chi' = \mathcal{R}\chi \;\; ; \;\; M' = \mathcal{R}^{-t}M\mathcal{R}^{-1} \;\; ; \;\; \mathcal{R}^t\Omega\mathcal{R} = \Omega \;\; ; \;\; \mathcal{R} \in O(D, D).$$

.

- The $O(D, D)$ invariant metric Ω is itself an element of $O(D, D)$.

Recovering the familiar T-duality invariance

- Define the duality transformation $\mathcal{R} = \Omega$ under which $X^\mu \to \tilde{X}_\mu$. The action, expressed in terms of X^μ and $\tilde{X}_\mu$, after this transformation, becomes:

$$S = -\frac{T}{2}\int d^2\sigma \Big[\partial_0 X^\mu \partial_1 \tilde{X}_\mu + \partial_0 \tilde{X}^\mu \partial_1 X_\mu$$
$$-(G - B\,G^{-1}B)_{\mu\nu}\partial_1 X^\mu \partial_1 X^\nu - (B\,G^{-1})_\mu^{\ \nu}\partial_1 X^\mu \partial_1 \tilde{X}_\nu$$
$$+ (G^{-1}\,B)^\mu_{\ \nu}\partial_1 \tilde{X}_\mu \partial_1 X^\nu - (G^{-1})^{\mu\nu}\partial_1 \tilde{X}_\mu \partial_1 \tilde{X}_\nu\Big]$$

and exhibits what in string theory is the familiar T-duality invariance, in presence of backgrounds, i.e. $X \leftrightarrow \tilde{X}$ together with a transformation of the generalized metric given by $M' = M^{-1}$, i.e.

$$G \leftrightarrow (G - BG^{-1}B)^{-1}$$
$$BG^{-1} \leftrightarrow -G^{-1}B$$

Correspondence with the Standard Formulation in Constant Backgrounds

- In order to understand the relation to the standard formulation, one can integrate over $\tilde{X}_\mu$ by eliminating it through the use of the equations of motion. In the case of G, B constant one gets the standard sigma-model action:

$$S[X] = -\frac{T}{2}\int d^2\sigma(\sqrt{G}G^{mm} + \epsilon^{mn})(G+B)_{\mu\nu}\partial_m X^\mu \partial_n X^\nu$$

which describes the toroidal compactification under proper periodicity conditions on X. If, instead, one eliminates X from its equation of motion one obtains the dual model for $\tilde{X}$:

$$S[\tilde{X}] = -\frac{T}{2}\int d^2\sigma(\sqrt{G}G^{mn} + \epsilon^{mn})(G+B)^{-1\mu\nu}\partial_m \tilde{X}^\mu \partial_n \tilde{X}^\nu$$

- The action $S[X, \tilde{X}]$ is therefore a first-order action which interpolates between $S[X]$ and $S[\tilde{X}]$ and is manifestly duality symmetric.

- From the above formulation it is easy to derive the action for free strings. This corresponds to the case in which:

$$C = -\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \text{ and } M = \begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix}$$

with $G_{\mu\nu}$ being the flat metric in the target space. One gets:

$$\begin{aligned} S_0 &= S[X^\mu, e] + S[\tilde{X}_\mu, e] \\ &= -\frac{1}{4\pi\alpha'}\int d^2\sigma e \Big[\nabla_0 X^\mu \nabla_1 \tilde{X}_\mu + \nabla_0 \tilde{X}^\mu \nabla_1 X_\mu \\ &\quad -G_{\mu\nu}\nabla_1 X^\mu \nabla_1 X^\nu - \tilde{G}^{\mu\nu}\nabla_1 \tilde{X}_\mu \nabla_1 \tilde{X}_\nu\Big] \\ &= S[X^\mu_+, e] + S[X^\mu_-, e] \end{aligned}$$

with $\tilde{G}^{\mu\nu} = G^{-1\mu\nu}$, $\nabla_a = e_a{}^\alpha \partial_\alpha$ and $\mu = 1, \cdots, D$. This is invariant under $X^\mu \leftrightarrow \tilde{X}_\mu$ together with $G_{\mu\nu} \leftrightarrow \tilde{G}^{\mu\nu}$.

Inserting Vertex Operators

- The free action S_0 still describes D and not $2D$ scalar degrees of freedom (only the zero mode of X and $\tilde{X}$ are independent on-shell).

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- S_0 can be perturbated by $S_{int}[X, \tilde{X}]$ with the insertion of vertex operators involving both X and $\tilde{X}$. If S_{int} does not depend on $\tilde{X}$ one can integrate $\tilde{X}$ out in the path integral of the theory and reproduce the usual results of the standard formulation.

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- Assuming that strings are compactified on a circle of radius R, one should expect that: at large scales $R >> \sqrt{\alpha'}$ the relevant interactions are $S_{int}(X)$; at intermediate scales $R \sim \sqrt{\alpha'}$ the relevant interactions involve both X and $\tilde{X}$ while at $R << \sqrt{\alpha'}$ the relevant interactions are $S_{int}(\tilde{X})$.

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- The duality symmetric formulation may be considered as a natural generalization of the standard one at the string scale.

Equivalence between non-covariant and covariant actions

- The action

$$S = -\frac{T}{2} \int d^2\sigma \, \left[C_{ij} \partial_0 \chi^i \, \partial_1 \chi^j + M_{ij} \partial_1 \chi^i \partial_1 \chi^j \right]$$

is candidate to provide a T-duality invariant sigma model. In particular, with C and M constant, describes bosonic closed strings on a toroidally compactified target space. It exhibits a manifest T-duality invariance $O(D, D)$ with the fields χ^i interpreted as string coordinates on the double torus T^{2D}.

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- It can be shown to be equivalent to the following covariant action (Hull, 2005):

$$S = -\frac{T}{2}\int d^2\sigma\partial^\alpha\chi^i M_{ij}\partial_\alpha\chi^j$$

with the self-duality relation imposed in order to halve the degrees of freedom from $2D$ to D (also Duff, 1987):

$$\partial_\alpha\chi^i = \Omega^{ij}M_{jk}(\partial^\beta\chi^k)$$

including both the eqs. of motion and the condition $\epsilon_{ab}t^{ab} = 0$.

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including both the eqs. of motion and the condition $\epsilon_{ab} t^{ab} = 0$.

Other covariant formulations

- Covariant action for a string in *doubled yet gauged* spacetime:

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\left[-\frac{1}{2}\sqrt{-h}h^{ij}D_i\chi^M D_j\chi^N M_{MN}(\chi) - \epsilon^{ij}D_i\chi^M A_{jM}\right]$$

  where h_{ij} is the world-sheet metric, the covariant derivative is given by

$$D_i\chi^M = \partial_i\chi^M - A_i^M$$

  and M_{MN} is the $O(D, D)$ generalized metric subject to the *section condition*:

$$\partial_A\partial^A = 0$$

  [K. Lee and J.H. Park, 2014]

- The Tseytlin model can be made covariant also through the Pasti-Sorokin-Tonin procedure [1995, 1996]

C constant and M only X or $\tilde{X}$-dependent

- Aim: to introduce interactions and understand if the local Lorentz constraint still holds under the form $C = MCM$ in case of non-constant backgrounds.

First case:

C constant and M only X-dependent (or only $\bar{X}$-dependent).

- In the case in which $C = \Omega$ and M only X-dependent, in deriving the equations of motion for X^μ and $\tilde{X}_\mu$ one has to keep in consideration the contribution coming from the term

$$\frac{1}{2}(\partial_i M_{jk})\partial_1 \chi^j \partial_1 \chi^k .$$

C and M only X [or $\tilde{X}$]-dependent

- The equations of motion for X^μ and $\tilde{X}_\mu$ respectively become:

$$\partial_1 \left[-\partial_0 \tilde{X}_\mu + (G - BG^{-1}B)_{\mu\nu} \partial_1 X^\nu + (BG^{-1})_\mu{}^\nu \partial_1 \tilde{X}_\nu \right] = \frac{1}{2} \partial_1 X^\nu \left[\partial_\mu (G - BG^{-1}B)_{\nu\rho} \partial_1 X^\rho + \partial_\mu (BG^{-1})^{\nu\rho} \partial_1 \tilde{X}_\rho \right]$$

and

$$\partial_1 \left[-\partial_0 X^\mu + (-G^{-1}B)^\mu{}_\nu \partial_1 X^\nu + (G^{-1})^{\mu\nu} \partial_1 \tilde{X}_\nu \right] = \frac{1}{2} \partial_1 \tilde{X}_\nu \left[\bar{\partial}^\mu (-G^{-1}B)^\nu{}_\rho \partial_1 X^\rho + \bar{\partial}^\mu (G^{-1})^{\nu\rho} \partial_1 \tilde{X}_\rho \right] = 0$$

where $\bar{\partial}^\mu$ denotes the derivative with respect to $\tilde{X}_\mu$.

- Also in this case, one can use the invariance of the equation of motion for $\tilde{X}_\mu$ under *shifts* for putting:

$$-\partial_0 X^\mu + (-G^{-1}B)^\mu{}_\nu \partial_1 X^\nu + (G^{-1})^{\mu\nu} \partial_1 \tilde{X}_\nu = 0.$$

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and

$$\partial_1 \left[-\partial_0 X^{\mu} + (-G^{-1}B)^{\mu}_{\ \nu}\partial_1 X^{\nu} + (G^{-1})^{\mu\nu}\partial_1 \tilde{X}_{\nu} \right] = \frac{1}{2}\partial_1 \tilde{X}_{\nu} \left[\bar{\partial}^{\mu}(-G^{-1}B)^{\nu}_{\ \rho}\partial_1 X^{\rho} + \bar{\partial}^{\mu}(G^{-1})^{\nu\rho}\partial_1 \tilde{X}_{\rho} \right] = 0$$

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$O(D,D)$ INVARIANCE STILL HOLDS

- When this expression is substituted in the condition $\epsilon_{ab}t^{ab} = 0$, that is valid for any kind of backgrounds:

$$\left[C_{ij}\partial_0\chi^j + M_{ij}\partial_1\chi^j\right](C^{-1})^{ik}\left[C_{kl}\partial_0\chi^l + M_{kl}\partial_1\chi^l)\right] + \left[C - MC^{-1}M\right]_{ij}\partial_1\chi^i\,\partial_1\chi^j = 0.$$

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one can easily see that the off-diagonal structure of C makes the first term vanish and so one gets again the condition $C = MCM$ characterizing the $O(D,D)$ invariance.

- The same result is obtained if one considers $C=\Omega$ and M only $\bar{X}$-dependent.
- In the case of $C=\Omega, M=M(X)$ the constraint $C=MCM$ is still valid and the expression for M keeps on being:

$$M_{ij} = \begin{pmatrix} (G - B\,G^{-1}B)_{\mu\nu} & (B\,G^{-1})_{\mu}^{\ \nu} \\ (-G^{-1}\,B)^{\mu}_{\ \nu} & (G^{-1})^{\mu\nu} \end{pmatrix}$$

but now with X-dependent G and B.

Further Observation on $C = \Omega$ and $M = M(X)$

- Starting from $S(X^\mu, \tilde{X}_\mu)$ and eliminating $\tilde{X}_\mu$ through the equation of motion, one can get the sigma model action for X^μ:

$$S[X] = -\frac{T}{2}\int d^2\sigma \left(\sqrt{g}g^{ab} + \epsilon^{ab}\right)(G+B)_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$$

  that corresponds to the usual formulation of the world sheet of the string in an arbitrary background (G, B).

- If X^μ is eliminated, then one gets the dual sigma model for $\tilde{X}_\mu$:

$$S[\tilde{X}] = -\frac{T}{2}\int d^2\sigma \left(\sqrt{g}g^{ab} + \epsilon^{ab}\right)(G+B)^{-1\mu\nu}\partial_a \tilde{X}^\mu \partial_b \tilde{X}^\nu$$

- This is the case that should reproduce the α'-corrections found in double field theory (Hohm and Zwiebach, 2014) with $C = \Omega$ and by suitably expanding M around flat background.
- Non-abelian T-duality? [Daniel's talk]

C and M dependent only on X or $\tilde{X}$

Second Case:

C and M both dependent only on X (or $\bar{X}$).

- In this case one has to consider, in the equation of motion for $\tilde{X}_\mu$, also the contribution coming from

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- When rewritten explicitly, this quantity vanishes when the index i runs over the one of the $\tilde{X}_\mu$ and therefore it does not give any contribution to the equation of motion of this coordinate.
- One can conclude that the condition $C = MCM$ still holds under the hypothesis that C and/or M are dependent only on X (or only on $\tilde{X}$).

C and M $(X, \tilde{X})$-dependent

Third case:

both C and M dependent on the coordinates χ^i.

- One can think to introduce a parameter $\epsilon \sim \alpha'$ and to expand C and M up to the second order according to:

$$C = C_0 + \epsilon C_1 + \epsilon^2 C_2$$
$$M = M_0 + \epsilon M_1 + \epsilon^2 M_2$$

with $C_0 = M_0 C_0^{-1} M_0$.

- By linearizing the condition $\epsilon_{ab} t^{ab} = 0$ and the equations of motion for the coordinates, one gets, at the order ϵ:

$$(\epsilon_{ab} t^{ab})_{\text{on-shell}} = -\frac{1}{2} Q_{ij} \partial_1 \chi^i \partial_1 \chi^j = 0$$
$$Q = C_1 - (C_0^{-1} M_0)^t M_1 - M_1 (C_0^{-1} M_0) + (C_0^{-1} M_0)^t C_1 (C_0^{-1} M_0)$$

Hence, the linearized condition on C_1 and M_1 is $Q = 0$.

Beyond $O(D,D)$?

- This condition can be actually derived by linearizing the condition $C = MCM$. So at this order the $O(D,D)$ condition keeps on holding, being the first term in the expression of the ϵ-trace order ϵ^2:

$$\left[C_{ij}\partial_0\chi^j + M_{ij}\partial_1\chi^j\right](C^{-1})^{ik}\left[C_{kl}\partial_0\chi^l + M_{kl}\partial_1\chi^l)\right] + \left[C - MC^{-1}M\right]_{ij}\partial_1\chi^i\,\partial_1\chi^j = 0.$$

- This means that the latter plays a role going to the order ϵ^2 and the contribution coming from it adds to the one coming from the term proportional to $C - MCM$. Starting from this order, it seems that the $O(D,D)$ invariance does not hold anymore or one can ask if the deformation is compatible with $O(D,D)$ (discussions with Olaf Hohm and Hai Lin)
- Poisson T-duality? [Daniel's talk].

Constraints of the FJ Lagrangians

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- In the case of a discrete number of degrees of freedom q^i with $i = 1, \cdots, N$ a FJ Lagrangian looks like:

$$L = \frac{1}{2} q^i c_{ij} \dot{q}^j - V(q) \quad \text{with} \quad \det(c_{ij}) \neq 0.$$

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$$L = \frac{1}{2} q^i c_{ij} \dot{q}^j - V(q) \quad \text{with} \quad \det(c_{ij}) \neq 0.$$

- It is first-order and is characterized by N primary second-class constraints:

$$T_j \equiv p_j - \frac{1}{2} q^i c_{ij}$$

with

$$\{T_i, T_j\} = c_{ij} \neq 0$$

Quantization of the FJ Lagrangians

- In order to quantize the theory, the Dirac quantization method has to be applied with the corresponding brackets:

$$\{f, g\}_{DB} \equiv \{f, T_j\}_{DB}\, c^{(-1)jk}\, \{T_k, g\}_{PB}$$

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- According to the usual transition rule $i\,\{f,g\}_{DB} \to \{f,g\}$ from the classical to the quantum theory, the following commutators are obtained:

$$[q_i, q_j] = ic_{ij}^{-1} \;\; ; \;\; [q_i, p_j] = \frac{1}{2} i\delta_{ij} \;\; ; \;\; [p_i, p_j] = -\frac{1}{4} ic_{ij}$$

Quantization of the Double Sigma Model

- Let us consider again:

$$S = -\frac{T}{2}\int d^2\xi\, e\left[C_{ij}\nabla_0\chi^i\,\nabla_1\chi^j + M_{ij}\nabla_1\chi^i\nabla_1\chi^j\right].$$

with the coordinates χ^i on a double torus T^{2D} defined by the identification $\mathcal{X} \equiv \mathcal{X} + 2\pi l\mathcal{L}$, $(l = \sqrt{\alpha'})$ being $\mathcal{L} = (w, lp)$ a vector spanning a Lorentzian lattice $\Lambda^{D,D}$. In components, the identification becomes:

$$X^\mu(\tau,\sigma+\pi) = X^\mu(\tau,\sigma) + 2\,\pi\, l\, w^\mu$$
$$\tilde{X}_\mu(\tau,\sigma+\pi) = \tilde{X}_\mu(\tau,\sigma) + 2\pi\, l^2\, p_\mu.$$

On the torus the $O(D,D;\mathbb{R})$ symmetry becomes an $O(D,D;\mathbb{Z})$ symmetry.

Diagonalization of C and M

- In order to reconduce the double action to a sum of Floreanini-Jackiw Lagrangians, it is necessary to put the matrices C and M simultaneously in a block-diagonal form. This is performed by the matrix

$$(\mathcal{T}^{-1})^{ij} = \frac{1}{\sqrt{2}} \begin{pmatrix} (G^{-1})^{\mu\nu} & (G^{-1})^{\mu\nu} \\ (-E^t\, G^{-1})_\mu^{\ \nu} & (E\, G^{-1})_\mu^{\ \nu} \end{pmatrix} ,$$

where $E \equiv G + B$. In fact, the matrix $\mathcal{T}^{-1}$ transforms C and M respectively into

$$\mathcal{T}^{-t} C \mathcal{T}^{-1} = \begin{pmatrix} G^{-1} & 0 \\ 0 & -G^{-1} \end{pmatrix} \equiv \mathcal{C}^{-1}$$

$$\mathcal{T}^{-t} M \mathcal{T}^{-1} = \begin{pmatrix} G^{-1} & 0 \\ 0 & G^{-1} \end{pmatrix} \equiv \mathcal{G}^{-1}$$

and introduces new coordinates $\Phi_i = \mathcal{T}_{ij}\mathcal{X}^j \equiv (X_{R\,\mu}, X_{L\,\mu})$, in terms of which the R and L sectors are completely decoupled also in the presence of the B-field. The matrix $\mathcal{G}^{-1}$ is the generalized metric in the chiral coordinates system.

Note on the matrix $\mathcal{T}$

- The matrix $\mathcal{T}$ is not an element of the group $O(D, D)$ because it changes the metric C in $\mathcal{C}^{-1}$. It has to be seen as leading to a field redefinition that makes the explicit dependence on the B-field disappear in the action.
- An $O(D, D)$ transformation leaves invariant the metric $\mathcal{C}$ but, in general, transforms $\mathcal{G}^{-1}$ in a non-diagonal matrix. Hence after an $O(D, D)$ transformation, such matrix will exhibit all the dependence on the fields G and B as any general symmetric $O(D, D)$ matrix.
- The transformations which leave invariant the two metrics $\mathcal{G}$ and $\mathcal{C}$, and hence the action, belong to the subgroup $O(D) \times O(D)$ of the original orthogonal group $O(D, D)$.

The double world-sheet action in chiral coordinates

- In the flat gauge the action becomes

$$S \equiv \int d^2\xi [\mathcal{L}_R + \mathcal{L}_L] \,,$$

with

$$\frac{1}{T}\mathcal{L}_{L;R} \equiv \pm\frac{1}{2}\partial_0 X^t_{L;R} G^{-1} \partial_1 X_{L;R} - \frac{1}{2}\partial_1 X^t_{L;R} G^{-1} \partial_1 X_{L;R}$$

which is just the realization in the double string theory of the Floreanini-Jackiw Lagrangians with a non-vanishing Kalb-Ramond field as background.

- The equations of motion become:

$$*dX_R = dX_R \quad ; \quad *dX_L = -dX_L \,.$$

Solution of the eqs. of motion

- The solution of the duality equations of motion, with identifications on the torus now rewritten as:

$$X_{R\,\mu}[\tau-(\sigma+\pi)] = X_{R\,\mu}(\tau-\sigma) - 2\pi\, l^2\, p_{R\,\mu}$$
$$X_{L\,\mu}[\tau+(\sigma+\pi)] = X_{L\,\mu}(\tau+\sigma) + 2\pi\, l^2\, p_{L\,\mu}$$

with

$$\begin{pmatrix} -lp_R \\ lp_L \end{pmatrix} = \mathcal{T} \begin{pmatrix} w \\ lp \end{pmatrix} ,$$

is given by the usual expansion of the right and left bosonic string coordinates.

$$X_R(\tau-\sigma) = x_R + 2\, l^2\, p_R(\tau-\sigma) + il \sum_{n\neq 0} \frac{\alpha_n}{n} e^{-2in(\tau-\sigma)}$$

$$X_L(\tau+\sigma) = x_L + 2\, l^2\, p_L(\tau+\sigma) + il \sum_{n\neq 0} \frac{\tilde{\alpha}_n}{n} e^{-2in(\tau+\sigma)}$$

Primary Constraints

- The linearity of the Lagrangian density in the time derivatives of the fields imply the presence of primary constraints:

$$\Psi_R(P_R, X_R) = P_R + \frac{T}{2} G^{-1} \partial_1 X_R \approx 0$$
$$\Psi_L(P_L, X_L) = P_L - \frac{T}{2} G^{-1} \partial_1 X_L \approx 0 \; .$$

satisfying the following equal 'time' algebra

$$\left\{ \Psi_{R;L}(\tau,\sigma), \, \Psi^t_{R;L}(\tau,\sigma') \right\}_{PB} = \mp T G^{-1} \delta'(\sigma - \sigma') \, ,$$

with $\delta'(x) = \partial_x \delta(x)$ and the upper [lower] sign on the right hand side of the previous identity refers to the label R [L] on the left of the same equation.

- This algebra implies that the primary constraints are second class.

Dirac Brackets

- These second class constraints have to be considered together with the usual string constraints coming from $t_{ab} = 0$. By analogy with the standard procedure followed in string theory, the constraints are evaluated here on the solution of the equation of motion for the fields $X_{R;\, L}$.
- The Dirac procedure yields no secondary constraints and leads to the following Dirac brackets:

$$
\begin{aligned}
\{X_{R;L}(\tau,\sigma), X_{R;L}(\tau,\sigma')\}_{DB} &= \mp\frac{G}{T}\epsilon(\sigma-\sigma') \\
\{P_{R;L}(\tau,\sigma), X_{R;L}(\tau,\sigma')\}_{DB} &= \frac{1}{2}\mathbb{I}\,\delta(\sigma-\sigma') \\
\{P_{R;L}(\tau,\sigma), P_{R;L}(\tau,\sigma')\}_{DB} &= \pm\frac{T}{4}G^{-1}\delta'(\sigma-\sigma')
\end{aligned}
$$

where $\epsilon(\sigma-\sigma')$ is the step function.

Oscillator algebra

- The double world-sheet sigma-model is now quantized by replacing the Dirac brackets with the corresponding commutator according to the well-known substitution:

$$\{\cdot\,,\,\cdot\}_{DB} \to -i[\cdot\,,\,\cdot]\;.$$

- The Dirac brackets of the constraints via the above usual substitution determine the following commutators for the Fourier modes:

$$[p_{R;L},\, x_{R;L}] = iG\;;\;\; [\alpha_m,\, \alpha_n] = mG\delta_{m,-n}\;;\;\; [\tilde{\alpha}_m,\, \tilde{\alpha}_n] = mG\delta_{m,-n}\,.$$

- One recovers, in the R,L-sectors, the Virasoro algebra with a vanishing conformal anomaly in the usual critical dimension.

Non-commutativity

- In terms of the coordinates X^μ and $\tilde{X}_\mu$, the Dirac quantization procedure leads, among the others, to a non-commutativity relation:

$$\left[X(\tau,\sigma),\tilde{X}(\tau,\sigma')\right] = \frac{i}{T}\mathbb{I}\epsilon(\sigma-\sigma')$$

with $\epsilon(\sigma) \equiv \frac{1}{2}\left[\theta(\sigma)-\theta(-\sigma)\right]$.

- The Dirac quantization method implies that X^μ and $\tilde{X}_\mu$ behave like non-commuting phase space type coordinates, even if their expressions in terms of Fourier modes generate the usual oscillator algebra of the standard formulation (De Angelis, Gionti, Marotta, FP - 2014).
- From this perspective, this non-commutativity may lead to the interpretation of high-energy scattering in the X-space as effectively "probing" the $\tilde{X}$-space.

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The End

Thank you for your attention.