Semi-Covariant Formulation of Double Field Theory: Review

IMTAK JEON

KIAS, Seoul

Duality and Novel Geometry in M-theory

26 January 2016

APCTP, Pohang

Based on works

in collaboration with Jeong-Hyuck Park, Kanghoon Lee, Yoonji Suh*

- **Differential geometry with a projection: Application to double field theory**
  JHEP 1104:014 (2011), arXiv:1011.1324
- **Double field formulation of Yang-Mills theory**
  Phys. Lett. B 701:260 (2011), arXiv:1102.0419
- **Stringy differential geometry, beyond Riemann**
  Phys. Rev. D 84:044022 (2011), arXiv:1105.6294
- **Incorporation of fermions into double field theory**
  JHEP 1111:025 (2011), arXiv:1109.2035
- **Supersymmetric Double Field Theory: Stringy Reformulation of Supergravity** Phys. Rev. D Rapid Comm. (2012), arXiv:1112.0069
- **Ramond-Ramond Cohomology and O(D,D) T-duality**
  JHEP 09 (2012) 079, arXiv:1206.3478
- **Stringy Unification of Type IIA and IIB Supergravities under $\mathcal{N}=2$ $D=10$ Supersymmetric Double Field Theory***
  PLB723 (2013) 245-250, arXiv:1210.5078

1. Introduction

$$\text{“Double”: } x \longrightarrow (x, \tilde{x})$$

- The fields in SFT on a toroidal background have dependance on p_i and w^i,

  $$p_i \longleftrightarrow x^i \qquad w^i \longleftrightarrow \tilde{x}_i$$

  so have natural dependance on both x and $\tilde{x}$.: $\Phi(x^i, \tilde{x}_i)$.

  T-duality freely exchanges the x^i and $\tilde{x}_i$.

  “String field theory is a double field theory”. [Kugo, Zwiebach]

- Our focus is on the "massless sectors" of the SFT.

1. Introduction

- In general, supergravities are well known to be as the string effective theories.
- However, the supergravity is based on **particle** description and its description is based on Riemannian geometry where the fundamental object is only metric $g_{\mu\nu}$.
- So, some **stringy effect** might be missing in the supergravity description, and it may not be the best description of the string low energy effective theory.

1. Introduction

- String theory requires that $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ should be treated on an equal footing , because they form a multiplet of T-duality.
- This suggests that there should be an unifying description of them, beyond the Riemannian geometry.
- Double Field Theory(DFT) has been suggested as an unifying description of string effective theory by manifesting the T-duality structure [Siegel, Tseytlin, Duff, Hull, Zwiebach, Hohm]

1. Introduction

- The goal of this talk is to explain about the underlying geometric description for this DFT, called "Semi-covariant formulation"

- It is **completely covariant approach** for DFT as it manifests
  - **O**(D, D) **T-duality**
  - **DFT-diffeomorphisms (generalized Lie derivative)**
  - **A pair of local Lorentz symmetries, Spin**$(1, D-1)_L \times$ **Spin**$(D-1, 1)_R$

- *cf.* Alternatives:[Siegel, Gwak, Hohm, Zwiebach; Waldram, Coimbra, Strickland-Constable (Generalized Geometry a la Hitchin).]

  cf. U-duality extension: [Hohm, Samtleben, Berman, Cederwall, Thompson, Park, Suh, Malek, Blair, Grana, Marques, Perry...]

1. Introduction

- The goal of this talk is to explain about the underlying geometric description for this DFT, called "Semi-covariant formulation"

- It is **completely covariant approach** for DFT as it manifests

    - **O**(D, D) **T-duality**
    - **DFT-diffeomorphisms (generalized Lie derivative)**
    - **A pair of local Lorentz symmetries, Spin**$(1, D-1)_L \times$ **Spin**$(D-1, 1)_R$

- *cf.* Alternatives:[Siegel, Gwak, Hohm, Zwiebach; Waldram, Coimbra, Strickland-Constable (Generalized Geometry a la Hitchin).]

  cf. U-duality extension: [Hohm, Samtleben, Berman, Cederwall, Thompson, Park, Suh, Malek, Blair, Grana, Marques, Perry...]

1. Introduction

- The goal of this talk is to explain about the underlying geometric description for this DFT, called "Semi-covariant formulation"

- It is **completely covariant approach** for DFT as it manifests

  - **O**(D,D) **T-duality**
  - **DFT-diffeomorphisms (generalized Lie derivative)**
  - **A pair of local Lorentz symmetries, Spin**$(1,D-1)_L \times$ **Spin**$(D-1,1)_R$

- *cf.* Alternatives:[Siegel, Gwak, Hohm, Zwiebach; Waldram, Coimbra, Strickland-Constable (Generalized Geometry a la Hitchin).]

  cf. U-duality extension: [Hohm, Samtleben, Berman, Cederwall, Thompson, Park, Suh, Malek, Blair, Grana, Marques, Perry...]

Contents

Bosonic Double Field Theory

- DFT manifests the T-duality by using $\mathbf{O}(D,D)$ tensors as its dynamical variables.
  e.g. NS-NS fields in DFT :

  dilaton , (scalar density) — 'generalized metric' (symmetric $\mathbf{O}(D,D)$ element)

$$e^{-2d} = \sqrt{-g}e^{-2\phi}\,, \qquad \mathcal{H}_{AB} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}$$

  cf. Non-geometric parametrization [Ko-Melby-Thompson-Meyer-Park 2015](Melby-Thompson's talk)

- The Busher's rule is realized as an subgroup of $\mathbf{O}(D,D)$ rotation, [Giveon, Rabinovici, Veneziano, Tseytlin, Siegel] :
  d is scalar and $\mathcal{H}_{AB}$ is rank 2 tensor.

- Metric in DFT : $\mathbf{O}(D,D)$ metric,

$$\mathcal{J}_{AB} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

  freely raises or lowers the $(D+D)$-dimensional vector indices, A, B.

2. Bosonic Double Field Theory

- DFT action for NS-NS sector: [Hull and Zwiebach , later with Hohm]

$$S_{\rm DFT} = \int {\rm d}y^{2D}\, e^{-2d} L_{\rm DFT}(\mathcal{H}, d)\,,$$

  where

$$L_{\rm DFT}(\mathcal{H}, d) = \quad \mathcal{H}^{AB}\left(4\partial_A\partial_B d - 4\partial_A d\partial_B d + \tfrac{1}{8}\partial_A\mathcal{H}^{CD}\partial_B\mathcal{H}_{CD} - \tfrac{1}{2}\partial_A\mathcal{H}^{CD}\partial_C\mathcal{H}_{BD}\right)$$
$$+4\partial_A\mathcal{H}^{AB}\partial_B d - \partial_A\partial_B\mathcal{H}^{AB}\,.$$

  - $\mathrm{O}(D, D)$ structure is manifest and background independent.
  - All spacetime dimension is 'formally doubled', $y^A = (\tilde{x}_\mu, x^\nu)$, $A = 1, 2, \cdots, D{+}D$.

Section condition

- DFT is a D-dimensional theory written in terms of $(D+D)$-dimensional language, i.e. tensors.
- Section condition (strong constraint) :
  The $\mathbf{O}(D,D)$ d'Alembert operator is trivial, acting on arbitrary fields or gauge parameters as well as their products:

$$\partial_A \partial^A = \mathcal{J}^{AB} \partial_A \partial_B = 2 \frac{\partial^2}{\partial \tilde{x}_\mu \partial x^\mu} \sim 0$$

- In DFT, the solution is unique. Up to $\mathbf{O}(D,D)$ rotation, we can choose a frame to set

$$\frac{\partial}{\partial \tilde{x}_\mu} \sim 0 \, .$$

- DFT action in Riemannian parametrization gives the effective action:

$$S_{\rm DFT} \Longrightarrow S_{\rm eff.} = \int {\rm d}x^D \sqrt{-g} e^{-2\phi} \left(R_g + 4(\partial\phi)^2 - \tfrac{1}{12} H^2 \right) .$$

Section condition

- Level matching condition for the massless sector,

$$p \cdot w = N - \bar{N} \equiv 0 \quad \Longleftrightarrow \quad \partial_A \partial^A = 2\frac{\partial^2}{\partial \tilde{x}_\mu \partial x^\mu} \equiv 0\,,$$

  for all fields. (weak constraint)

  So,

$$\partial_A \partial^A \Phi = 0\,, \qquad \text{but } \partial_A \Phi_1 \partial^A \Phi_2 \neq 0$$

- Section condition(strong constraint) seems necessary to write a complete theory, because of action invariance and closedness of symmetry algebra .

Gauge symmetry: 'DFT-diffeomorphism'

Unification of diffeomorphism and B-field gauge symmetry , expressed *via*

- 'generalized Lie derivative' [Siegel, Courant, Grana ...]

$$\hat{\mathcal{L}}_X T_{\omega A} := X^B \partial_B T_{\omega A} + \omega \partial_B X^B T_{\omega A} + \partial_A X^B T_{\omega B} - \partial^B X_A T_{\omega B}.$$

- X^A is an unifying gauge parameter (B-field gauge symmetry+diffeomorphism),

$$X^A = (\Lambda_\mu, \delta x^\nu)$$

- $\mathcal{H}_{AB}$ is a rank 2 DFT tensor, and e^{-2d} is a weight 1 DFT scalar,

Algebra of the gauge symmetry

- Commutator of the generalized Lie derivatives is closed, up to the section condition, by using **c**-bracket,

$$[\hat{\mathcal{L}}_X, \hat{\mathcal{L}}_Y] \sim \hat{\mathcal{L}}_{[X,Y]_{\mathbf{C}}} ,$$

where $[X, Y]_{\mathbf{C}}$ denotes C-bracket

$$[X, Y]^A_{\mathbf{C}} := X^B \partial_B Y^A - Y^B \partial_B X^A + \frac{1}{2} Y^B \partial^A X_B - \frac{1}{2} X^B \partial^A Y_B ,$$

Remarks 1 : section condition

- Understanding the level matching condition (weak constraint) in DFT is crucial but very subtle. (Kanghoon's talk)
- However, "relaxing" the section condition to some extent in case of dimensional reduction has been understood. [Aldazabal, Baron, Nunez, Grana, Marqués, Geissbühler Berman, Lee]
  The section condition is sufficient but not necessary condition for the algebra closure and action invariance. (next week's talk)
- This talk will be restricted on the section condition(strong constraint)

Remarks 1 : section condition

- Understanding the level matching condition (weak constraint) in DFT is crucial but very subtle. (Kanghoon's talk)
- However, "relaxing" the section condition to some extent in case of dimensional reduction has been understood. [Aldazabal, Baron, Nunez, Grana, Marqués, Geissbühler Berman, Lee]
  The section condition is sufficient but not necessary condition for the algebra closure and action invariance. (next week's talk)
- This talk will be restricted on the section condition(strong constraint)

Remarks 1 : section condition

- Understanding the level matching condition (weak constraint) in DFT is crucial but very subtle. (Kanghoon's talk)

- However, "relaxing" the section condition to some extent in case of dimensional reduction has been understood. [Aldazabal, Baron, Nunez, Grana, Marqués, Geissbühler Berman, Lee]
  The section condition is sufficient but not necessary condition for the algebra closure and action invariance. (next week's talk)

- This talk will be restricted on the section condition(strong constraint)

Remarks 2 : doubled-yet-gauged coordinates

- *Doubled-yet-gauged coordinates* [Park]

The D-dimensional section is better understood in terms of doubled-yet-gauged (D+D)-dimensional coordinates.
: We start with $D + D$ coordinates, and impose an **equivalence relation**,

$$x^A \sim x^A + \phi\partial^A\varphi ,$$

where ϕ and φ are arbitrary functions in DFT.

Each gauge orbit parametrized by this shift functions represents a single physical point.

which we call **'Coordinate Gauge Symmetry'**.

Remarks 2 : doubled-yet-gauged coordinates

- *Realization of the coordinate gauge symmetry.*

  We enforcing that arbitrary functions and their arbitrary derivatives, denoted here collectively by Φ, are invariant under the coordinate gauge symmetry *shift*,

$$\Phi(x+\Delta)=\Phi(x)\,,\qquad\qquad \Delta^{A}=\phi\partial^{A}\varphi\,.$$

- *Section condition.*

  The invariance under the coordinate gauge symmetry can be shown to be equivalent to the section condition:

$$\textbf{Coordinate Gauge Symmetry}\quad\Longleftrightarrow\quad\partial_{A}\partial^{A}\equiv 0\,.$$

Park, Lee-Park 2013

Remarks 2 : doubled-yet-gauged coordinates

- *Realization of the coordinate gauge symmetry.*

We enforcing that arbitrary functions and their arbitrary derivatives, denoted here collectively by Φ, are invariant under the coordinate gauge symmetry *shift*,

$$\Phi(x+\Delta)=\Phi(x)\,,\qquad\qquad \Delta^{A}=\phi\partial^{A}\varphi\,.$$

- *Section condition.*

The invariance under the coordinate gauge symmetry can be shown to be equivalent to the section condition:

$$\textbf{Coordinate Gauge Symmetry}\quad\Longleftrightarrow\quad \partial_{A}\partial^{A}\equiv 0\,.$$

Park, Lee-Park 2013

Remarks 3: Double sigma model

- The coordinate gauge symmetry can be naturally realized on the worldsheet as a conventional gauge symmetry of a string action.
- Introducing the gauge field for the coordinate gauge symmetry and defining the covariant derivative, $D_iX^M = \partial_i X^M - \mathcal{A}_i^M$, **DFT sigma model action** is written Park-Lee 2013 ,

$$\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\ \mathcal{L}_{\mathrm{string}}\,, \qquad \mathcal{L}_{\mathrm{string}} = -\tfrac{1}{2}\sqrt{-h}\,h^{ij}D_iX^MD_jX^N\mathcal{H}_{MN}(X) - \epsilon^{ij}D_iX^M\mathcal{A}_{jM}\,,$$

- Under the Riemaniann parametrization, **the DFT sigma model reduces to the standard string action,**

$$\frac{1}{4\pi\alpha'}\mathcal{L}_{\mathrm{string}} \equiv \frac{1}{2\pi\alpha'}\left[-\tfrac{1}{2}\sqrt{-h}h^{ij}\partial_iX^\mu\partial_jX^\nu G_{\mu\nu}(X) + \tfrac{1}{2}\epsilon^{ij}\partial_iX^\mu\partial_jX^\nu B_{\mu\nu}(X) + \tfrac{1}{2}\epsilon^{ij}\partial_i\tilde{X}_\mu\partial_j X\right.$$

with the topological term introduced by Giveon-Rocek; Hull .

Remarks 3: Double sigma model

- The coordinate gauge symmetry can be naturally realized on the worldsheet as a conventional gauge symmetry of a string action.
- Introducing the gauge field for the coordinate gauge symmetry and defining the covariant derivative, $D_i X^M = \partial_i X^M - \mathcal{A}_i^M$, **DFT sigma model action** is written Park-Lee 2013 ,

$$\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\ \mathcal{L}_{\text{string}}\,, \qquad \mathcal{L}_{\text{string}} = -\tfrac{1}{2}\sqrt{-h}\,h^{ij} D_i X^M D_j X^N \mathcal{H}_{MN}(X) - \epsilon^{ij} D_i X^M \mathcal{A}_{jM}\,,$$

- Under the Riemaniann parametrization, **the DFT sigma model reduces to the standard string action,**

$$\frac{1}{4\pi\alpha'}\mathcal{L}_{\text{string}} \equiv \frac{1}{2\pi\alpha'}\left[-\tfrac{1}{2}\sqrt{-h}h^{ij}\partial_i X^\mu \partial_j X^\nu G_{\mu\nu}(X) + \tfrac{1}{2}\epsilon^{ij}\partial_i X^\mu \partial_j X^\nu B_{\mu\nu}(X) + \tfrac{1}{2}\epsilon^{ij}\partial_i \tilde{X}_\mu \partial_j X\right.$$

with the topological term introduced by Giveon-Rocek; Hull .

Remarks 3: Double sigma model

- The coordinate gauge symmetry can be naturally realized on the worldsheet as a conventional gauge symmetry of a string action.
- Introducing the gauge field for the coordinate gauge symmetry and defining the covariant derivative, $D_i X^M = \partial_i X^M - \mathcal{A}_i^M$, **DFT sigma model action** is written Park-Lee 2013 ,

$$\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\ \mathcal{L}_{\text{string}}\,, \qquad \mathcal{L}_{\text{string}} = -\tfrac{1}{2}\sqrt{-h}\,h^{ij}D_iX^M D_jX^N\mathcal{H}_{MN}(X) - \epsilon^{ij}D_iX^M\mathcal{A}_{jM}\,,$$

- Under the Riemaniann parametrization, **the DFT sigma model reduces to the standard string action,**

$$\frac{1}{4\pi\alpha'}\mathcal{L}_{\text{string}} \equiv \frac{1}{2\pi\alpha'}\left[-\tfrac{1}{2}\sqrt{-h}h^{ij}\partial_iX^\mu\partial_jX^\nu G_{\mu\nu}(X) + \tfrac{1}{2}\epsilon^{ij}\partial_iX^\mu\partial_jX^\nu B_{\mu\nu}(X) + \tfrac{1}{2}\epsilon^{ij}\partial_i\tilde{X}_\mu\partial_j X$$

with the topological term introduced by Giveon-Rocek; Hull .

Remarks 4: finite transformation

- *Diffeomorphisms.*
  - Hohm-Zwiebach ansatz for finite transformations:

$$F := \tfrac{1}{2}\left(L\bar{L}^{-1} + \bar{L}^{-1}L\right)\,, \qquad \bar{F} := \mathcal{J}F^t\mathcal{J}^{-1} = \tfrac{1}{2}\left(L^{-1}\bar{L} + \bar{L}L^{-1}\right) = F^{-1}\,,$$

where

$$L_M{}^N := \partial_M x'^N\,, \qquad \bar{L} := \mathcal{J}L^t\mathcal{J}^{-1}\,.$$

  - Though nice and compact, F does not precisely coincide with $\mathbf{exp}(\hat{\mathcal{L}}_X)$.
  - Yet, **up to coordinate gauge symmetry** it is possible to show Park 2013

$$F \equiv \mathbf{exp}(\hat{\mathcal{L}}_X)$$

c.f. Berman-Cederwall-Perry, Hull, Papadopoulos, Sakatani, Rey
(Sakatani's talk)

Remarks 4: finite transformation

- *Diffeomorphisms.*
  - Hohm-Zwiebach ansatz for finite transformations:

$$F := \tfrac{1}{2}\left(L\bar{L}^{-1} + \bar{L}^{-1}L\right) , \qquad \bar{F} := \mathcal{J}F^{t}\mathcal{J}^{-1} = \tfrac{1}{2}\left(L^{-1}\bar{L} + \bar{L}L^{-1}\right) = F^{-1} ,$$

where

$$L_M{}^N := \partial_M x'^N , \qquad \bar{L} := \mathcal{J}L^{t}\mathcal{J}^{-1} .$$

  - Though nice and compact, F does not precisely coincide with $\mathbf{exp}(\hat{\mathcal{L}}_X)$.
  - Yet, **up to coordinate gauge symmetry** it is possible to show Park 2013

$$F \equiv \mathbf{exp}(\hat{\mathcal{L}}_X)$$

c.f. Berman-Cederwall-Perry, Hull, Papadopoulos, Sakatani, Rey (Sakatani's talk)

Remarks 4: finite transformation

- *Diffeomorphisms.*

  - Hohm-Zwiebach ansatz for finite transformations:

    $$F := \tfrac{1}{2}\left(L\bar{L}^{-1} + \bar{L}^{-1}L\right) \,, \qquad \bar{F} := \mathcal{J}F^t\mathcal{J}^{-1} = \tfrac{1}{2}\left(L^{-1}\bar{L} + \bar{L}L^{-1}\right) = F^{-1} \,,$$

    where

    $$L_M{}^N := \partial_M x'^N \,, \qquad \bar{L} := \mathcal{J}L^t\mathcal{J}^{-1} \,.$$

  - Though nice and compact, F does not precisely coincide with $\mathbf{exp}(\hat{\mathcal{L}}_X)$.

  - Yet, **up to coordinate gauge symmetry** it is possible to show Park 2013

    $$F \equiv \mathbf{exp}(\hat{\mathcal{L}}_X)$$

c.f. Berman-Cederwall-Perry, Hull, Papadopoulos, Sakatani, Rey
(Sakatani's talk)

Remarks 4: finite transformation

- *Diffeomorphisms.*

  - Hohm-Zwiebach ansatz for finite transformations:

$$F := \tfrac{1}{2}\left(L\bar{L}^{-1} + \bar{L}^{-1}L\right)\,, \qquad \bar{F} := \mathcal{J}F^{t}\mathcal{J}^{-1} = \tfrac{1}{2}\left(L^{-1}\bar{L} + \bar{L}L^{-1}\right) = F^{-1}\,,$$

    where

$$L_M{}^N := \partial_M x'^N\,, \qquad \bar{L} := \mathcal{J}L^{t}\mathcal{J}^{-1}\,.$$

  - Though nice and compact, F does not precisely coincide with $\mathbf{exp}(\hat{\mathcal{L}}_X)$.

  - Yet, **up to coordinate gauge symmetry** it is possible to show Park 2013

$$F \;\equiv\; \mathbf{exp}(\hat{\mathcal{L}}_X)$$

c.f. Berman-Cederwall-Perry, Hull, Papadopoulos, Sakatani, Rey
(Sakatani's talk)

3. Semi-covariant formlation

Connection

$$\Gamma^0_{CAB} = \quad 2\left(P\partial_C P\bar{P}\right)_{[AB]} + 2\left(\bar{P}_{[A}{}^D\bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E\right)\partial_D P_{EC}$$

$$-\tfrac{4}{D-1}\left(\bar{P}_{C[A}\bar{P}_{B]}{}^D + P_{C[A}P_{B]}{}^D\right)\left(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\right) \ ,$$

Curvature

$$(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD}$$

Semi-covariant formulation

- Basic geometric objects are

$$\mathcal{J}_{AB}\,, \qquad \mathcal{H}_{AB}\,, \qquad d\,.$$

- Note that

$$\mathcal{H}_A{}^C\mathcal{H}_C{}^B = \delta_A{}^B\,, \qquad \mathcal{H}_{AB} = \mathcal{H}_{BA}\,,$$

- We can define 'projection' which is related to $\mathcal{H}$ by

$$P_{AB} = \tfrac{1}{2}(\mathcal{J}_{AB} + \mathcal{H}_{AB})\,, \quad \bar{P}_{AB} = \tfrac{1}{2}(\mathcal{J}_{AB} - \mathcal{H}_{AB})$$

which satisfy the property of the projections,

$$P_A{}^B P_B{}^C = P_A{}^C\,, \quad P_{AB} = P_{BA}\,, \quad \bar{P}_A{}^B\bar{P}_B{}^C = \bar{P}_A{}^C\,, \quad \bar{P}_{AB} = \bar{P}_{BA}$$

- Projection will the characteristic property of DFT geometry.
- The basic geometric objects, which should be treated equally, are

$$(d\,, P_{AB}\,, \bar{P}_{AB})\,.$$

Semi-covariant formulation

- Basic geometric objects are

$$\mathcal{J}_{AB}\,,\qquad \mathcal{H}_{AB}\,,\qquad d\,.$$

- Note that

$$\mathcal{H}_A{}^C\mathcal{H}_C{}^B=\delta_A{}^B\,,\qquad \mathcal{H}_{AB}=\mathcal{H}_{BA}\,,$$

- We can define 'projection' which is related to $\mathcal{H}$ by

$$P_{AB}=\tfrac{1}{2}(\mathcal{J}_{AB}+\mathcal{H}_{AB})\,,\quad \bar{P}_{AB}=\tfrac{1}{2}(\mathcal{J}_{AB}-\mathcal{H}_{AB})$$

which satisfy the property of the projections,

$$P_A{}^BP_B{}^C=P_A{}^C\,,\quad P_{AB}=P_{BA}\,,\quad \bar{P}_A{}^B\bar{P}_B{}^C=\bar{P}_A{}^C\,,\quad \bar{P}_{AB}=\bar{P}_{BA}$$

- Projection will the characteristic property of DFT geometry.
- The basic geometric objects, which should be treated equally, are

$$(d\,,P_{AB}\,,\bar{P}_{AB})\,.$$

Semi-covariant formulation

- Basic geometric objects are

$$\mathcal{J}_{AB}\,, \qquad \mathcal{H}_{AB}\,, \qquad d\,.$$

- Note that

$$\mathcal{H}_A{}^C\mathcal{H}_C{}^B = \delta_A{}^B\,, \qquad \mathcal{H}_{AB} = \mathcal{H}_{BA}\,,$$

- We can define 'projection' which is related to $\mathcal{H}$ by

$$P_{AB} = \tfrac{1}{2}(\mathcal{J}_{AB} + \mathcal{H}_{AB})\,, \quad \bar{P}_{AB} = \tfrac{1}{2}(\mathcal{J}_{AB} - \mathcal{H}_{AB})$$

  which satisfy the property of the projections,

$$P_A{}^B P_B{}^C = P_A{}^C\,, \quad P_{AB} = P_{BA}\,, \quad \bar{P}_A{}^B \bar{P}_B{}^C = \bar{P}_A{}^C\,, \quad \bar{P}_{AB} = \bar{P}_{BA}$$

- Projection will the characteristic property of DFT geometry.
- The basic geometric objects, which should be treated equally, are

$$(d\,, P_{AB}\,, \bar{P}_{AB})\,.$$

- We further define a pair of six-index projectors,

$$\begin{aligned}&\mathcal{P}_{CAB}{}^{DEF} := P_C{}^D P_{[A}{}^{[E} P_{B]}{}^{F]} + \tfrac{2}{D-1} P_{C[A} P_{B]}{}^{[E} P^{F]D}\,, \qquad \mathcal{P}_{CAB}{}^{DEF}\mathcal{P}_{DEF}{}^{GHI} = \mathcal{P}_{CAB}{}^{GHI}\,,\\ &\bar{\mathcal{P}}_{CAB}{}^{DEF} := \bar{P}_C{}^D \bar{P}_{[A}{}^{[E} \bar{P}_{B]}{}^{F]} + \tfrac{2}{D-1} \bar{P}_{C[A} \bar{P}_{B]}{}^{[E} \bar{P}^{F]D}\,, \qquad \bar{\mathcal{P}}_{CAB}{}^{DEF}\bar{\mathcal{P}}_{DEF}{}^{GHI} = \bar{\mathcal{P}}_{CAB}{}^{GHI}\,,\end{aligned}$$

which satisfy the following properties, symmetric and traceless,

$$\begin{aligned}&\mathcal{P}_{CABDEF} = \mathcal{P}_{DEFCAB} = \mathcal{P}_{C[AB]D[EF]}\,, \qquad \bar{\mathcal{P}}_{CABDEF} = \bar{\mathcal{P}}_{DEFCAB} = \bar{\mathcal{P}}_{C[AB]D[EF]}\,,\\ &\mathcal{P}^A{}_{ABDEF} = 0\,, \quad P^{AB}\mathcal{P}_{ABCDEF} = 0\,, \qquad \bar{\mathcal{P}}^A{}_{ABDEF} = 0\,, \quad \bar{P}^{AB}\bar{\mathcal{P}}_{ABCDEF} = 0\,.\end{aligned}$$

- These projectors will govern the DFT-diffeomorphic anomaly in the semi-covariant formalism, which can be easily projected out.

Semi-covariant formulation

- Postulate a “semi-covariant" derivative, ∇_A on DFT tensor T_A with weight ω,

$$\nabla_C T_A = \partial_C T_A - \omega \Gamma^B{}_{BC} T_A + \Gamma_{CA}{}^B T_B \,,$$

- We demand the following compatibility conditions,

$$\nabla_A \bar{P}_{BC} = 0 \,, \qquad \nabla_A P_{BC} = 0 \,, \qquad \nabla_A d = 0 \,,$$

(*cf.* $\nabla_\lambda g_{\mu\nu} = 0$ in Riemannian geometry)

Semi-covariant formulation

- Postulate a "semi-covariant" derivative, ∇_A on DFT tensor T_A with weight ω,

$$\nabla_C T_A = \partial_C T_A - \omega \Gamma^B{}_{BC} T_A + \Gamma_{CA}{}^B T_B \,,$$

- We demand the following compatibility conditions,

$$\nabla_A \bar{P}_{BC} = 0 \,, \qquad \nabla_A P_{BC} = 0 \,, \qquad \nabla_A d = 0 \,,$$

(*cf.* $\nabla_\lambda g_{\mu\nu} = 0$ in Riemannian geometry)

- Torsion free conection is uniquely determined in terms of basic geometrical variables, [IJ, Lee, Park '11]

$$\Gamma^{0}_{CAB} = 2\,(P\partial_C P\bar{P})_{[AB]} + 2\left(\bar{P}_{[A}{}^{D}\bar{P}_{B]}{}^{E} - P_{[A}{}^{D}P_{B]}{}^{E}\right)\partial_D P_{EC}$$
$$-\tfrac{4}{D-1}\left(\bar{P}_{C[A}\bar{P}_{B]}{}^{D} + P_{C[A}P_{B]}{}^{D}\right)\left(\partial_D d + (P\partial^{E}P\bar{P})_{[ED]}\right)\,,$$

satisfying the torsion free condition,

$$\Gamma^{0}_{[ABC]} = 0\,, \qquad (\Leftrightarrow \quad \hat{\mathcal{L}}^{\partial}_{X} = \hat{\mathcal{L}}^{\nabla}_{X})$$

and further satisfying

$$\mathcal{P}_{CAB}{}^{DEF}\Gamma^{0}_{DEF} = 0\,, \qquad \bar{\mathcal{P}}_{CAB}{}^{DEF}\Gamma^{0}_{DEF} = 0\,.$$

Stringy differential geometry 1105.6294 (1011.1324)

- Under $\delta_X \mathcal{P}_{AB} = \hat{\mathcal{L}}_X \mathcal{P}_{AB}$ $\delta_X d = \hat{\mathcal{L}}_X d$ (DFT-diffeomorphism) , the variation of $\nabla_C T_A$ contains an anomalous non-covariant part,

$$(\delta_X - \hat{\mathcal{L}}_X)\nabla_C T_A \sim 2(\mathcal{P}+\bar{\mathcal{P}})_{CA}{}^{BFDE}\partial_F\partial_{[D}X_{E]}T_B\,.$$

- Hence, these are not DFT-diffeomorphism covariant,

- However, the anomalous term are controlled by the rank six projectors , so they can be projected out by combining the projection matrices P_{AB} and $\bar{P}_{AB}$.

Stringy differential geometry 1105.6294 (1011.1324)

- Under $\delta_X\mathcal{P}_{AB}=\hat{\mathcal{L}}_X\mathcal{P}_{AB}\ \ \delta_X d=\hat{\mathcal{L}}_X d$ (DFT-diffeomorphism) , the variation of $\nabla_C T_A$ contains an anomalous non-covariant part,

$$(\delta_X-\hat{\mathcal{L}}_X)\nabla_C T_A \sim 2(\mathcal{P}+\bar{\mathcal{P}})_{CA}{}^{BFDE}\partial_F\partial_{[D}X_{E]}T_B\,.$$

- Hence, these are not DFT-diffeomorphism covariant,

- However, the anomalous term are controlled by the rank six projectors , so they can be projected out by combining the projection matrices P_{AB} and $\bar{P}_{AB}$.

Projection-aided covariant derivatives

"semi-covariant derivative" :

combined with the projections , we can generate various covariant quantities:

Examples:

- **For $\mathbf{O}(D, D)$ tensors:**

$$P_C{}^D \bar{P}_A{}^B \nabla_D T_B \,, \qquad \bar{P}_C{}^D P_A{}^B \nabla_D T_B \,,$$

$$P^{AB} \nabla_A T_B \,, \qquad \bar{P}^{AB} \nabla_A T_B \,, \qquad \text{Divergences} \,,$$

$$P^{AB} \bar{P}_C{}^D \nabla_A \nabla_B T_D \,, \qquad \bar{P}^{AB} P_C{}^D \nabla_A \nabla_B T_D \,. \qquad \text{Laplacians}$$

- Patern: need opposite chirality or contraction

Curvatures 1105.6294

- The usual diffeomorhphism field strength defined by

$$R_{CDAB} = \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}{}^{E} \Gamma_{BED} - \Gamma_{BC}{}^{E} \Gamma_{AED} \,,$$

  is NOT covariant.

- Instead, we define *semi-covariant four-index curvature* , as for a **key quantity** in our formalism, *cf*. [Siegel; Waldram; Hohm, Zwiebach]

$$S_{ABCD} := \tfrac{1}{2}\left(R_{ABCD} + R_{CDAB} - \Gamma^{E}{}_{AB}\Gamma_{ECD}\right) \,.$$

- It satisfies
  - just like the Riemann curvature,

$$S_{ABCD} = \tfrac{1}{2}(S_{[AB][CD]} + S_{[CD][AB]}) \,,$$

$$S_{A[BCD]} = 0 \quad : \quad \text{Bianchi identity} \,,$$

  - and with projectors,

$$(P^{AB}P^{CD} + \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} \sim 0 \,,$$

$$P_I{}^{A}P_J{}^{B}\bar{P}_K{}^{C}\bar{P}_L{}^{D}S_{ABCD} \sim 0 \,,$$

$$P_I{}^{A}\bar{P}_J{}^{B}P_K{}^{C}\bar{P}_L{}^{D}S_{ABCD} \sim 0 \,, \quad etc.$$

Curvatures 1105.6294

- The usual diffeomorhphism field strength defined by

$$R_{CDAB} = \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}{}^{E}\Gamma_{BED} - \Gamma_{BC}{}^{E}\Gamma_{AED} \,,$$

  is NOT covariant.

- Instead, we define *semi-covariant four-index curvature* , as for a **key quantity** in our formalism, *cf*. [Siegel; Waldram; Hohm, Zwiebach]

$$S_{ABCD} := \tfrac{1}{2}\left(R_{ABCD} + R_{CDAB} - \Gamma^{E}{}_{AB}\Gamma_{ECD}\right) \,.$$

- It satisfies
  - just like the Riemann curvature,

$$S_{ABCD} = \tfrac{1}{2}\left(S_{[AB][CD]} + S_{[CD][AB]}\right) \,,$$

$$S_{A[BCD]} = 0 \quad : \quad \text{Bianchi identity} \,,$$

  - and with projectors,

$$(P^{AB}P^{CD} + \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} \sim 0 \,,$$

$$P_I{}^A P_J{}^B \bar{P}_K{}^C \bar{P}_L{}^D S_{ABCD} \sim 0 \,,$$

$$P_I{}^A \bar{P}_J{}^B P_K{}^C \bar{P}_L{}^D S_{ABCD} \sim 0 \,, \quad etc.$$

Curvatures 1105.6294

- The usual diffeomorhphism field strength defined by

$$R_{CDAB} = \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}{}^{E}\Gamma_{BED} - \Gamma_{BC}{}^{E}\Gamma_{AED}\,,$$

  is NOT covariant.
- Instead, we define *semi-covariant four-index curvature* , as for a **key quantity** in our formalism, *cf*. [Siegel; Waldram; Hohm, Zwiebach]

$$S_{ABCD} := \tfrac{1}{2}\left(R_{ABCD} + R_{CDAB} - \Gamma^{E}{}_{AB}\Gamma_{ECD}\right)\,.$$

- It satisfies
  - just like the Riemann curvature,

$$S_{ABCD} = \tfrac{1}{2}(S_{[AB][CD]} + S_{[CD][AB]})\,,$$

$$S_{A[BCD]} = 0 \quad : \quad \text{Bianchi identity}\,,$$

  - and with projectors,

$$(P^{AB}P^{CD} + \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} \sim 0\,,$$

$$P_I{}^A P_J{}^B \bar{P}_K{}^C \bar{P}_L{}^D S_{ABCD} \sim 0\,,$$

$$P_I{}^A \bar{P}_J{}^B P_K{}^C \bar{P}_L{}^D S_{ABCD} \sim 0\,, \quad etc.$$

Curvatures I [1105.6294]

- This is still Not covariant tensor, but contracting with projection operators, we can obtain covariant quatities.

  - Rank two-tensor:

$$P_I{}^A \bar{P}_J{}^B S_{AB}\,, \qquad \text{where } S_{AB} := S^C{}_{ACB}$$

  - Scalar curvature: defines the Lagrangian for NS-NS sector

$$(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD}$$

- The above scalar curvature exactly reproduces the bosonic Lagrangian by Hull, Zwiebach,Hohm.
- There is no covariant rank 4 tensor.

Further completely covariant example

- **Yang-Mills field strength in DFT** is given by two opposite projections,

$$P_A{}^M \bar{P}_B{}^N \mathcal{F}_{MN}\,,$$

where $\mathcal{F}_{MN}$ is the semi-covariant field strength of a YM potential, $\mathcal{V}_M$,

$$\mathcal{F}_{MN} := \nabla_M \mathcal{V}_N - \nabla_N \mathcal{V}_M - i\,[\mathcal{V}_M, \mathcal{V}_N]\ .$$

Unlike the Riemannian case, the Γ connections are not canceled out.

IJ-Lee-Park 2011, Choi-Park 2015

Choi's talk

- **Completely covariant Killing equations** of DFT:

$$\hat{\mathcal{L}}_X \mathcal{H}_{MN} = 0 \quad \Longleftrightarrow \quad (P\nabla)_M (\bar{P}X)_N - (\bar{P}\nabla)_N (PX)_M = 0\,,$$
$$\hat{\mathcal{L}}_X d = 0 \quad \Longleftrightarrow \quad \nabla_M X^M = 0\,.$$

Park-Rey-Rim-Sakatani 2015

Rim's talk

Further completely covariant example

- **Yang-Mills field strength in DFT** is given by two opposite projections,

$$P_A{}^M \bar{P}_B{}^N \mathcal{F}_{MN}\,,$$

where $\mathcal{F}_{MN}$ is the semi-covariant field strength of a YM potential, $\mathcal{V}_M$,

$$\mathcal{F}_{MN} := \nabla_M \mathcal{V}_N - \nabla_N \mathcal{V}_M - i\left[\mathcal{V}_M, \mathcal{V}_N\right]\,.$$

Unlike the Riemannian case, the Γ connections are not canceled out.

IJ-Lee-Park 2011, Choi-Park 2015

Choi's talk

- **Completely covariant Killing equations** of DFT:

$$\begin{array}{lll}\hat{\mathcal{L}}_X \mathcal{H}_{MN} = 0 & \Longleftrightarrow & (P\nabla)_M (\bar{P}X)_N - (\bar{P}\nabla)_N (PX)_M = 0\,,\\ \hat{\mathcal{L}}_X d = 0 & \Longleftrightarrow & \nabla_M X^M = 0\,.\end{array}$$

Park-Rey-Rim-Sakatani 2015

Rim's talk

Reproduction of DFT

- Natural DFT action for NS-NS sector is

$$S_{\text{DFT}} = \int_{\Sigma^D} e^{-2d} \left[(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} - 2\Lambda \right] ,$$

  where the integral is taken over a section, Σ^D, and the DFT-cosmological constant term has been inserted.

- The curvature term agrees with Hull, Zwiebach and Hohm ,

$$\mathcal{H}^{AB}\left(4\partial_A\partial_B d - 4\partial_A d\partial_B d + \tfrac{1}{8}\partial_A\mathcal{H}^{CD}\partial_B\mathcal{H}_{CD} - \tfrac{1}{2}\partial_A\mathcal{H}^{CD}\partial_C\mathcal{H}_{BD}\right)$$

$$+4\partial_A\mathcal{H}^{AB}\partial_B d - \partial_A\partial_B\mathcal{H}^{AB} .$$

- The DFT-cosmological constant term becomes an exponential potential, $e^{-2\phi}$, in term Riemannian geometry. The cosmological constant problem is clearly reformulated in DFT

Reproduction of DFT

- Natural DFT action for NS-NS sector is

$$S_{\rm DFT} = \int_{\Sigma^D} e^{-2d} \left[(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} - 2\Lambda \right] ,$$

  where the integral is taken over a section, Σ^D, and the DFT-cosmological constant term has been inserted.

- The curvature term agrees with Hull, Zwiebach and Hohm ,

$$\mathcal{H}^{AB} \left(4\partial_A\partial_B d - 4\partial_A d\partial_B d + \tfrac{1}{8}\partial_A\mathcal{H}^{CD}\partial_B\mathcal{H}_{CD} - \tfrac{1}{2}\partial_A\mathcal{H}^{CD}\partial_C\mathcal{H}_{BD}\right)$$

$$+4\partial_A\mathcal{H}^{AB}\partial_B d - \partial_A\partial_B\mathcal{H}^{AB} .$$

- The DFT-cosmological constant term becomes an exponential potential, $e^{-2\phi}$, in term Riemannian geometry. The cosmological constant problem is clearly reformulated in DFT

Reproduction of DFT

- Natural DFT action for NS-NS sector is

$$S_{\text{DFT}} = \int_{\Sigma^D} e^{-2d} \left[(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} - 2\Lambda \right] ,$$

  where the integral is taken over a section, Σ^D, and the DFT-cosmological constant term has been inserted.

- The curvature term agrees with Hull, Zwiebach and Hohm ,

$$\mathcal{H}^{AB} \left(4\partial_A\partial_B d - 4\partial_A d\partial_B d + \tfrac{1}{8}\partial_A\mathcal{H}^{CD}\partial_B\mathcal{H}_{CD} - \tfrac{1}{2}\partial_A\mathcal{H}^{CD}\partial_C\mathcal{H}_{BD} \right)$$

$$+4\partial_A\mathcal{H}^{AB}\partial_B d - \partial_A\partial_B\mathcal{H}^{AB} .$$

- The DFT-cosmological constant term becomes an exponential potential, $e^{-2\phi}$, in term Riemannian geometry. The cosmological constant problem is clearly reformulated in DFT

4. Supersymmetric extension of double field theory

Symmetries of SDFT

Semi-covariant formulation manifest all the bosonic symmetries

- **O**(D,D) **T-duality:**
- **DFT-diffeomorphism (generalized Lie derivative)**
  - **Diffeomorphism**
  - B**-field gauge symmetry**
- **A pair of local Lorentz symmetries, Spin**$(1,D{-}1)_L \times$ **Spin**$(D{-}1,1)_R$
- $D = 10$ **maximal Local SUSY**

Field contents of $D = 10$ Maximal SDFT

- **Bosons**
  - **NS-NS sector** $\begin{cases} \textbf{DFT-dilaton:} & d \\ \textbf{Double-vielbeins:} & V_{Ap}\,, \quad \bar{V}_{A\bar{p}} \end{cases}$
  - **R-R potential:** $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$

- **Fermions (NS-R, R-NS)**
  - **DFT-dilatinos:** $\rho^{\alpha}\,, \quad \rho'^{\bar{\alpha}}$
  - **Gravitinos:** $\psi^{\alpha}_{\bar{p}}\,, \quad \psi'^{\bar{\alpha}}_{p}$

Field contents of $D = 10$ Maximal SDFT

- **Bosons**
  - **NS-NS sector** $\left\{ \begin{array}{ll} \textbf{DFT-dilaton:} & d \\ \textbf{Double-vielbeins:} & V_{Ap}\,, \quad \bar{V}_{A\bar{p}} \end{array} \right.$
  - **R-R potential:** $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$
- **Fermions (NS-R, R-NS)**
  - **DFT-dilatinos:** $\rho^{\alpha}\,, \quad \rho'^{\bar{\alpha}}$
  - **Gravitinos:** $\psi^{\alpha}_{\bar{p}}\,, \quad \psi'^{\bar{\alpha}}_{p}$

Index	Representation	Metric (raising/lowering indices)
$A, B, \cdots$	$\mathbf{O}(D,D)$ vector	$\mathcal{J}_{AB}$
$p, q, \cdots$	$\mathbf{Spin}(1, D-1)_L$ vector	$\eta_{pq} = \mathbf{diag}(-++\cdots+)$
$\alpha, \beta, \cdots$	$\mathbf{Spin}(1, D-1)_L$ spinor	$C_{+\alpha\beta}\,, \quad (\gamma^p)^T = C_+\gamma^p C_+^{-1}$
$\bar{p}, \bar{q}, \cdots$	$\mathbf{Spin}(D-1, 1)_R$ vector	$\bar{\eta}_{\bar{p}\bar{q}} = \mathbf{diag}(+--\cdots-)$
$\bar{\alpha}, \bar{\beta}, \cdots$	$\mathbf{Spin}(D-1, 1)_R$ spinor	$\bar{C}_{+\bar{\alpha}\bar{\beta}}\,, \quad (\bar{\gamma}^{\bar{p}})^T = \bar{C}_+\bar{\gamma}^{\bar{p}}\bar{C}_+^{-1}$

Field contents of $D = 10$ Maximal SDFT

- **Bosons**
  - **NS-NS sector** $\left\{ \begin{array}{ll} \textbf{DFT-dilaton:} & d \\ \textbf{Double-vielbeins:} & V_{Ap}\,, \quad \bar{V}_{A\bar{p}} \end{array} \right.$
  - **R-R potential:** $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$

- **Fermions (NS-R, R-NS)**
  - **DFT-dilatinos:** $\rho^{\alpha}\,, \quad \rho'^{\bar{\alpha}}$
  - **Gravitinos:** $\psi^{\alpha}_{\bar{p}}\,, \quad \psi'^{\bar{\alpha}}_{p}$

All NS-NS fields, d, V_{Ap}, $\bar{V}_{A\bar{p}}$, will be equally treated as basic geometric objects.

Field contents of $D = 10$ Maximal SDFT

- **Bosons**
  - **NS-NS sector** $\begin{cases} \textbf{DFT-dilaton:} & d \\ \textbf{Double-vielbeins:} & V_{Ap}\,, \quad \bar{V}_{A\bar{p}} \end{cases}$
  - **R-R potential:** $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$
- **Fermions (NS-R, R-NS)**
  - **DFT-dilatinos:** $\rho^{\alpha}\,, \quad \rho'^{\bar{\alpha}}$
  - **Gravitinos:** $\psi^{\alpha}_{\bar{p}}\,, \quad \psi'^{\bar{\alpha}}_{p}$

R-R potential is bi-fundamental spinor representation as a democratic description.

$cf.$ $\mathbf{O}(D,D)$ spinor representation Fukuma, Oota Tanaka; Hohm, Kwak, Zwiebach

Field contents of $D = 10$ Maximal SDFT

- **Bosons**
  - **NS-NS sector** $\left\{ \begin{array}{ll} \textbf{DFT-dilaton:} & d \\ \textbf{Double-vielbeins:} & V_{Ap}\,, \quad \bar{V}_{A\bar{p}} \end{array} \right.$
  - **R-R potential:** $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$
- **Fermions (NS-R, R-NS)**
  - **DFT-dilatinos:** $\rho^{\alpha}\,, \quad \rho'^{\bar{\alpha}}$
  - **Gravitinos:** $\psi^{\alpha}_{\bar{p}}\,, \quad \psi'^{\bar{\alpha}}_{p}$

Field contents of $D = 10$ Maximal SDFT

- **Bosons**
  - **NS-NS sector** $\begin{cases} \textbf{DFT-dilaton:} & d \\ \textbf{Double-vielbeins:} & V_{Ap}\,, \quad \bar{V}_{A\bar{p}} \end{cases}$
  - **R-R potential:** $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$
- **Fermions (NS-R, R-NS)**
  - **DFT-dilatinos:** $\rho^{\alpha}\,, \quad \rho'^{\bar{\alpha}}$
  - **Gravitinos:** $\psi_{\bar{p}}^{\alpha}\,, \quad \psi_{p}'^{\bar{\alpha}}$

$cf.$ **Relation to the fields in the ordinary supergravity**

$$\rho \sim \lambda - \gamma^{a}\psi_{a}$$

$$d = \phi - \tfrac{1}{2}\ln\sqrt{-g}$$

$cf.$ **Hassan 99'**

Field contents of $D = 10$ Maximal SDFT

- **Bosons**
  - **NS-NS sector** $\left\{\begin{array}{ll}\textbf{DFT-dilaton:} & d \\ \textbf{Double-vielbeins:} & V_{Ap}\,, \quad \bar{V}_{A\bar{p}}\end{array}\right.$
  - **R-R potential:** $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$
- **Fermions (NS-R, R-NS)**
  - **DFT-dilatinos:** $\rho^{\alpha}\,, \quad \rho'^{\bar{\alpha}}$
  - **Gravitinos:** $\psi^{\alpha}_{\bar{p}}\,, \quad \psi'^{\bar{\alpha}}_{p}$

***A priori*, $\mathbf{O}(D, D)$ rotates only the $\mathbf{O}(D, D)$ vector indices (capital Roman), and the R-R sector and all the fermions are $\mathbf{O}(D, D)$ T-duality singlet.**

The usual IIA $\Leftrightarrow$ IIB exchange will be realized only after fixing a gauge.

Field contents of $D = 10$ Maximal SDFT

- **Bosons**
  - **NS-NS sector** $\begin{cases} \textbf{DFT-dilaton:} & d \\ \textbf{DFT-vielbeins:} & V_{Ap}\,, \quad \bar{V}_{A\bar{p}} \end{cases}$
  - **R-R potential:** $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$

- **Fermions (NS-R, R-NS)**
  - **DFT-dilatinos:** $\rho^{\alpha}\,, \quad \rho'^{\bar{\alpha}}$
  - **Gravitinos:** $\psi^{\alpha}_{\bar{p}}\,, \quad \psi'^{\bar{\alpha}}_{p}$

- **Set the chiralities**

$$\gamma^{(D+1)}\mathcal{C}\bar{\gamma}^{(D+1)} = cc'\mathcal{C}\,. \qquad \begin{array}{l} \gamma^{(D+1)}\psi_{\bar{p}} = c\psi_{\bar{p}}\,, \\ \bar{\gamma}^{(D+1)}\psi'_{p} = c'\psi'_{p}\,, \end{array} \qquad \begin{array}{l} \gamma^{(D+1)}\rho = -c\rho\,, \\ \bar{\gamma}^{(D+1)}\rho' = -c'\rho'\,. \end{array}$$

c and c' are sign factors, and equivalent up to a Pin$(1,9) \times$ Pin$(9,1)$.
So we may fix $c = c' = +1$ without loss of generality.
However, the theory contains two 'types' of solutions, i.e. IIA and IIB.

Double-vielbein 1105.6294, 1109.2035

- **Double-vielbein** simultaneously diagonalizes $\mathcal{J}_{AB}$ and $\mathcal{H}_{AB}$,

$$\mathcal{J}=\begin{pmatrix} V & \bar{V} \end{pmatrix}\begin{pmatrix} \eta^{-1} & 0 \\ 0 & \bar{\eta} \end{pmatrix}\begin{pmatrix} V & \bar{V} \end{pmatrix}^{T}, \quad \mathcal{H}=\begin{pmatrix} V & \bar{V} \end{pmatrix}\begin{pmatrix} \eta^{-1} & 0 \\ 0 & -\bar{\eta} \end{pmatrix}\begin{pmatrix} V & \bar{V} \end{pmatrix}^{T}.$$

- It follows the defining properties

$$V_{Ap}V^{A}{}_{q}=\eta_{pq}\,, \qquad V_{Ap}\bar{V}^{A}{}_{\bar{p}}=0 \qquad V_{A}{}^{p}V_{Bp}=P_{AB}\,,$$

$$\bar{V}_{A\bar{p}}\bar{V}^{A}{}_{\bar{q}}=\bar{\eta}_{\bar{p}\bar{q}}\,, \qquad \bar{V}_{A}{}^{\bar{p}}\bar{V}_{B\bar{p}}=\bar{P}_{AB}\,,$$

$P_{AB}, \bar{P}_{AB}$ are projection matrices('left and right'),

$$P_{A}{}^{B}P_{B}{}^{C}=P_{A}{}^{C}\,, \qquad \bar{P}_{A}{}^{B}\bar{P}_{B}{}^{C}=\bar{P}_{A}{}^{C}\,, \qquad P_{A}{}^{B}\bar{P}_{B}{}^{C}=0$$

which are related to $\mathcal{H}$ and $\mathcal{J}$,

$$P_{AB}+\bar{P}_{AB}=\mathcal{J}_{AB}\,, \qquad P_{AB}-\bar{P}_{AB}=\mathcal{H}_{AB}$$

- The basic geometric objects, which should be treated equally, are

$$(d\,,V_{Ap}\,,\bar{V}_{A\bar{p}})\,, \qquad \text{or} \qquad (d\,,P_{AB}\,,\bar{P}_{AB})\,.$$

Semi-covariant derivatives

- We introduce **master 'semi-covariant' derivative**

$$\mathcal{D}_A = \partial_A + \Gamma_A + \Phi_A + \bar{\Phi}_A \,.$$

- It is also useful to set

$$\nabla_A = \partial_A + \Gamma_A \,,$$

- The 'semi-covariant' derivative for the DFT-diffeomorphism is

$$\nabla_C T_{\omega A_1 A_2 \cdots A_n} := \partial_C T_{\omega A_1 A_2 \cdots A_n} - \omega \Gamma^B{}_{BC} T_{\omega A_1 A_2 \cdots A_n} + \sum_{i=1}^{n} \Gamma_{CA_i}{}^B T_{\omega A_1 \cdots A_{i-1} B A_{i+1} \cdots A_n} \,.$$

Semi-covariant derivatives

- We introduce **master 'semi-covariant' derivative**

$$\mathcal{D}_A = \partial_A + \Gamma_A + \Phi_A + \bar{\Phi}_A \,.$$

- It is also useful to set

$$\nabla_A = \partial_A + \Gamma_A \,,$$

- The 'semi-covariant' derivative for the DFT-diffeomorphism is

$$\nabla_C T_{\omega A_1 A_2 \cdots A_n} := \partial_C T_{\omega A_1 A_2 \cdots A_n} - \omega \Gamma^B{}_{BC} T_{\omega A_1 A_2 \cdots A_n} + \sum_{i=1}^{n} \Gamma_{CA_i}{}^B T_{\omega A_1 \cdots A_{i-1} B A_{i+1} \cdots A_n} \,.$$

- **compatibility for the whole NS-NS sector**

$$\mathcal{D}_A d = 0\,, \qquad \mathcal{D}_A V_{Bp} = 0\,, \qquad \mathcal{D}_A \bar{V}_{B\bar{p}} = 0\,. \qquad (cf.\ \mathcal{D}_\mu e_\nu{}^a = 0)$$

together with

$$\mathcal{D}_A \eta_{pq} = \mathcal{D}_A \bar{\eta}_{\bar{p}\bar{q}} = \mathcal{D}_A (\gamma^p)^\alpha{}_\beta = \mathcal{D}_A (\bar{\gamma}^{\bar{p}})^{\bar{\alpha}}{}_{\bar{\beta}} = \mathcal{D}_A C_{+\alpha\beta} = \mathcal{D}_A \bar{C}_{+\bar{\alpha}\bar{\beta}} = 0\,.$$

It follows that

$$\nabla_A d = 0\,, \qquad \nabla_A P_{BC} = 0\,, \qquad \nabla_A \bar{P}_{BC} = 0\,, \qquad (cf. \nabla_\mu g_{\nu\lambda} = 0)$$

- Spin connections

$$\Phi_{Apq} = V^B{}_p \nabla_A V_{Bq}\,, \qquad \bar{\Phi}_{A\bar{p}\bar{q}} = \bar{V}^B{}_{\bar{p}} \nabla_A \bar{V}_{B\bar{q}}\,,$$

- Torsion free conection is uniquely determined in terms of basic geometrical variables, [IJ, Lee, Park '11]

$$\Gamma^0_{CAB} = 2\left(P\partial_C P\bar{P}\right)_{[AB]} + 2\left(\bar{P}_{[A}{}^D\bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E\right)\partial_D P_{EC}$$

$$-\tfrac{4}{D-1}\left(\bar{P}_{C[A}\bar{P}_{B]}{}^D + P_{C[A}P_{B]}{}^D\right)\left(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\right),$$

satisfying the torsion free condition,

$$\Gamma^0_{[ABC]} = 0\,, \qquad (\Leftrightarrow \quad \hat{\mathcal{L}}^{\partial}_X = \hat{\mathcal{L}}^{\nabla}_X)$$

and further satisfying

$$\mathcal{P}_{CAB}{}^{DEF}\Gamma^0_{DEF} = 0\,, \qquad \bar{\mathcal{P}}_{CAB}{}^{DEF}\Gamma^0_{DEF} = 0\,.$$

Covariant derivatives

"semi-covariant derivative" :

combined with the projections , we can get various covariant quantities:

Examples:

- **For $\mathbf{O}(D,D)$ tensors:**

$$P_C{}^D \bar{P}_A{}^B \nabla_D T_B\,, \qquad \bar{P}_C{}^D P_A{}^B \nabla_D T_B\,,$$

$$P^{AB}\nabla_A T_B\,, \qquad \bar{P}^{AB}\nabla_A T_B\,, \qquad \text{Divergences}\,,$$

$$P^{AB}\bar{P}_C{}^D \nabla_A \nabla_B T_D\,, \qquad \bar{P}^{AB} P_C{}^D \nabla_A \nabla_B T_D\,. \qquad \text{Laplacians}$$

- Rule: need opposite chirality or contraction

Covariant derivatives

- **For Spin$(1, D-1)_L \times$ Spin$(D-1, 1)_R$ tensors:**

$$\mathcal{D}_p T_{\bar{q}} \,, \qquad \mathcal{D}_{\bar{p}} T_q \,,$$

$$\mathcal{D}^p T_p \,, \qquad \mathcal{D}^{\bar{p}} T_{\bar{p}} \,,$$

$$\mathcal{D}_p \mathcal{D}^p T_{\bar{q}} \,, \qquad \mathcal{D}_{\bar{p}} \mathcal{D}^{\bar{p}} T_q \,,$$

where we set

$$\mathcal{D}_p := V^A{}_p \mathcal{D}_A \,, \qquad \mathcal{D}_{\bar{p}} := \bar{V}^A{}_{\bar{p}} \mathcal{D}_A \,.$$

These are the pull-back of the previous results using the double-vielbeins.

Covariant derivatives

- **Dirac operators for fermions,** ρ^{α}, $\psi_{\bar{p}}^{\alpha}$, $\rho'^{\bar{\alpha}}$, $\psi_{p}'^{\bar{\alpha}}$ **:** [IJ, Lee, Park '11]

$$\gamma^{p}\mathcal{D}_{p}\rho=\gamma^{A}\mathcal{D}_{A}\rho\,,\qquad\qquad \gamma^{p}\mathcal{D}_{p}\psi_{\bar{p}}=\gamma^{A}\mathcal{D}_{A}\psi_{\bar{p}}\,,$$

$$\mathcal{D}_{\bar{p}}\rho\,,\qquad\qquad \mathcal{D}_{\bar{p}}\psi^{\bar{p}}=\mathcal{D}_{A}\psi^{A}\,,$$

$$\bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\rho'=\bar{\gamma}^{A}\mathcal{D}_{A}\rho'\,,\qquad\qquad \bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\psi'_{p}=\bar{\gamma}^{A}\mathcal{D}_{A}\psi'_{p}\,,$$

$$\mathcal{D}_{p}\rho'\,,\qquad\qquad \mathcal{D}_{p}\psi'^{p}=\mathcal{D}_{A}\psi'^{A}\,,$$

Covariant derivatives

- **For Spin$(1, D-1)_L \times$ Spin$(D-1, 1)_R$ bi-fundamental spinors, $\mathcal{C}^{\alpha}{}_{\bar{\beta}}$:**
  [IJ, Lee, Park '12]

$$\gamma^A \mathcal{D}_A \mathcal{C} \,, \qquad\qquad \mathcal{D}_A \mathcal{C} \bar{\gamma}^A \,.$$

- Further define

$$\mathcal{D}_+ \mathcal{C} := \gamma^A \mathcal{D}_A \mathcal{C} + \gamma^{(D+1)} \mathcal{D}_A \mathcal{C} \bar{\gamma}^A \,,$$

$$\mathcal{D}_- \mathcal{C} := \gamma^A \mathcal{D}_A \mathcal{C} - \gamma^{(D+1)} \mathcal{D}_A \mathcal{C} \bar{\gamma}^A \,.$$

- Especially for the torsionless case, the corresponding operators are **nilpotent** up to the section condition

$$(\mathcal{D}^0_+)^2 \mathcal{C} \sim 0 \,, \qquad\qquad (\mathcal{D}^0_-)^2 \mathcal{C} \sim 0 \,,$$

- **The field strength of the R-R potential, $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$, is then defined by**

$$\mathcal{F} := \mathcal{D}^0_+ \mathcal{C} \,.$$

Covariant derivatives

- **For Spin$(1, D-1)_L \times$ Spin$(D-1, 1)_R$ bi-fundamental spinors, $\mathcal{C}^{\alpha}{}_{\bar{\beta}}$:**
  [IJ, Lee, Park '12]

$$\gamma^A \mathcal{D}_A \mathcal{C}\,, \qquad\qquad \mathcal{D}_A \mathcal{C} \bar{\gamma}^A\,.$$

- Further define

$$\mathcal{D}_+ \mathcal{C} := \gamma^A \mathcal{D}_A \mathcal{C} + \gamma^{(D+1)} \mathcal{D}_A \mathcal{C} \bar{\gamma}^A\,,$$

$$\mathcal{D}_- \mathcal{C} := \gamma^A \mathcal{D}_A \mathcal{C} - \gamma^{(D+1)} \mathcal{D}_A \mathcal{C} \bar{\gamma}^A\,.$$

- Especially for the torsionless case, the corresponding operators are **nilpotent** up to the section condition

$$(\mathcal{D}^0_+)^2 \mathcal{C} \sim 0\,, \qquad\qquad (\mathcal{D}^0_-)^2 \mathcal{C} \sim 0\,,$$

- **The field strength of the R-R potential, $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$, is then defined by**

$$\mathcal{F} := \mathcal{D}^0_+ \mathcal{C}\,.$$

$D = 10$ Maximal SDFT

- **Lagrangian** (full order of fermions):

$$\mathcal{L}_{\text{Type II}} = e^{-2d}\Big[\tfrac{1}{8}(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} + \tfrac{1}{2}\mathrm{Tr}(\mathcal{F}\bar{\mathcal{F}}) - i\bar{\rho}\mathcal{F}\rho' + i\bar{\psi}_{\bar{p}}\gamma_q\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^q$$
$$+ i\tfrac{1}{2}\bar{\rho}\gamma^p\mathcal{D}^\star_p\rho - i\bar{\psi}^{\bar{p}}\mathcal{D}^\star_{\bar{p}}\rho - i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^q\mathcal{D}^\star_q\psi_{\bar{p}} - i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}'^\star_{\bar{p}}\rho' + i\bar{\psi}'^p\mathcal{D}'^\star_p\rho' + i\tfrac{1}{2}\bar{\psi}'^p\bar{\gamma}^{\bar{q}}\mathcal{D}'^\star_{\bar{q}}\psi'_p\Big]$$

where $\bar{\mathcal{F}}^{\bar{\alpha}}{}_{\alpha}$ denotes the charge conjugation, $\bar{\mathcal{F}} := \bar{C}_+^{-1}\mathcal{F}^T C_+$.

- $\mathcal{D}_A$ in S_{ACBD}, $\mathcal{D}^\star_A$ and $\mathcal{D}'^\star_A$ are defined by their own torsionful connection ,
- The torsions are determined to satisfy usual 1.5 formalism ,

$$\delta\mathcal{L}_{\text{SDFT}} = \delta\Gamma_{ABC} \times 0\,.$$

- The Lagrangian is **pseudo** **:** **self-duality** of the R-R field strength needs to be imposed by hand, just like the 'democratic' type II SUGRA Bergshoeff, *et al.*

$$\left(1 - \gamma^{(D+1)}\right)\left(\mathcal{F} - i\tfrac{1}{2}\rho\bar{\rho}' + i\tfrac{1}{2}\gamma^p\psi_{\bar{q}}\bar{\psi}'_p\bar{\gamma}^{\bar{q}}\right) \sim 0\,.$$

$D = 10$ Maximal SDFT

- **Local SUSY** (full order of fermions):

$$\delta_\varepsilon d = -i\tfrac{1}{2}(\bar{\varepsilon}\rho + \bar{\varepsilon}'\rho')\,,$$

$$\delta_\varepsilon V_{Ap} = i\bar{V}_A{}^{\bar{q}}(\bar{\varepsilon}'\bar{\gamma}_{\bar{q}}\psi'_p - \bar{\varepsilon}\gamma_p\psi_{\bar{q}})\,,$$

$$\delta_\varepsilon \bar{V}_{A\bar{p}} = iV_A{}^q(\bar{\varepsilon}\gamma_q\psi_{\bar{p}} - \bar{\varepsilon}'\bar{\gamma}_{\bar{p}}\psi'_q)\,,$$

$$\delta_\varepsilon \mathcal{C} = i\tfrac{1}{2}(\gamma^p\varepsilon\bar{\psi}'_p - \varepsilon\bar{\rho}' - \psi_{\bar{p}}\bar{\varepsilon}'\bar{\gamma}^{\bar{p}} + \rho\bar{\varepsilon}') + \mathcal{C}\delta_\varepsilon d - \tfrac{1}{2}(\bar{V}^A{}_{\bar{q}}\,\delta_\varepsilon V_{Ap})\gamma^{(d+1)}\gamma^p\mathcal{C}\bar{\gamma}^{\bar{q}}\,,$$

$$\delta_\varepsilon \rho = -\gamma^p\hat{\mathcal{D}}_p\varepsilon + i\tfrac{1}{2}\gamma^p\varepsilon\,\bar{\psi}'_p\rho' - i\gamma^p\psi^{\bar{q}}\bar{\varepsilon}'\bar{\gamma}_{\bar{q}}\psi'_p\,,$$

$$\delta_\varepsilon \rho' = -\bar{\gamma}^{\bar{p}}\hat{\mathcal{D}}'_{\bar{p}}\varepsilon' + i\tfrac{1}{2}\bar{\gamma}^{\bar{p}}\varepsilon'\,\bar{\psi}_{\bar{p}}\rho - i\bar{\gamma}^{\bar{q}}\psi'_p\bar{\varepsilon}\gamma^p\psi_{\bar{q}}\,,$$

$$\delta_\varepsilon \psi_{\bar{p}} = \hat{\mathcal{D}}_{\bar{p}}\varepsilon + (\mathcal{F} - i\tfrac{1}{2}\gamma^q\rho\,\bar{\psi}'_q + i\tfrac{1}{2}\psi^{\bar{q}}\,\bar{\rho}'\bar{\gamma}_{\bar{q}})\bar{\gamma}_{\bar{p}}\varepsilon' + i\tfrac{1}{4}\varepsilon\bar{\psi}_{\bar{p}}\rho + i\tfrac{1}{2}\psi_{\bar{p}}\bar{\varepsilon}\rho\,,$$

$$\delta_\varepsilon \psi'_p = \hat{\mathcal{D}}'_p\varepsilon' + (\bar{\mathcal{F}} - i\tfrac{1}{2}\bar{\gamma}^{\bar{q}}\rho'\bar{\psi}_{\bar{q}} + i\tfrac{1}{2}\psi'^q\bar{\rho}\gamma_q)\gamma_p\varepsilon + i\tfrac{1}{4}\varepsilon'\bar{\psi}'_p\rho' + i\tfrac{1}{2}\psi'_p\bar{\varepsilon}'\rho'\,.$$

$\hat{\mathcal{D}}$ is also defined by its own torsionful connection.

- **The action is invariant up to the self-duality.**

Relation to the ordinary supergravity,

how SDFT unifies the IIA and IIB,

mechanism to exchange IIA and IIB by $\mathbf{O}(D,D)$.

Parametrization: Reduction to Generalized Geometry

- We have used the DFT-variables. We may parametrize them in terms of Riemannian variables.
- Assuming that the upper half blocks are non-degenerate, the double-vielbein takes the most general form,

$$V_{Ap} = \tfrac{1}{\sqrt{2}}\begin{pmatrix} (e^{-1})_{p}{}^{\mu} \\ (B+e)_{\nu p} \end{pmatrix}, \qquad \bar{V}_{A\bar{p}} = \tfrac{1}{\sqrt{2}}\begin{pmatrix} (\bar{e}^{-1})_{\bar{p}}{}^{\mu} \\ (B+\bar{e})_{\nu\bar{p}} \end{pmatrix}.$$

Here $e_{\mu}{}^{p}$ and $\bar{e}_{\nu}{}^{\bar{p}}$ are two copies of the D-dimensional vielbein corresponding to the same spacetime metric,

$$e_{\mu}{}^{p}e_{\nu}{}^{q}\eta_{pq} = -\bar{e}_{\mu}{}^{\bar{p}}\bar{e}_{\nu}{}^{\bar{q}}\bar{\eta}_{\bar{p}\bar{q}} = g_{\mu\nu}\,.$$

and $B_{\mu\nu}$ corresponds to the Kalb-Ramond two-form gauge field, with $B_{\mu p} = B_{\mu\nu}(e^{-1})_{p}{}^{\nu}$, $B_{\mu\bar{p}} = B_{\mu\nu}(\bar{e}^{-1})_{\bar{p}}{}^{\nu}$.

Parametrization: Reduction to Generalized Geometry

- Take this parametrization and impose $\frac{\partial}{\partial \tilde{x}_\mu} \sim 0$.

- This reduces (S)DFT to generalized geometry

Hitchin; Grana, Minasian, Petrini, Waldram

- For example, the $\mathbf{O}(D,D)$ covariant Dirac operators become

$$\sqrt{2}\gamma^A \mathcal{D}_A \rho \sim \gamma^m \left(\partial_m \rho + \tfrac{1}{4}\omega_{mnp}\gamma^{np}\rho + \tfrac{1}{24}H_{mnp}\gamma^{np}\rho - \partial_m\phi\rho \right),$$

$$\sqrt{2}\gamma^A \mathcal{D}_A \psi_{\bar{p}} \sim \gamma^m \left(\partial_m \psi_{\bar{p}} + \tfrac{1}{4}\omega_{mnp}\gamma^{np}\psi_{\bar{p}} + \bar{\omega}_{m\bar{p}\bar{q}}\psi^{\bar{q}} + \tfrac{1}{24}H_{mnp}\gamma^{np}\psi_{\bar{p}} + \tfrac{1}{2}H_{m\bar{p}\bar{q}}\psi^{\bar{q}} - \partial_m\phi\psi_{\bar{p}} \right)$$

$$\sqrt{2}\bar{V}^A{}_{\bar{p}}\mathcal{D}_A \rho \sim \partial_{\bar{p}}\rho + \tfrac{1}{4}\omega_{\bar{p}qr}\gamma^{qr}\rho + \tfrac{1}{8}H_{\bar{p}qr}\gamma^{qr}\rho\,,$$

$$\sqrt{2}\mathcal{D}_A\psi^A \sim \partial^{\bar{p}}\psi_{\bar{p}} + \tfrac{1}{4}\omega_{\bar{p}qr}\gamma^{qr}\psi^{\bar{p}} + \bar{\omega}^{\bar{p}}{}_{\bar{p}\bar{q}}\psi^{\bar{q}} + \tfrac{1}{8}H_{\bar{p}qr}\gamma^{qr}\psi^{\bar{p}} - 2\partial_{\bar{p}}\phi\psi^{\bar{p}}\,.$$

$\omega_\mu \pm \frac{1}{2}H_\mu$ and $\omega_\mu \pm \frac{1}{6}H_\mu$ naturally appear as spin connections. Liu, Minasian

Unification of type IIA and IIB SUGRAs

- In general, two zehnbeins $e_\mu{}^p$ and $\bar{e}_\mu{}^{\bar{p}}$ are different, so there can be different Riemaniann solution for each zehnbeins.
- To relate with the supergravity solution, we need to relate two zehnbeins equal to each other

$$e_\mu{}^p \equiv \bar{e}_\mu{}^{\bar{p}}$$

by a Lorentz rotation,

$$(e^{-1}\bar{e})_p{}^{\bar{p}}(e^{-1}\bar{e})_q{}^{\bar{q}}\bar{\eta}_{\bar{p}\bar{q}} = -\eta_{pq}\,.$$

- This rotation also rotates the RR field, and depending on the signature of $\det(e^{-1}\bar{e})$ the chirality may or may not flipped.
- Depending on the resulting chirality of the RR filed, the solution is of IIA and IIB.
- In this way, a single chiral theory can contains two types of solution IIA and IIB , i.e. the maximal SDFT unifies the IIA and IIB supereravities .

Unification of type IIA and IIB SUGRAs

- In general, two zehnbeins $e_\mu{}^p$ and $\bar{e}_\mu{}^{\bar{p}}$ are different, so there can be different Riemaniann solution for each zehnbeins.
- To relate with the supergravity solution, we need to relate two zehnbeins equal to each other

$$e_\mu{}^p \equiv \bar{e}_\mu{}^{\bar{p}}$$

by a Lorentz rotation,

$$(e^{-1}\bar{e})_p{}^{\bar{p}}(e^{-1}\bar{e})_q{}^{\bar{q}}\bar{\eta}_{\bar{p}\bar{q}} = -\eta_{pq} \,.$$

- This rotation also rotates the RR field, and depending on the signature of $\det(e^{-1}\bar{e})$ the chirality may or may not flipped.
- Depending on the resulting chirality of the RR filed, the solution is of IIA and IIB.
- In this way, a single chiral theory can contains two types of solution IIA and IIB , i.e. the maximal SDFT unifies the IIA and IIB supereravities .

Unification of type IIA and IIB SUGRAs

- In general, two zehnbeins ${e_\mu}^p$ and ${\bar{e}_\mu}^{\bar{p}}$ are different, so there can be different Riemaniann solution for each zehnbeins.
- To relate with the supergravity solution, we need to relate two zehnbeins equal to each other

$$ {e_\mu}^p \equiv {\bar{e}_\mu}^{\bar{p}} $$

by a Lorentz rotation,

$$ (e^{-1}\bar{e})_p{}^{\bar{p}}(e^{-1}\bar{e})_q{}^{\bar{q}}\bar{\eta}_{\bar{p}\bar{q}} = -\eta_{pq} \,. $$

- This rotation also rotates the RR field, and depending on the signature of $\det(e^{-1}\bar{e})$ the chirality may or may not flipped.
- Depending on the resulting chirality of the RR filed, the solution is of IIA and IIB.
- In this way, a single chiral theory can contains two types of solution IIA and IIB , i.e. the maximal SDFT unifies the IIA and IIB supereravities .

Diagonal gauge fixing and Reduction to SUGRA

- Once Identifying two zhenbeins

$$e_{\mu}{}^{p} \equiv \bar{e}_{\mu}{}^{\bar{p}}$$

the local Lorentz symmetries are broken to the diagonal gauge symmetry

$$\mathbf{Spin}(1,D-1)_L \times \mathbf{Spin}(D-1,1)_R \quad \Longrightarrow \quad \mathbf{Spin}(1,D-1)_D\,.$$

- ordinary SUGRA $\equiv$ diagonal gauge-fixed SDFT,

Diagonal gauge fixing and Reduction to SUGRA

- After the diagonal gauge fixing, we may parameterize the R-R potential as

$$\mathcal{C} \equiv \left(\tfrac{1}{2}\right)^{\frac{D+2}{4}} \sum\nolimits_p' \tfrac{1}{p!}\, \mathcal{C}_{a_1a_2\cdots a_p}\gamma^{a_1a_2\cdots a_p}$$

and obtain the field strength,

$$\mathcal{F} := \mathcal{D}^0_+\mathcal{C} \equiv \left(\tfrac{1}{2}\right)^{\frac{D}{4}} \sum\nolimits_p' \tfrac{1}{(p+1)!}\, \mathcal{F}_{a_1a_2\cdots a_{p+1}}\gamma^{a_1a_2\cdots a_{p+1}}$$

where $\sum_p'$ denotes the odd p sum for Type IIA and even p sum for Type IIB, and

$$\mathcal{F}_{a_1a_2\cdots a_p} = p\left(D_{[a_1}\mathcal{C}_{a_2\cdots a_p]} - \partial_{[a_1}\phi\, \mathcal{C}_{a_2\cdots a_p]}\right) + \tfrac{p!}{3!(p-3)!}\, H_{[a_1a_2a_3}\mathcal{C}_{a_4\cdots a_p]}$$

- The pair of nilpotent differential operators, $\mathcal{D}^0_+$ and $\mathcal{D}^0_-$, reduce to an exterior derivative and its dual,

$$\begin{array}{ccc} \mathcal{D}^0_+ & \Longrightarrow & \mathrm{d} + (H - \mathrm{d}\phi)\wedge \\ \mathcal{D}^0_- & \Longrightarrow & *\left[\,\mathrm{d} + (H - \mathrm{d}\phi)\wedge\,\right]* \end{array}$$

Diagonal gauge fixing and Reduction to SUGRA

- After the diagonal gauge fixing, we may parameterize the R-R potential as

$$\mathcal{C} \equiv \left(\tfrac{1}{2}\right)^{\frac{D+2}{4}} \textstyle\sum_p^{\prime} \frac{1}{p!}\, \mathcal{C}_{a_1 a_2 \cdots a_p} \gamma^{a_1 a_2 \cdots a_p}$$

  and obtain the field strength,

$$\mathcal{F} := \mathcal{D}^0_+ \mathcal{C} \equiv \left(\tfrac{1}{2}\right)^{\frac{D}{4}} \textstyle\sum_p^{\prime} \frac{1}{(p+1)!}\, \mathcal{F}_{a_1 a_2 \cdots a_{p+1}} \gamma^{a_1 a_2 \cdots a_{p+1}}$$

  where $\sum_p^{\prime}$ denotes the odd p sum for Type IIA and even p sum for Type IIB, and

$$\mathcal{F}_{a_1 a_2 \cdots a_p} = p\left(D_{[a_1} \mathcal{C}_{a_2 \cdots a_p]} - \partial_{[a_1} \phi\, \mathcal{C}_{a_2 \cdots a_p]}\right) + \tfrac{p!}{3!(p-3)!}\, H_{[a_1 a_2 a_3} \mathcal{C}_{a_4 \cdots a_p]}$$

- The pair of nilpotent differential operators, $\mathcal{D}^0_+$ and $\mathcal{D}^0_-$, reduce to an exterior derivative and its dual,

$$\begin{array}{ccc} \mathcal{D}^0_+ & \Longrightarrow & \mathrm{d} + (H - \mathrm{d}\phi)\wedge \\ \mathcal{D}^0_- & \Longrightarrow & *\left[\,\mathrm{d} + (H - \mathrm{d}\phi)\wedge\,\right]* \end{array}$$

Modified $\mathbf{O}(D,D)$ IIA $\leftrightarrow$ IIB

- In order to preserve the diagonal gauge, $e_\mu{}^p \equiv \bar{e}_\mu{}^{\bar{p}}$, the $\mathbf{O}(D,D)$ transformation rule is modified.
- A compensating local Lorentz transformation, $\bar{L}_{\bar{q}}{}^{\bar{p}}\,, S_{\bar{L}}{}^{\bar{\alpha}}{}_{\bar{\beta}} \in \mathbf{Pin}(D{-}1,1)_R$, must be accompanied:

$$\bar{V}_A{}^{\bar{p}} \quad\longrightarrow\quad M_A{}^B \bar{V}_B{}^{\bar{q}} \bar{L}_{\bar{q}}{}^{\bar{p}}\,, \qquad\qquad \bar{\gamma}^{\bar{q}} \bar{L}_{\bar{q}}{}^{\bar{p}} = S_{\bar{L}}^{-1} \bar{\gamma}^{\bar{p}} S_{\bar{L}}\,,$$

where

$$\bar{L} = \bar{e}^{-1}\left[\mathbf{a}^t - (g+B)\mathbf{b}^t\right]\left[\mathbf{a}^t + (g-B)\mathbf{b}^t\right]^{-1}\bar{e}\,,$$

in the parametrization of the generic $\mathbf{O}(D,D)$ group element,

$$M_A{}^B = \begin{pmatrix} \mathbf{a}^\mu{}_\nu & \mathbf{b}^{\mu\sigma} \\ \mathbf{c}_{\rho\nu} & \mathbf{d}_\rho{}^\sigma \end{pmatrix}.$$

Modified $\mathbf{O}(D,D)$ Transformation Rule After The Diagonal Gauge Fixing

d	$\longrightarrow$	d
$V_A{}^p$	$\longrightarrow$	$M_A{}^B\,V_B{}^p$
$\bar{V}_A{}^{\bar{p}}$	$\longrightarrow$	$M_A{}^B\,\bar{V}_B{}^{\bar{q}}\,\bar{L}_{\bar{q}}{}^{\bar{p}}$
$\mathcal{C}^{\alpha}{}_{\bar{\alpha}}\,,\quad \mathcal{F}^{\alpha}{}_{\bar{\alpha}}$	$\longrightarrow$	$\mathcal{C}^{\bar{\alpha}}{}_{\bar{\beta}}(S_{\bar{L}}^{-1})^{\bar{\beta}}{}_{\bar{\alpha}}\,,\quad \mathcal{F}^{\bar{\alpha}}{}_{\bar{\beta}}(S_{\bar{L}}^{-1})^{\bar{\beta}}{}_{\bar{\alpha}}$
ρ^{α}	$\longrightarrow$	ρ^{α}
$\rho'^{\bar{\alpha}}$	$\longrightarrow$	$(S_{\bar{L}})^{\bar{\alpha}}{}_{\bar{\beta}}\rho'^{\bar{\beta}}$
$\psi^{\alpha}_{\bar{p}}$	$\longrightarrow$	$(\bar{L}^{-1})_{\bar{p}}{}^{\bar{q}}\,\psi^{\alpha}_{\bar{q}}$
$\psi'^{\bar{\alpha}}_{p}$	$\longrightarrow$	$(S_{\bar{L}})^{\bar{\alpha}}{}_{\bar{\beta}}\psi'^{\bar{\beta}}_{p}$

- **All the barred indices are now to be rotated.**

Consistent with Hassan

Modified $\mathbf{O}(D,D)$: IIA $\Leftrightarrow$ IIB

- **If and only if** $\det(\bar{L})=-1$**, the modified** $\mathbf{O}(D,D)$ **rotation flips the chirality of the theory, since**

$$\bar{\gamma}^{(D+1)}S_{\bar{L}}=\det(\bar{L})\,S_{\bar{L}}\bar{\gamma}^{(D+1)}\,.$$

- This is the mechanism of exchanging of type IIA and IIB supergravities under $\mathbf{O}(D,D)$ T-duality.

Summary

- Having the DFT extension of the Christoffel connection,

  $$\Gamma_{CAB} = 2(P\partial_C P\bar{P})_{[AB]} + 2\left(\bar{P}_{[A}{}^{D}\bar{P}_{B]}{}^{E} - P_{[A}{}^{D}P_{B]}{}^{E}\right)\partial_D P_{EC} - \tfrac{4}{D-1}\left(\bar{P}_{C[A}\bar{P}_{B]}{}^{D} + P_{C[A}P_{B]}{}^{D}\right)\left(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\right),$$

  the semi-covariant formalism provides the geometrical description for Double Field Theory
- It manifests all the bosonic symmetries and succesfully provides the supersymmetric extention of DFT in full order of fermions
- It is the unifyng description of the type IIA and IIB: a single theory contains two types of solutions.
- Parametrization and diagoal gauge fixed SDFT is ordinary supergravities.
- After diagonal gauge IIA and IIB exchange is realized.

Conclusion

Thank you.

Semi-covariant formulation of Double Field Theory

Remark: Failure of the Equivalence Principle

Unlike the Christoffel symbol, the DFT-diffeomorphisms cannot transform our connection to vanish point-wise:

$$\begin{aligned}\Gamma_{CAB} &= 2\left(P\partial_C P\bar{P}\right)_{[AB]} + 2\left(\bar{P}_{[A}{}^{D}\bar{P}_{B]}{}^{E} - P_{[A}{}^{D}P_{B]}{}^{E}\right)\partial_D P_{EC} \\ &\quad -\tfrac{4}{D-1}\left(\bar{P}_{C[A}\bar{P}_{B]}{}^{D} + P_{C[A}P_{B]}{}^{D}\right)\left(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\right) \\ &\neq 0\,.\end{aligned}$$

That is to say, there is no normal coordinate in DFT. This can be viewed as the failure of the equivalence principle applied to an extended object, *i.e.* string.