Finite Transformations in Doubled and Exceptional Space

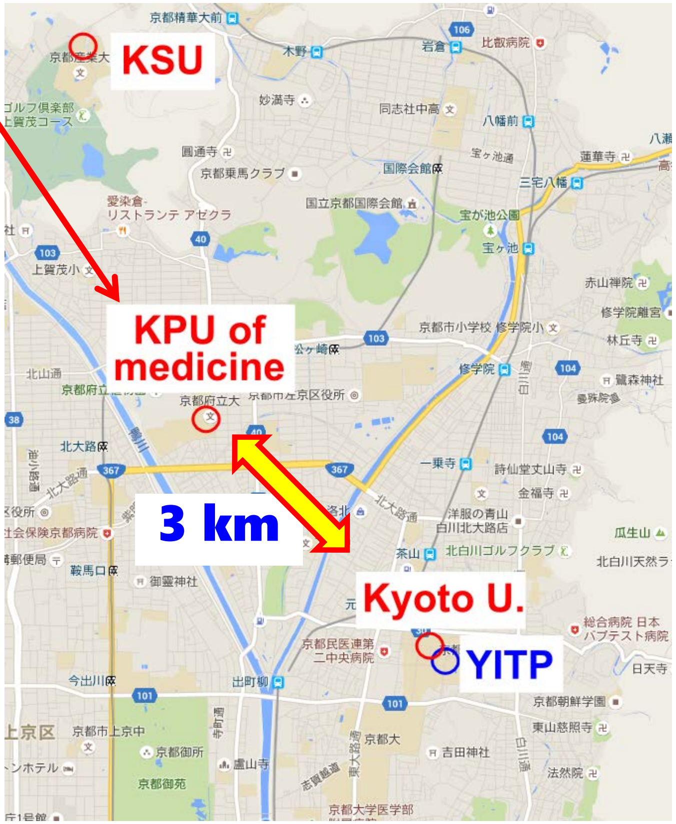
Yuho Sakatani

Seoul National University (SNU),
Institute for Basic Sciences (IBS)

"Duality and Novel Geometry in M-theory"
4 Feb. 2016.

Based on arXiv:1510.06735v2,
in collaboration with Soo-Jong Rey

From the coming **April**, I will transfer to **Kyoto Prefectural University of Medicine**



Kyoto city

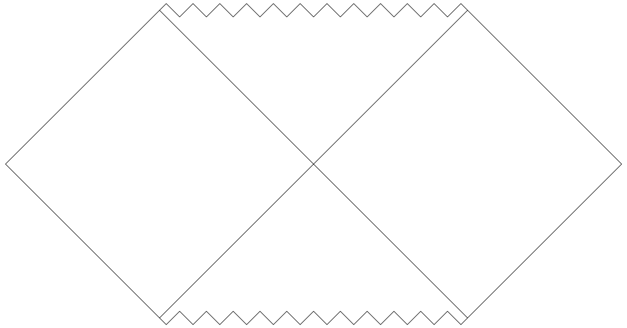
Finite coordinate transformation

In General Relativity

Under a finite coordinate transformation, $x^m \to x'^m(x)$ **,**

a vector field $w^m(x)$ **transforms as** $\boxed{w'^m(x') = \frac{\partial x'^m}{\partial x^n} w^n(x) \,.}$

Finite transformation law is important,
in order to investigate properties of spacetimes :



Schwarzschild coords. g_{mn}

Kruskal coords. $g'_{mn} = \frac{\partial x^k}{\partial x'^m} \frac{\partial x^l}{\partial x'^n} g_{kl}$

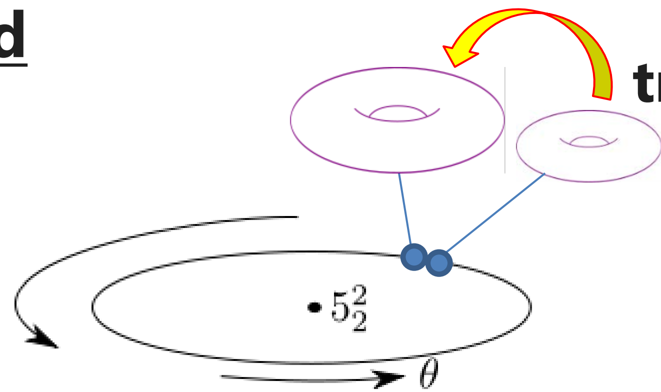
Finite coordinate transformation in DFT

In DFT, there exists a larger gauge symmetry:
(part of O(*d*,*d*) symmetry).

We can consider more general patching of doubled space.

E.g. T-fold

Finite O(*d*,*d*) transformation



In general, the transition function is given by a finite gauge transformation in DFT, and we need to find the finite transformation law of a (generalized) tensor field.

Plan

1. **Double Field Theory** (quick review, 2 pages)

2. **Finite transformations in DFT** (review)

   O. Hohm and B. Zwiebach, JHEP 1302, 075 (2013),
   J-H. Park, JHEP 1306, 098 (2013),
   D. S. Berman, M. Cederwall, M. J. Perry, JHEP 1409, 066 (2014).
   C. M. Hull, JHEP 1504, 109 (2015).

3. **Finite transformations in DFT** (our approach)

   [S-J. Rey, YS, arXiv:1510.06735]

4. **Exceptional Field Theory**

   Review: SL(5) EFT

   Finite transformations in SL(5) EFT

   Generalization is straightforward

Aim of this talk

C. M. Hull, arXiv:1406.7794

DFT

similar but different approach

S-J. Rey, YS, arXiv:1510.06735 .... October 22th.

DFT + SL(5) EFT

N. Chaemjumrus, C.M. Hull, arXiv:1512.03837 .... December 11th.

SL(5) + SO(5,5) + E_6 EFT

same approach

their comment about our paper

our paper

cases $E_4 = SL(5,\mathbb{R})$, $E_5 = SO(5,5)$ and E_6; we expect similar results will hold for E_7. While this paper was in preparation, the paper [44] appeared addressing the same question, but using an approach which appears to suffer from the same issues as the approach of [38].

Hohm-Zwiebach's approach

I'd like to explain what is the issue

and argue that there is no issue in our approach.

§1. Double Field Theory

(quick review)

Double Field Theory (DFT)

DFT … **manifestly T-duality covariant reformulation of supergravity.**

- **To make the covariance manifest, Dims. of space are doubled.**

$$x^M = (x^m,\ \tilde{x}_m)$$

- **Gauge symmetry: generated by Generalized Lie derivative**

($\supset$ *d*-diml diffeomorphism + B-field gauge symmetry)

$$\hat{\pounds}_V W^M \equiv V^N \partial_N W^M - \left(\partial_N V^M - \partial^M V_N\right) W^N .$$

- **Consistency** ($\delta_V W^M$ behaves as a generalized vector !)

We require [the strong constraint]

$\Rightarrow$ field can depend only on the half of the doubled coords.

$$\frac{\partial}{\partial \tilde{x}_m} * = 0 .$$

$$\Rightarrow \left\{ \begin{array}{l} S_{\mathrm{DFT}} \to \dfrac{1}{2\kappa^2} \displaystyle\int \mathrm{d}^d x \sqrt{-G}\, \mathrm{e}^{-2\phi} \left(R + 4\,|\partial\phi|^2 - \frac{1}{2}\,|H|^2\right) . \\ \textbf{gauge sym.} = \mathrm{Diff}_d \ltimes \textbf{(B-field gauge sym.)} \end{array} \right.$$

Our Convention

$$x^M = (x^m,\ \tilde{x}_m)$$

original coordinates

$$\Longleftrightarrow \quad x^M = (\tilde{x}_m,\ x^m)$$

(standard)

[In **Exceptional Field Theory (EFT),** $x^M = (x^i,\ y_{ij}, \ldots)$]

O(*d*,*d*) metric :

$$\eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \eta^{MN}\,, \quad \hat{\pounds}_V \eta_{MN} = 0\,.$$

(the O(*d*,*d*) indices, *M*, *N*, are raised/lowered)

Generalized metric:

$$\mathcal{H}_{MN} \equiv \begin{pmatrix} G - B\,G^{-1}\,B & B\,G^{-1} \\ -G^{-1}\,B & G^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \delta_m^p & B_{mp} \\ 0 & \delta_p^m \end{pmatrix} \begin{pmatrix} G_{pq} & 0 \\ 0 & G^{pq} \end{pmatrix} \begin{pmatrix} \delta_n^q & 0 \\ -B_{qn} & \delta_q^n \end{pmatrix}$$

(today we will not consider another parameterization)

$$\mathcal{H}_{MN} \equiv \cancel{\begin{pmatrix} \delta_m^p & 0 \\ \beta^{mp} & \delta_p^m \end{pmatrix} \begin{pmatrix} G_{pq} & 0 \\ 0 & G^{pq} \end{pmatrix} \begin{pmatrix} \delta_n^q & -\beta^{qn} \\ 0 & \delta_q^n \end{pmatrix}}$$

(please look at our paper)

§2. Finite transformations in DFT (review)

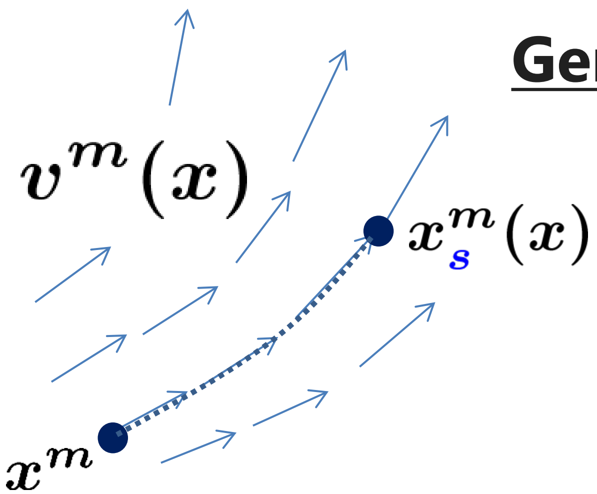
F.T. in DFT $\left\{\right.$
O. Hohm and B. Zwiebach, JHEP 1302, 075 (2013),
J-H. Park, JHEP 1306, 098 (2013),
D. S. Berman, M. Cederwall, M. J. Perry, JHEP 1409, 066 (2014).

(S-J. Rey, YS, arXiv:1510.06735.)

GG -like $\tilde{\partial}^m = 0$ $\left\{\right.$
C. M. Hull, JHEP 1504, 109 (2015).

Notations

General Relativity



Solution of $\dfrac{\mathrm{d}}{\mathrm{d}s} x^m_s = v^m \,.$

Finite diffeo. $x^m \underset{\varphi_s}{\longrightarrow} x^m_s(x) = \mathrm{e}^{s\,v} x^m$

(integral curve)

Vector field $w^m(x)$ **transforms as**

(pull-back $\varphi_s^* w^m$ **)**

$$\boxed{w^m_s(x) = \frac{\partial x^m}{\partial x^n_s} w^n(x_s)}$$

★ { • **active diffeomorphism** … (**Park '13; Rey, YS '15**)
• **passive coordinate transf.**

$$\boxed{\forall \text{ finite } s, \quad \frac{\mathrm{d}}{\mathrm{d}s} w^m_s = \pounds_v w^m_s}$$

$$\Rightarrow \quad w^m_s(x) = \frac{\partial x^m}{\partial x^n_s} w^n(x_s) \boxed{= \mathrm{e}^{s\pounds_v} w^m(x) \,.}$$

Review: Finite transf. in DFT (1/4)

In DFT, infinitesimal transf. is generated by the gen. Lie deriv.;

$$\delta_V W^M(x) = \hat{\mathcal{L}}_V W^M(x) \,.$$

finite $\Rightarrow$

$$W_{sM}(x) \equiv \mathrm{e}^{s\,\hat{\mathcal{L}}_V}\, W^M(x) \equiv \underline{G^M{}_N}\, W^N(x_s) \,.$$

complicated funct. of V^M **and** s

A proposal [Hohm, Zwiebach (2012)] :

"Gauge transf." = "Generalized diffeo. in the doubled space" ;

$$\delta_V W^M(x) \qquad \delta x^M(x) = V^M(x)$$

$$W_{sM}(x) = \mathcal{F}_M{}^N\, W_N(x_s) \,, \quad \mathcal{F}_M{}^N(x, x_s) \equiv \frac{1}{2}\left(\frac{\partial x_s^K}{\partial x^M}\frac{\partial x_K}{\partial x_{sN}} + \frac{\partial x_M}{\partial x_{sK}}\frac{\partial x_s^N}{\partial x^K}\right) .$$

$$w_{sm}(x) = \frac{\partial x_s^n}{\partial x^m}\, w_n(x_s) \qquad\qquad x_s^M \equiv \mathrm{e}^{sV}\, x^M$$

Review: Finite transf. in DFT (2/4)

Hohm-Zwiebach's formula gives

$$\boxed{W_s{}^M(x) = \mathcal{F}_M{}^N\, W_N(x_s)} = W^M(x) + s\,\hat{\pounds}_V W^M(x) + \mathcal{O}(s^2)\,.$$

(at an infinitesimal level)

<u>However,</u> other than $s = 0$, $\dfrac{\mathrm{d}}{\mathrm{d}s} W_{sM}(x) \neq \hat{\pounds}_V W_{sM}(x)\,.$

$$\Longrightarrow \quad \boxed{W_{sM}(x) \equiv \mathcal{F}_M{}^N(x, x_s)\, W_N(x_s)} \neq \mathrm{e}^{\,s\,\hat{\pounds}_V}\, W^M(x)\,.$$

<u>[Hohm, Zwiebach (2012)]</u>

$$x^M_{(s=1)} \equiv \mathrm{e}^{\,V}\, x^M \quad \xrightarrow[\text{correction}]{} \quad x^M_{(s=1)} \equiv \mathrm{e}^{\,V + \sum_i \rho_i \partial \chi_i}\, x^M = \begin{pmatrix} \mathrm{e}^{v}\, x^m \\ \mathrm{e}^{V + \sum_i \rho_i \partial \chi_i}\, \tilde{x}_m \end{pmatrix}$$

$$W_{(s=1)M}(x) \equiv \mathcal{F}_M{}^N(x, x_{(s=1)})\, W_N(x_{(s=1)}) \overset{!}{=} \mathrm{e}^{\,\hat{\pounds}_V}\, W^M(x)\,.$$

$$\sum_i \rho_i \partial^M \chi_i = \tfrac{1}{12}\,(V \cdot V^N)(x)\, \partial^M V_N(x) + \cdots\,.$$

$\Longrightarrow$ Full order correction [U. Naseer, JHEP 1506, 002 (2015)]

Review: Finite transf. in DFT (3/4)

[J-H. Park, JHEP 1306, 098 (2013)]

$$W_{sM}(x) = \mathcal{F}_M{}^N(x, x_s)\, W_N(x_s)\,, \qquad x_s^M \equiv \mathrm{e}^{sV}\, x^M\,.$$

$$\frac{\mathrm{d}}{\mathrm{d}s} W_{sM}(x) = \hat{\mathcal{L}}_{\mathcal{V}} W_{sM}(x)$$ (for arbitrary s)

$$\mathcal{V}^M \equiv V^M + \underbrace{\frac{1}{2} V_N\, \partial^M f_s^N}_{\phi^i\, \partial^M \varphi_i}\,, \qquad f_s^M \equiv \sum_{n=1} \frac{s^n}{n!} \left(V^N \partial_N\right)^{n-1} V^M\,,$$

$\phi^i\, \partial^M \varphi_i$ ⇒ [does not generate a translation]

$$(\phi^i\, \partial^M \varphi_i)\, \partial_M * = 0 \qquad (\partial^M * \partial_M * = 0)$$

generate a B-field gauge transf.

["coordinate gauge symmetry": $x^M \sim x^M + \phi^i\, \partial^M \varphi_i$]

⇒ Up to coord. gauge sym., $\dfrac{\mathrm{d}}{\mathrm{d}s} W_{sM} = \hat{\mathcal{L}}_{\mathcal{V}} W_{sM} \sim \hat{\mathcal{L}}_V W_{sM}$,

Hohm-Zwiebach's proposition is correct !

$$W_{sM}(x) \equiv \mathcal{F}_M{}^N(x, x_s)\, V_N(x_s) \sim \mathrm{e}^{s\,\hat{\mathcal{L}}_V}\, W_M(x)\,.$$

Review: Finite transf. in DFT (4/4)

[D. Berman, M. Cederwall, M. Perry (2014)] showed

$$W'_M(x) \equiv e^{\hat{\mathcal{L}}_V} W_M(x) = (\mathcal{F} \cdot e^{\Delta})_M{}^N W_N(x) \,.$$

$$e^{\Delta} \equiv \prod_{n=2}^{\infty} \prod_{k=0}^{n-1} \Bigl(1 + \frac{1}{2} \frac{(-1)^n (n-2k-1)}{(n+1)(k+1)!(n-k)!} M_{n,k}\Bigr) \,,$$

$$(M_{n,k})_M{}^N \equiv \partial_M (V^k \cdot V^L)\, \partial^N (V^{n-k-1} \cdot V_L) \,.$$

$$\tilde{\partial}^m = 0$$

$$e^{\Delta} = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \,.$$

B-field transf.

Hohm-Zwiebach's $\mathcal{F}_M{}^N$ is equal to the exponential of the generalized Lie deriv., up to a B-field transformation.

[Hohm, Zwiebach (2012)] will be correct :

$$V_{sM}(x) = \mathcal{F}_M{}^N V_N(x_s) \,, \quad \mathcal{F}_M{}^N(x, x_s) \equiv \frac{1}{2} \Bigl(\frac{\partial x_s^K}{\partial x^M} \frac{\partial x_K}{\partial x_{sN}} + \frac{\partial x_M}{\partial x_{sK}} \frac{\partial x_s^N}{\partial x^K}\Bigr) \,.$$

Some issues in Hohm-Zwiebach's proposal

1. ~~Composition law;~~
[Hohm, Zwiebach, JHEP 1302, 075 (2013);
Hohm, Lüst, Zwiebach, arXiv:1309.2977]

2. Patching condition with H-Z is restrictive;
[G. Papadopoulos, JHEP 1410, 089 (2014);
C. Hull, JHEP 04 (2015) 109]

Issue 1: Composition law

GR :

$$x^m \xrightarrow{\textcircled{1}} x_1^m \xrightarrow{\textcircled{2}} x_2^m\,, \qquad w''^m(x_2) \overset{\textcircled{2}}{=} \frac{\partial x_2^m}{\partial x_1^n}\boxed{w'^n(x_1)} \overset{\textcircled{1}}{=} \frac{\partial x_2^m}{\partial x_1^n}\boxed{\frac{\partial x_1^n}{\partial x^k}\,w^k(x)}$$

$$= \frac{\partial x_2^m}{\partial x^n}\,w^n(x)\,. \quad \textbf{(chain rule)}$$

Composition law is satisfied.

DFT :

$$\boxed{x^M \to x'^M(x)\,, \qquad V'_M(x') = \mathcal{F}_M{}^N(x', x)\,V_N(x)\,.}$$

$$x^M \to x_1^M \to x_2^M\,, \qquad W''_M(x_2) = \mathcal{F}_M{}^K(x_2, x_1)\mathcal{F}_K{}^N(x_1, x)\,W_N(x)$$

$$\neq \mathcal{F}_M{}^N(x_2, x)\,W^N(x)\,.$$

Composition law is NOT satisfied.

[Hohm, Zwiebach '12]

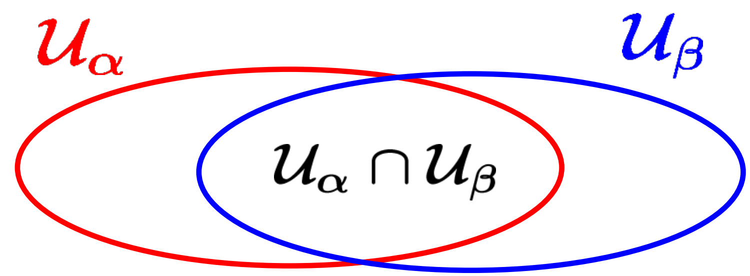
We cannot have a good geometric interpretation of the finite transformation in the doubled space.

Issue 2: H-flux is trivial (1/2)

[G. Papadopoulos, JHEP 1410, 089 (2014)]

Let us assume Hohm-Zwiebach's proposal.

On an overlap, $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$,
we have 2 local coordinates;



$$\begin{cases} x^m_{(\alpha)} = x^m_{(\alpha\beta)}\big(x_{(\beta)}\big)\,, \\ \tilde{x}_{(\alpha)m} = \tilde{x}_{(\beta)m} - \zeta_{(\alpha\beta)m}\,. \end{cases}$$

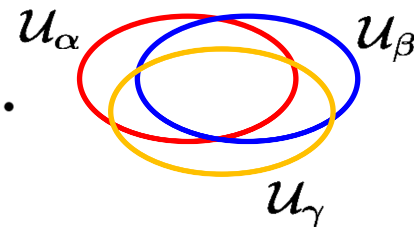
Regarding this as a Finite coord. transf.
the generalized metric on each patch should be related as

$$\mathcal{H}^{(\alpha)}_{MN} = \mathcal{F}_M{}^K\big(x_{(\alpha)}, x_{(\beta)}\big)\,\mathcal{F}_N{}^L\big(x_{(\alpha)}, x_{(\beta)}\big)\,\mathcal{H}^{(\beta)}_{KL}\,.$$

(i.e. we use $\mathcal{F}_M{}^N$ as the transition function)

Issue 2: H-flux is trivial (2/2)

$$x^m_{(\alpha)} = x^m_{(\alpha\beta)}(x_{(\beta)})\,, \qquad \tilde{x}_{(\alpha)m} = \tilde{x}_{(\beta)m} - \zeta_{(\alpha\beta)m}\,.$$



triple overlap

Consistency requires $\quad \zeta_{(\alpha\beta)m} + \zeta_{(\beta\gamma)m} + \zeta_{(\gamma\alpha)m} = 0\,.$

H-Z's proposal gives

$$B_{(\alpha)mn} = \frac{\partial x^k_{(\beta)}}{\partial x^m_{(\alpha)}}\frac{\partial x^l_{(\beta)}}{\partial x^n_{(\alpha)}}\left(B_{(\beta)kl} + \partial^{(\beta)}_{[k}\zeta_{(\alpha\beta)l]}\right) + \partial^{(\alpha)}_{[m}\zeta_{(\alpha\beta)n]}\,.$$

From these two, it was proven that we can find a 1-form $\lambda_{(\alpha)}$

that makes $\widetilde{B}_{(\alpha)} \equiv B_{(\alpha)} + \mathrm{d}\lambda_{(\alpha)}$ is globally defined; $\widetilde{B}_{(\alpha)} = \widetilde{B}_{(\beta)}\,.$

[G. Papadopoulos, JHEP 1410, 089 (2014)]

⇨ $H_3 \equiv \mathrm{d}B_2 = \mathrm{d}\widetilde{B}$ **is globally an exact form.**

⇨ **We cannot obtain a non-trivial H-flux;** $\int_{\partial M_4} H_3 = 0\,.$

Patching with H-Z's proposal is restrictive!

How can we obtain a non-trivial H-flux ?

In the conventional SUGRA (Generalized Geometry);

Patching condition
$$\begin{cases} x^m_{(\alpha)} = x^m_{(\alpha\beta)}(x_{(\beta)})\,, & \cancel{\tilde{x}_{(\alpha)m} = \tilde{x}_{(\beta)m} - \zeta_{(\alpha\beta)m}\,.} \\ B_{(\alpha)} = B_{(\beta)} + \mathrm{d}\zeta_{(\alpha\beta)} & \end{cases}$$

$$\zeta_{(\alpha\beta)m} + \zeta_{(\beta\gamma)m} + \zeta_{(\gamma\alpha)m} = 0\,.$$

Consistency require,

$$\mathrm{d}\big(\zeta_{(\alpha\beta)} + \zeta_{(\beta\gamma)} + \zeta_{(\gamma\alpha)}\big) = 0\,.$$

(weaker condition)

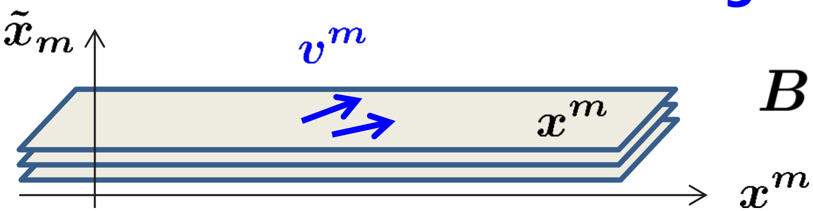
⇨ **We cannot find a globally defined** $\widetilde{B}_{(\alpha)} \equiv B_{(\alpha)} + \mathrm{d}\lambda_{(\alpha)}$

[G. Papadopoulos (2014)]

We should not realize the B-field gauge transf.
as the dual-coord. transf. $\tilde{x}_m \to \tilde{x}_m - \zeta_m$ **!?**

[C. Hull, JHEP 04 (2015) 109]

$$x^m \to x^m + v^m$$
$$\tilde{x}_m \to \tilde{x}_m$$



gauge parameter

$$B \to B + \mathrm{d}\tilde{v}\,.$$

Hull's proposal (1/2)

[C. Hull, JHEP 04 (2015) 109]

■ $\tilde{\partial}^m = 0$

Untwisted form of a generalized vector W^M :

$$\hat{W}^M(x) = \begin{pmatrix} w^m(x) \\ \hat{w}_m(x) \end{pmatrix} = \begin{pmatrix} \delta^m_n & 0 \\ -B_{mn}(x) & \delta^n_m \end{pmatrix} \begin{pmatrix} w^n(x) \\ \tilde{w}_n(x) \end{pmatrix} .$$

(untwisted vector) untwisting (generalized vector)

■ The untwisted vector transforms as

$$\delta_V \hat{W}^M(x) = \begin{pmatrix} \pounds_v w^m(x) \\ \pounds_v \hat{w}_m(x) \end{pmatrix} \begin{matrix} \longleftarrow \text{vector} \\ \longleftarrow \text{1-form !} \end{matrix} \qquad \delta_V W^M = \hat{\pounds}_V W^M$$

Invariant under B-transf.

(c.f. $\delta_V \tilde{w}_m = \pounds_v \tilde{w}_m + 2\,\partial_{[m}\tilde{v}_{n]}$)

■ We can easily obtain the finite transf. for the untwisted vector:

$$\hat{W}_s^M(x) = \begin{pmatrix} w_s^m(x) \\ \hat{w}_{sm}(x) \end{pmatrix} = \begin{pmatrix} \frac{\partial x^m}{\partial x_s^n} & 0 \\ 0 & \frac{\partial x_s^n}{\partial x^m} \end{pmatrix} \begin{pmatrix} w^n(x_s) \\ \hat{w}_n(x_s) \end{pmatrix} . \qquad (x_s^m = \mathrm{e}^{sv}\, x^m)$$

⇒ To obtain a Finite transf. law for gen. vector, $W_s^M(x) = \mathrm{e}^{s\hat{\pounds}_V}\, W^M$, we need to know a Finite transf. law for $B_{mn}(x)$.

Hull's proposal (2/2)

[C. Hull, JHEP 04 (2015) 109]

Under a finite gauge transformation, B-field should transform as

$$B^{s}_{mn}(x) = \frac{\partial x_s^{p}}{\partial x^{m}}\,\frac{\partial x_s^{q}}{\partial x^{n}}\,\big(B_{pq} + \partial_p \tilde{\mathrm{v}}_q - \partial_q \tilde{\mathrm{v}}_p\big)(x_s)\,.$$

diffeo.

a certain finite parameter

From these, he obtain a Finite transf. law for generalized vector :

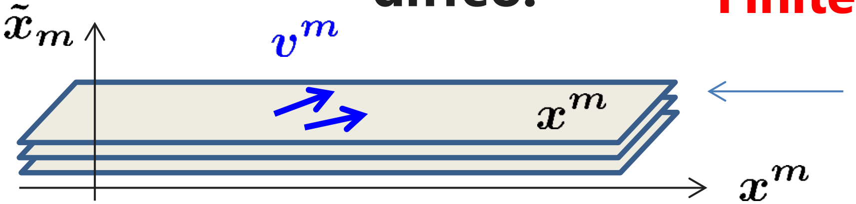
$$W_s{}^{M}(x) = \mathrm{e}^{\,s\tilde{\pounds}_{\mathrm{V}}}\, W^{M}(x) = \begin{pmatrix} \delta^m_n & 0 \\ B^{s}_{mn}(x) & \delta^n_m \end{pmatrix} \begin{pmatrix} w_s^{n}(x) \\ \hat{w}_{sn}(x) \end{pmatrix} = R^{M}{}_{N}\, W^{N}(x_s)$$

~~"up to B-field trsf."~~ **Exact!**

with $$R^{M}{}_{N} = \begin{pmatrix} \frac{\partial x^m}{\partial x_s^k} & 0 \\ 0 & \frac{\partial x_s^k}{\partial x^m} \end{pmatrix} \begin{pmatrix} \delta^k_n & 0 \\ 2\,\partial_{[k}\tilde{\mathrm{v}}_{n]}(x) & \delta^n_k \end{pmatrix}.$$

diffeo.

Finite gauge parameter.



d-dimensional null plane (section Σ_d)

Summary: Previous approaches

[Hohm, Zwiebach '12]

Finite gauge transf. = Finite gen. coord. transf; $x^M \to \mathrm{e}^{sV} x^M$

$$W_{sM}(x) = \mathcal{F}_M{}^N W_N(x_s)\,, \quad \mathcal{F}_M{}^N(x, x_s) \equiv \frac{1}{2}\Bigl(\frac{\partial x_s^K}{\partial x^M}\frac{\partial x_K}{\partial x_{sN}} + \frac{\partial x_M}{\partial x_{sK}}\frac{\partial x_s^N}{\partial x^K}\Bigr)\,,$$

Issue 1: ~~composition law~~ , Issue 2: ~~non-trivial H-flux~~ [Papadopoulos '14]

It was shown that $W_{sM}(x) \sim \mathrm{e}^{s\hat{\mathcal{L}}_V} W_M\,,$ **[J-H. Park '13; Berman, Cederwall, Perry '14]**

up to a B-field gauge transf.

[Hull '15] We should not use the equiv. relation, "up to B-field trsf."

$$W_s^M(x) = \mathrm{e}^{s\hat{\mathcal{L}}_V} W^M(x)\,.$$

certain parameters

$$\begin{cases} x^m \to x_s^m = \mathrm{e}^{sv} x^m \\ \tilde{x}_m \to \tilde{x}_m \end{cases} + \quad B^s_{mn}(x) = \frac{\partial x_s^p}{\partial x^m}\frac{\partial x_s^q}{\partial x^n}\bigl(B_{pq} + \partial_p \tilde{\mathrm{v}}_q - \partial_q \tilde{\mathrm{v}}_p\bigr)(x_s)\,.$$

$$W_s^M(x) = R^M{}_N W^N(x_s)\,, \qquad R^M{}_N = \begin{pmatrix} \frac{\partial x^m}{\partial x_s^k} & 0 \\ 0 & \frac{\partial x_s^k}{\partial x^m} \end{pmatrix}\begin{pmatrix} \delta_n^k & 0 \\ 2\,\partial_{[k}\tilde{\mathrm{v}}_{n]}(x) & \delta_k^n \end{pmatrix}.$$

Relation between $\tilde{\mathrm{v}}_m$ and V^M was not obtained!

§3. Finite transformations in DFT (our approach)

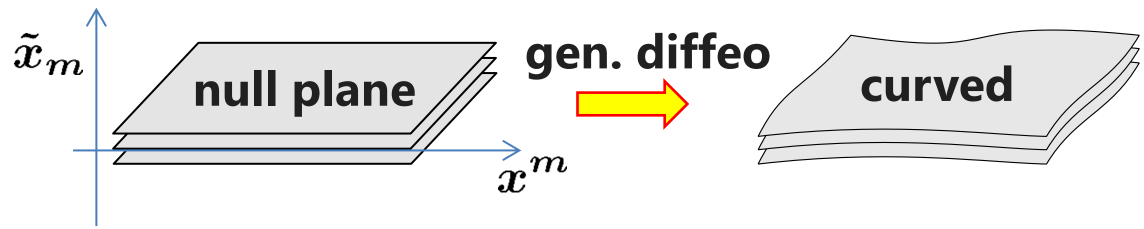
S-J. Rey, YS, arXiv:1510.06735.

- Composition law ✓
- non-trivial H-flux ✓

Our approach

☆ Similar to H-Z's approach, we consider the dual-coord. transf.

$$\begin{cases} x^m \to \mathrm{e}^{s\,v}\, x^m \\ \tilde{x}_m \to \tilde{x}_m \boxed{+\, \zeta_m^{(s,V)}(x)} \end{cases}$$



I will explain later how to get rid of the Papadopoulos problem.

☆ As with Hull's approach, we use $\tilde{\partial}^m = 0$ at the beginning.
We adopt Hull's idea to use the untwisted vector:

$$\hat{W}^M(x) \equiv \begin{pmatrix} w^m(x) \\ \hat{w}_m(x) \end{pmatrix} = \begin{pmatrix} \delta^m_n & 0 \\ -B_{mn}(x) & \delta^n_m \end{pmatrix} \begin{pmatrix} w^n(x) \\ \tilde{w}_n(x) \end{pmatrix} ,$$

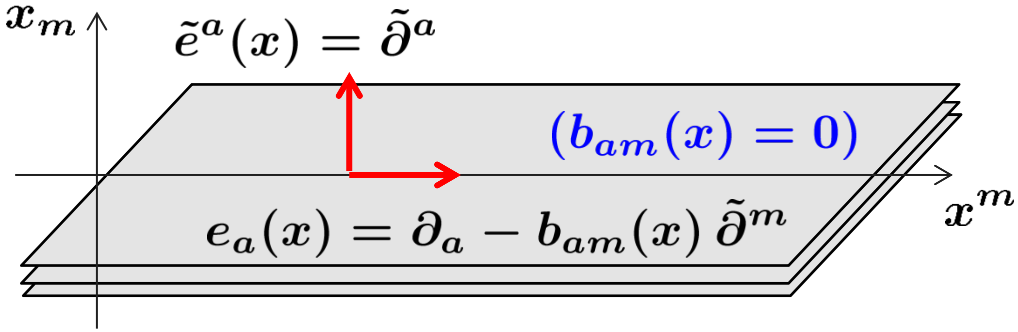
that transforms as

$$\hat{W}^M_{(s,V)}(x) = \begin{pmatrix} w^m_s(x) \\ \hat{w}_{sm}(x) \end{pmatrix} = \begin{pmatrix} \frac{\partial x^m}{\partial x^n_s} & 0 \\ 0 & \frac{\partial x^n_s}{\partial x^m} \end{pmatrix} \begin{pmatrix} w^n(x_s) \\ \hat{w}_n(x_s) \end{pmatrix} .$$

☆ Using our result, we can explicitly show that
the composition law is satisfied as in GR.

Setup : define b_{mn}

$\tilde{\partial}^m = 0$



☆ **null condition**

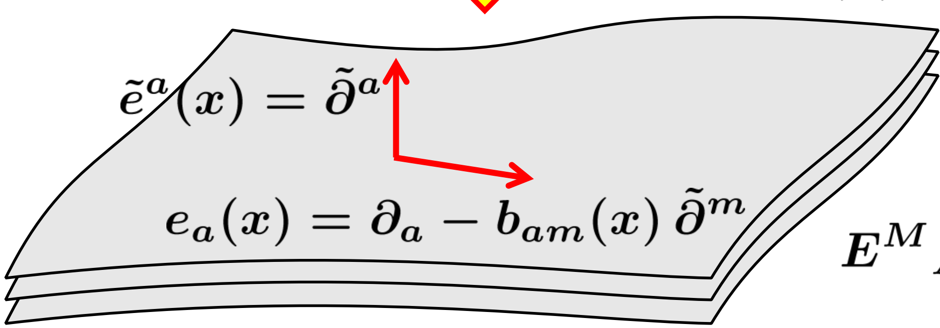
$$\eta_{MN}\, e^M{}_a\, e^N{}_b = 0$$

$$\Downarrow$$

$$b_{am} = -b_{ma}$$

$$E^M{}_A(x) \equiv (e^M{}_a(x),\, \tilde{e}^{Ma}) = \begin{pmatrix} \boxed{\delta^m_a} & 0 \\ b_{ma}(x) & \boxed{\delta^a_m} \end{pmatrix}$$

☆ **gauge fixing**



$$E^M{}_A(x) \xrightarrow{\hat{\pounds}_V} \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \xrightarrow{\text{GL}(d)} \begin{pmatrix} \delta^m_a & 0 \\ b'_{ma}(x) & \delta^a_m \end{pmatrix}$$

$$\Rightarrow \quad \boxed{\delta_V b_{mn}(x) = \pounds_v b_{mn}(x) + \partial_m \tilde{v}_n(x) - \partial_n \tilde{v}_m(x)\,.}$$

same as the gauge transf. for B-field

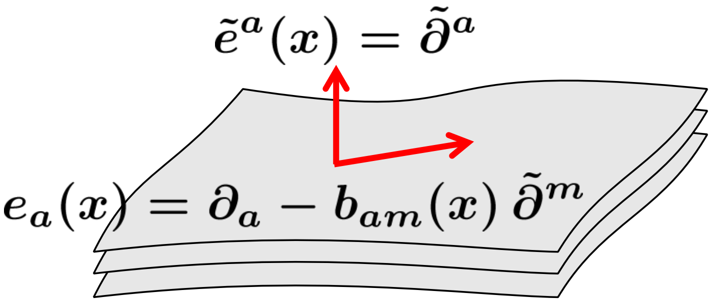
☆ **Involutive property**

$$\delta_V W^M = e^M{}_a\, z^a\,. \quad \Rightarrow \quad \boxed{\partial_{[l} b_{mn]} = 0\,.}$$

tangent vector

b_{mn} **is closed!**

Definition b_{mn} (summary)



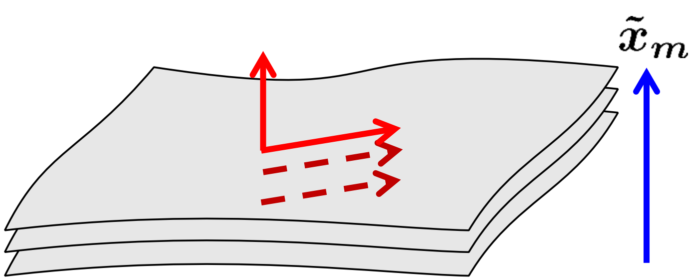
$$\boxed{\begin{aligned} &b_{mn} = b_{[mn]}\,, \qquad \partial_{[l} b_{mn]} = 0\,, \\ &\delta_V b_{mn}(x) = \pounds_v b_{mn}(x) + 2\,\partial_{[m}\tilde{v}_{n]}(x)\,. \end{aligned}}$$

☆ b_{mn} **is different from** B_{mn} **!**

dynamical

we can freely choose

☆ b_{mn} **is defined on the entire doubled space.**

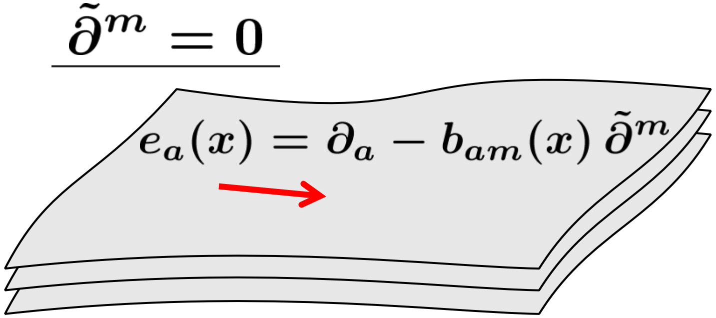


∃ Isometries in the dual direction.

$$\tilde{\partial}^k b_{mn}(x) = 0\,.$$

Foliation is uniform in the dual direction.

Our finite transformation (1/2)



Infinitesimal transf.

$$\begin{cases} x^m \to x^m + v^m(x) \\ \delta_V b_{mn} = \pounds_v b_{mn} + 2\,\partial_{[m}\tilde{v}_{n]}(x)\,. \end{cases}$$

We solved

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}\, x_s^m = v^m(x)\,, \\ \dfrac{\mathrm{d}}{\mathrm{d}s} b_{mn}^{(s,V)}(x) = \pounds_v b_{mn}^{(s,V)}(x) + 2\,\partial_{[m}\tilde{v}_{n]}(x)\,. \end{cases}$$

integrate

Finite transf.

$$\begin{cases} x_s^m = \mathrm{e}^{\,s\,v}\, x^m\,, \\ b_{mn}^{(s,V)}(x) = b_{mn}(x) + 2\,\partial_{[m}\zeta_{n]}^{(s,V)}(x)\,. \end{cases}$$

$$\begin{cases} \zeta_m^{(s,V)}(x) \equiv \displaystyle\int_0^s \mathrm{d}s'\,\hat{\mathbf{v}}_m^{(s',V)}(x)\,. \\ \hat{\mathbf{v}}_m^{(s,V)}(x) \equiv \dfrac{\partial x_s^n}{\partial x^m}\,\hat{\mathbf{v}}_n(x_s)\,. \end{cases}$$

$(\hat{\mathbf{v}}_m \equiv \tilde{v}_m - b_{mn}\, v^n)$

describes the "shape" of the subspace after the finite transf.

Our finite transformation (2/2)

■ Let us obtain finite transf. law for B-field

$$\begin{cases} \delta_V B_{mn} = \pounds_v B_{mn} + 2\,\partial_{[m}\tilde{v}_{n]}(x)\,. \\ \delta_V b_{mn} = \pounds_v b_{mn} + 2\,\partial_{[m}\tilde{v}_{n]}(x)\,. \end{cases}$$

$$\mathbb{B}_{mn}(x) \equiv B_{mn}(x) - b_{mn}(x)\,.$$

$$\Rightarrow \quad \delta_V \mathbb{B}_{mn}(x) = \pounds_v \mathbb{B}_{mn}(x)\,.$$

$$\mathbb{B}^{(s,V)}_{mn}(x) = \frac{\partial x^k_s}{\partial x^m}\frac{\partial x^l_s}{\partial x^n}\,\mathbb{B}_{kl}(x_s)\,. \qquad b^{(s,V)}_{mn}(x) = b_{mn}(x) + 2\,\partial_{[m}\zeta^{(s,V)}_{n]}(x)\,.$$

$$\boxed{B^{(s,V)}_{mn}(x) = \frac{\partial x^k_s}{\partial x^m}\frac{\partial x^l_s}{\partial x^n}\,(B_{kl} - b_{kl})(x_s) + b_{mn}(x) + 2\,\partial_{[m}\zeta^{(s,V)}_{n]}(x)\,.}$$

■ Finite transf. for a generalized vector

$$\begin{aligned} W^M_{(s,V)}(x) &= \begin{pmatrix} \delta^m_k & 0 \\ B^{(s,V)}_{mk}(x) & \delta^k_m \end{pmatrix} \begin{pmatrix} \frac{\partial x^k}{\partial x^l_s} & 0 \\ 0 & \frac{\partial x^l_s}{\partial x^k} \end{pmatrix} \begin{pmatrix} \delta^l_n & 0 \\ -B_{ln}(x_s) & \delta^n_l \end{pmatrix} \begin{pmatrix} w^n(x_s) \\ \tilde{w}_n(x_s) \end{pmatrix} \\ &= \underbrace{\boxed{\begin{pmatrix} \delta^m_k & 0 \\ b^{(s,V)}_{mk}(x) & \delta^k_m \end{pmatrix} \begin{pmatrix} \frac{\partial x^k}{\partial x^l_s} & 0 \\ 0 & \frac{\partial x^l_s}{\partial x^k} \end{pmatrix} \begin{pmatrix} \delta^l_n & 0 \\ -b_{ln}(x_s) & \delta^n_l \end{pmatrix}}}_{\mathcal{S}^M{}_N} \begin{pmatrix} w^n(x_s) \\ \tilde{w}_n(x_s) \end{pmatrix}. \end{aligned}$$

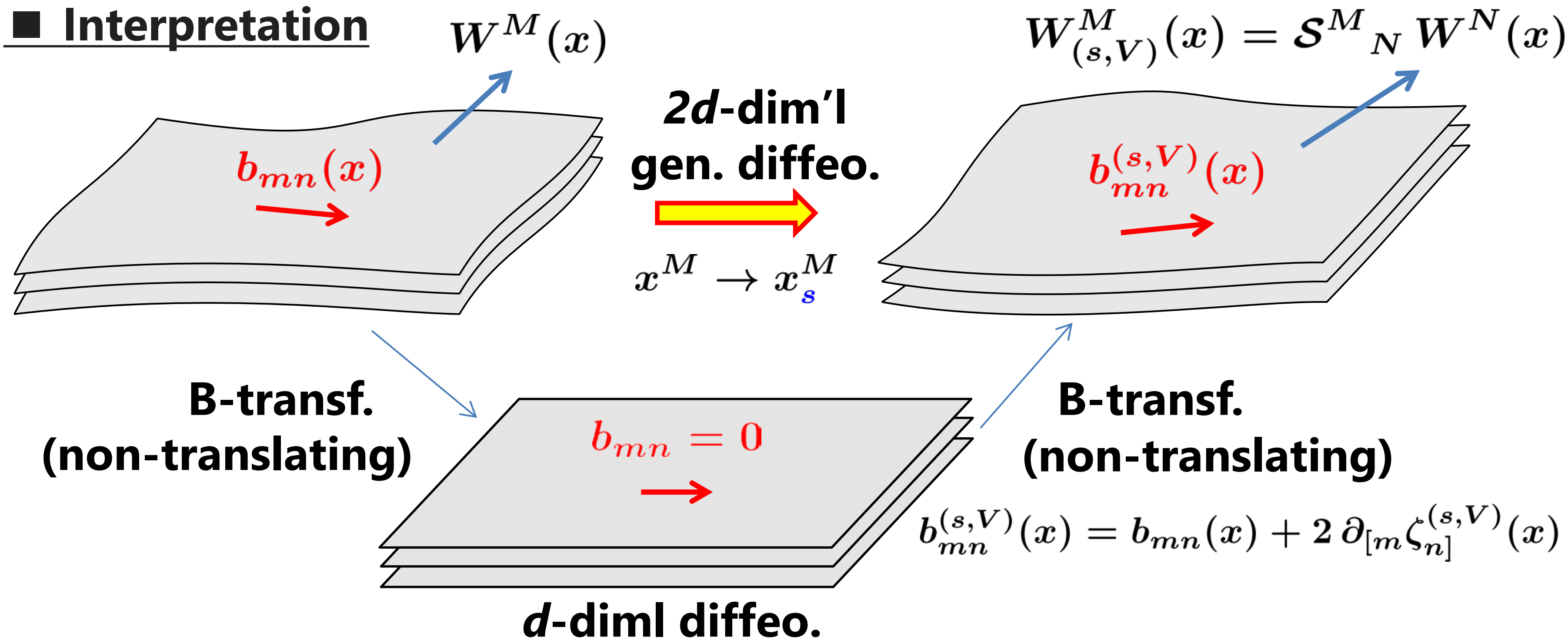
Interpretation

■ Finite transformations for a generalized tensor

$$(T_{(s,V)})^{M_1\cdots M_k}_{N_1\cdots N_l}(x) = \mathcal{S}^{M_1}{}_{K_1}\cdots\mathcal{S}^{M_k}{}_{K_k}\,\mathcal{S}_{N_1}{}^{L_1}\cdots\mathcal{S}_{N_l}{}^{L_l}\,T^{K_1\cdots K_k}_{L_1\cdots L_l}(x_s)\,.$$

$$\boxed{\mathcal{S}^M{}_N \equiv \begin{pmatrix} \delta^m_k & 0 \\ b^{(s,V)}_{mk}(x) & \delta^k_m \end{pmatrix}\begin{pmatrix} \frac{\partial x^k}{\partial x^l_s} & 0 \\ 0 & \frac{\partial x^l_s}{\partial x^k} \end{pmatrix}\begin{pmatrix} \delta^l_n & 0 \\ -b_{ln}(x_s) & \delta^n_l \end{pmatrix}\,.}$$

■ Interpretation

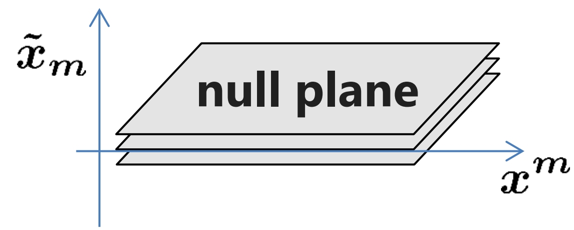


Comparison with Hull's result

☆ **Our transformation matrix**

$$\boxed{\mathcal{S}^{M}{}_{N} \equiv \begin{pmatrix} \delta_k^m & 0 \\ b^{(s,V)}_{mk}(x) & \delta_m^k \end{pmatrix} \begin{pmatrix} \frac{\partial x^k}{\partial x_s^l} & 0 \\ 0 & \frac{\partial x_s^l}{\partial x^k} \end{pmatrix} \begin{pmatrix} \delta_n^l & 0 \\ -b_{ln}(x_s) & \delta_l^n \end{pmatrix}.}$$

☆ **To compare with Hull's result, let us choose $b_{mn}(x) = 0$.**



$$\mathcal{S}^{M}{}_{N} = \begin{pmatrix} \delta_k^m & 0 \\ 2\,\partial_{[m}\zeta^{(s,V)}_{k]}(x) & \delta_m^k \end{pmatrix} \begin{pmatrix} \frac{\partial x^k}{\partial x_s^n} & 0 \\ 0 & \frac{\partial x_s^n}{\partial x^k} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial x^m}{\partial x_s^k} & 0 \\ 0 & \frac{\partial x_s^k}{\partial x^m} \end{pmatrix} \begin{pmatrix} \delta_n^k & 0 \\ 2\,\partial_{[k}\zeta^{(s,V)}_{n]}(x_s) & \delta_k^n \end{pmatrix}.$$

same form

$$R^{M}{}_{N} = \begin{pmatrix} \frac{\partial x^m}{\partial x_s^k} & 0 \\ 0 & \frac{\partial x_s^k}{\partial x^m} \end{pmatrix} \begin{pmatrix} \delta_n^k & 0 \\ 2\,\partial_{[k}\tilde{\mathrm{v}}_{n]}(x) & \delta_k^n \end{pmatrix}.$$

Hull

$\partial_{[k}\zeta^{(s,V)}_{n]}$

identify

$\partial_{[k}\tilde{\mathrm{v}}_{n]}$

Dual coordinates (1/3)

We define the variation of the dual coordinates as $\frac{\mathrm{d}}{\mathrm{d}s}\tilde{x}^s_m = \hat{\mathbf{v}}^{(s,V)}_m(x)$.

$$\Longrightarrow \quad \tilde{x}^s_m = \tilde{x}_m + \int_0^s \mathrm{d}s' \, \hat{\mathbf{v}}^{(s',V)}_m(x) = \tilde{x}_m + \zeta^{(s,V)}_m(x)$$

Under a finite generalized diffeo.

$$\begin{cases} x^m \to \mathrm{e}^{s\,v}\, x^m \\ \tilde{x}_m \to \tilde{x}_m + \zeta^{(s,V)}_m(x) \end{cases}$$

$$\boxed{b^{(s,V)}_{mn}(x)} = b_{mn}(x) + \boxed{2\,\partial_{[m}\zeta^{(s,V)}_{n]}(x)} = b_{mn}(x) + \boxed{2\,\partial_{[m}\tilde{x}^s_{n]}(x)} \,.$$

$$\mathcal{S}^M{}_N \equiv \begin{pmatrix} \delta^m_k & 0 \\ \boxed{b^{(s,V)}_{mk}(x)} & \delta^k_m \end{pmatrix} \begin{pmatrix} \frac{\partial x^k}{\partial x^l_s} & 0 \\ 0 & \frac{\partial x^l_s}{\partial x^k} \end{pmatrix} \begin{pmatrix} \delta^l_n & 0 \\ -b_{ln}(x_s) & \delta^n_l \end{pmatrix} \,.$$

Once a generalized diffeomorphism is given,
we can calculate the corresponding Finite transformation matrix.

(similar to Hohm-Zwiebach's proposal)

$\Longrightarrow$ **Papadopoulos problem??**

Dual coordinates (2/3)

In a doubled space, there always exist a "trivial Killing vector":

$$V^M(x) = \partial^M f(x) \qquad (\, f(x) : \text{ arbitrary function of } \; x^m \;)$$

Indeed, every tensor is inv. along the flow of trivial Killing vector:

$$\hat{\pounds}_{\vec{\partial} f} W^M(x) = 0\,.$$

We identify physical points with $x^M \sim x^M + \underbrace{\partial^M f(x)}_{\text{zero vector}}\,.$

weaker version of the coordinate gauge symmetry [Park '13]

$$x^M \sim x^M + \phi(x)\,\partial^M \varphi(x)\,.$$

We can understand the identification as follows:

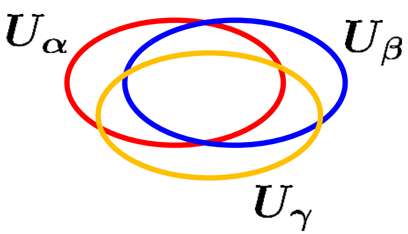
(Off-shell) degrees of freedom of $B_{mn}(x)$ is $\frac{(d-1)(d-2)}{2}$.

$$\frac{(d-1)(d-2)}{2} = \frac{d(d-1)}{2} - (d-1) \qquad (\tilde{v}_m \sim \tilde{v}_m + \partial_m f)$$

anti-sym. $\qquad B_{mn} \sim B_{mn} + 2\,\partial_{[m}\tilde{v}_{n]}$

Dual coordinates (3/3)

$$\begin{cases} x_s^m = \mathrm{e}^{s\,v}\, x^m \\ \tilde{x}^s_m = \tilde{x}_m + \zeta_m^{(s,V)}(x) \end{cases} \quad \Rightarrow \quad \begin{cases} x^m_{(\alpha)} = x^m_{(\alpha\beta)}(x_{(\beta)}) \\ \tilde{x}_{(\alpha)m} = \tilde{x}_{(\beta)m} - \zeta_{(\alpha\beta)m} \end{cases}$$



Consistency

[Papadopoulos '14]

$$\zeta_{(\alpha\beta)m} + \zeta_{(\beta\gamma)m} + \zeta_{(\gamma\alpha)m} = \partial_m f_{(\alpha\beta\gamma)}\,.$$

(up to the total deriv.)

[$\tilde{x}_m$ is defined only up to the relation, $\tilde{x}_m \sim \tilde{x}_m + \partial_m f\,.$]

$$\Leftrightarrow \quad \boxed{\mathrm{d}\big(\zeta_{(\alpha\beta)} + \zeta_{(\beta\gamma)} + \zeta_{(\gamma\alpha)}\big) = 0\,.}$$

Same condition with the Generalized Geometry!

Keeping in mind the equivalence relation,

⇒ even if we consider a diffeo. along the dual direction, we can consider a background with non-trivial H-flux!

Composition law (1/2)

$$W^M_{[s,V]}(x_s) = \begin{pmatrix} \delta^m_n & 0 \\ b^{(s,V)}_{mn}(x_s) & \delta^n_m \end{pmatrix} \begin{pmatrix} \frac{\partial x^n_s}{\partial x^k} & 0 \\ 0 & \frac{\partial x^k}{\partial x^n_s} \end{pmatrix} \begin{pmatrix} \delta^k_l & 0 \\ -b_{kl}(x) & \delta^l_k \end{pmatrix} \begin{pmatrix} w^l(x) \\ \tilde{w}_l(x) \end{pmatrix} .$$

① $x^M \to x^M_1 \to x^M_2$

$b_{mn} \to b^{(1)}_{mn} \to b^{(2;1)}_{mn}$

chain rule

$$W^M_{(2;1)}(x_2) = \begin{pmatrix} \delta^m_n & 0 \\ b^{(2;1)}_{mn}(x_2) & \delta^n_m \end{pmatrix} \boxed{\begin{pmatrix} \frac{\partial x^n_2}{\partial x^k_1} & 0 \\ 0 & \frac{\partial x^k_1}{\partial x^n_2} \end{pmatrix}} \cancel{\begin{pmatrix} \delta^k_l & 0 \\ -b^{(1)}_{kl}(x_1) & \delta^l_k \end{pmatrix}} \cdot$$

$$\cdot \cancel{\begin{pmatrix} \delta^l_p & 0 \\ b^{(1)}_{lp}(x_1) & \delta^p_l \end{pmatrix}} \boxed{\begin{pmatrix} \frac{\partial x^p_1}{\partial x^q} & 0 \\ 0 & \frac{\partial x^q}{\partial x^p_1} \end{pmatrix}} \begin{pmatrix} \delta^q_r & 0 \\ -b_{qr}(x) & \delta^r_q \end{pmatrix} \begin{pmatrix} w^r(x) \\ \tilde{w}_r(x) \end{pmatrix}$$

$$= \begin{pmatrix} \delta^m_n & 0 \\ b^{(2;1)}_{mn}(x_2) & \delta^n_m \end{pmatrix} \begin{pmatrix} \frac{\partial x^n_2}{\partial x^k} & 0 \\ 0 & \frac{\partial x^k}{\partial x^n_2} \end{pmatrix} \begin{pmatrix} \delta^k_l & 0 \\ -b_{kl}(x) & \delta^l_k \end{pmatrix} \begin{pmatrix} w^l(x) \\ \tilde{w}_l(x) \end{pmatrix} .$$

② $x^M \longrightarrow x^M_2$

$b_{mn} \to b^{(21)}_{mn}$

$$W^M_{(21)}(x_2) = \begin{pmatrix} \delta^m_n & 0 \\ b^{(21)}_{mn}(x_2) & \delta^n_m \end{pmatrix} \begin{pmatrix} \frac{\partial x^n_2}{\partial x^k} & 0 \\ 0 & \frac{\partial x^k}{\partial x^n_2} \end{pmatrix} \begin{pmatrix} \delta^k_l & 0 \\ -b_{kl}(x) & \delta^l_k \end{pmatrix} \begin{pmatrix} w^l(x) \\ \tilde{w}_l(x) \end{pmatrix} .$$

If $\boxed{b^{(2;1)}_{mn}(x) = b^{(21)}_{mn}(x) .}$ ⇨ **the composition law is satisfied!**

Composition law (2/2)

① $x^M \to x_1^M \to x_2^M$

$b_{mn} \to b^{(1)}_{mn} \to b^{(2;1)}_{mn}$

$$b^{(2;1)}_{mn}(x_1) = b^{(1)}_{mn}(x_1) + 2\,\partial_{[m}\zeta^{(s=1,V_2;V_1)}_{n]}(x_1)$$

pullback

$$\zeta^{(s,V_2;V_1)}_m \equiv \sum_{k=1}^{\infty} \frac{s^k}{k!}\left(\pounds^{k-1}_{v_2}(\tilde{v}_2 + \iota_{v_2} b^{(1)})\right)_m(x_1)$$

$$b^{(2;1)}_{mn}(x) = b_{mn}(x) + 2\,\partial_{[m}\left(\zeta^{(s=1,V_1)}_{n]} + \zeta^{(s=1,V_2;V_1)}_{n]}\right)(x)$$

Trajectory:

$x_1^M \to x_2^M$

$$\tilde{x}^s_m(x_1) = \tilde{x}^1_m + \zeta^{(s,V_2;V_1)}_m(x_1)$$

pullback

$$\tilde{x}^s_m(x) = \tilde{x}^1_m(x) + \frac{\partial x_1^n}{\partial x^m}\,\zeta^{(s,V_2;V_1)}_n(x_1)$$

$$= \tilde{x}_m + \zeta^{(s=1,V_1)}_m(x) + \zeta^{(s,V_2;V_1)}_m(x)$$

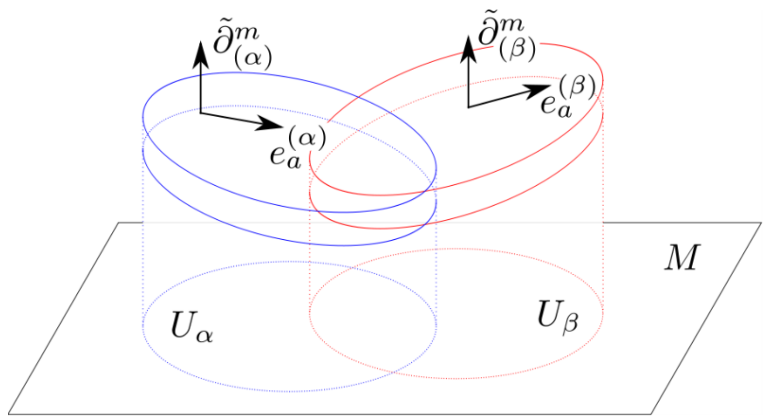
② $x^M \longrightarrow x_2^M$

$$\tilde{x}^2_m(x) = \tilde{x}_m + \underbrace{\left(\zeta^{(s=1,V_1)}_m + \zeta^{(s=1,V_2;V_1)}_m\right)(x)}_{\text{combined displacement}}$$

$$\Rightarrow \quad b^{(21)}_{mn}(x) = b_{mn}(x) + 2\,\partial_{[m}\left(\zeta^{(s=1,V_1)}_{n]} + \zeta^{(s=1,V_2;V_1)}_{n]}\right)(x)$$

$\therefore \quad \boxed{b^{(2;1)}_{mn}(x) = b^{(21)}_{mn}(x)\,.} \quad \Rightarrow$ **the composition law is satisfied!**

Patching condition



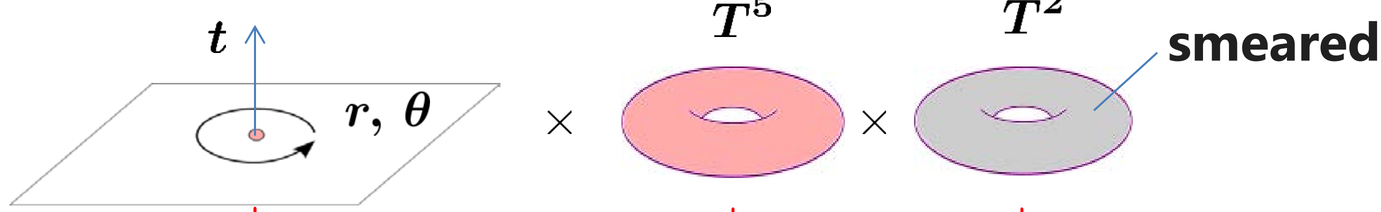
$$\begin{cases} x^m_{(\alpha)} = x^m_{(\alpha\beta)}(x_{(\beta)}) \\ b_{(\alpha)mn} = b_{(\beta)mn} + 2\,\partial_{[m}\zeta_{(\alpha\beta)n]} \end{cases}$$

In general, we need to patch open sets with different foliations .

$$W^M_{(\alpha)} = \underbrace{\begin{pmatrix} \delta^m_k & 0 \\ b^{(\alpha)}_{mk} & \delta^k_m \end{pmatrix} \begin{pmatrix} \frac{\partial x^k_{(\alpha)}}{\partial x^l_{(\beta)}} & 0 \\ 0 & \frac{\partial x^l_{(\beta)}}{\partial x^k_{(\alpha)}} \end{pmatrix} \begin{pmatrix} \delta^l_n & 0 \\ -b_{(\beta)ln} & \delta^n_l \end{pmatrix}}_{\mathcal{S}^M{}_N} W^N_{(\beta)}(x_{(\beta)})\,.$$

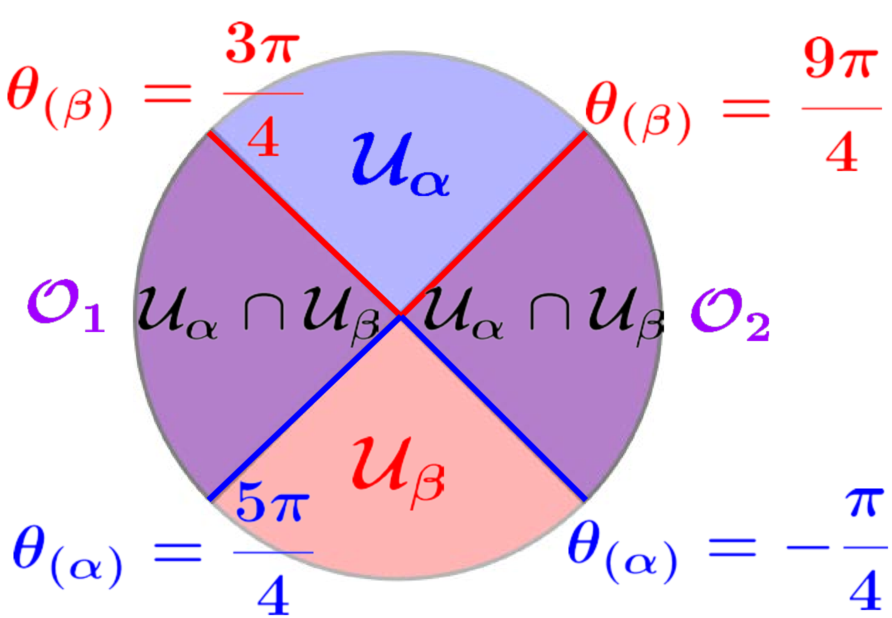
Example: (smeared) NS5-brane (1/2)

Background of (smeared) NS5 branes



$$ds^2 = \overbrace{-dt^2 + H(r)\left(dr^2 + r^2\, d\theta^2\right)} + \overbrace{dx^2_{3\cdots 7}} + \overbrace{H(r)\, dx^2_{89}}\,,$$

$$B^{(2)} = \frac{\sigma\,\theta}{2\pi}\, dx^8 \wedge dx^9\,, \quad e^{2\phi} = H(r)\,, \quad H(r) \equiv \frac{\sigma}{2\pi}\log(r_c/r)\,. \qquad \sigma \equiv \frac{l_s^2}{R_8 R_9}$$



$$B^{(2)}_{(\alpha)} = \frac{\sigma\,\theta_{(\alpha)}}{2\pi}\, dx^8 \wedge dx^9$$

$-\pi/4$ $\quad \mathcal{U}_\alpha \quad$ $5\pi/4$ $\;\theta_{(\alpha)}$ ·······

$\mathcal{O}_2$ $\quad \mathcal{O}_1 \quad$ $\mathcal{O}_2$ $\;\theta_{(\beta)}$

$3\pi/4$ $\quad \mathcal{U}_\beta \quad$ $9\pi/4$

$$B^{(2)}_{(\beta)} = \frac{\sigma\left(\theta_{(\beta)} - 2\pi\right)}{2\pi}\, dx^8 \wedge dx^9$$

Example: (smeared) NS5-brane (2/2)

$$\mathcal{O}_1 \begin{cases} x^m_{(\alpha)} = x^m_{(\beta)} \\ b_{(\alpha)89} = b_{(\beta)89} + \sigma \\ (\tilde{x}_{(\alpha)9} = \tilde{x}_{(\beta)9} + \sigma x^8_{(\beta)}) \end{cases}$$

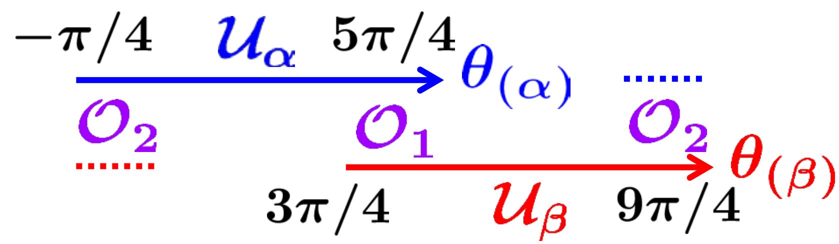
$$\mathcal{O}_2 \begin{cases} \theta_{(\alpha)} = \theta_{(\beta)} - 2\pi \\ x^{(\text{other})}_{(\alpha)} = x^{(\text{other})}_{(\beta)} \\ b_{(\alpha)89} = b_{(\beta)89} + \sigma \end{cases}$$

gen. Killing vector

$$\mathrm{e}^{\hat{\pounds}_\xi} \qquad \xi = 2\pi\partial_\theta - \sigma x^8\,\tilde{\partial}^9$$

$$B^{(2)}_{(\alpha)} = B^{(2)}_{(\beta)}$$

$$B^{(2)}_{(\alpha)} = \frac{\sigma\,\theta_{(\alpha)}}{2\pi}\,\mathrm{d}x^8 \wedge \mathrm{d}x^9$$



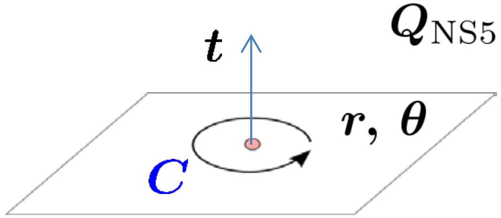
$$B^{(2)}_{(\beta)} = \frac{\sigma\,(\theta_{(\beta)} - 2\pi)}{2\pi}\,\mathrm{d}x^8 \wedge \mathrm{d}x^9$$

$$b_{(\beta)89} = 0 \;\rightarrow\; b_{(\alpha)89} = \sigma$$

In the presence of H-flux, we cannot chose a global section (foliation).

(such as $b_{mn} = 0$)

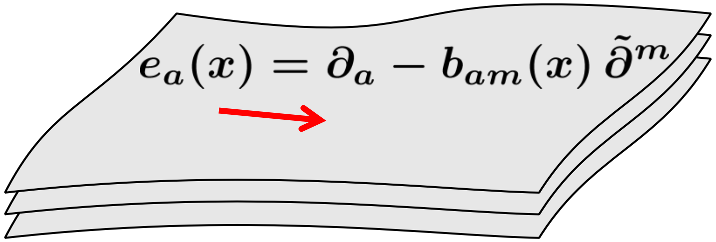
NS5-brane charge



$$Q_{\text{NS5}} \equiv \frac{2\pi}{\sigma}\int_C \mathrm{d}B_{89} = \frac{2\pi}{\sigma}\Big[\int_{\theta=0}^{\theta=\pi} \mathrm{d}B_{(\alpha)89} + \int_{\theta=\pi}^{\theta=2\pi} \mathrm{d}B_{(\beta)89}\Big]$$

$$= \frac{2\pi}{\sigma}\,\big[B_{(\alpha)89} - B_{(\beta)89}\big]^{\theta=\pi}_{\theta=0} = 2\pi\,.$$

Summary: our approach



$$\boxed{\begin{aligned} &b_{mn} = b_{[mn]}\,, \qquad \partial_{[l}b_{mn]} = 0\,, \\ &\delta_V b_{mn}(x) = \pounds_v b_{mn}(x) + 2\,\partial_{[m}\tilde{v}_{n]}(x)\,. \end{aligned}}$$

$$W^M_{(s,V)}(x) \equiv \mathrm{e}^{\,s\hat{\pounds}_V}\,W^M(x) = \mathcal{S}^M{}_N\,W^N(x_s)\,, \qquad x^m_s \equiv \mathrm{e}^{\,s\,v}\,x^m\,.$$

$$\mathcal{S}^M{}_N \equiv \begin{pmatrix} \delta^m_k & 0 \\ b^{(s,V)}_{mk}(x) & \delta^k_m \end{pmatrix} \begin{pmatrix} \frac{\partial x^k}{\partial x^l_s} & 0 \\ 0 & \frac{\partial x^l_s}{\partial x^k} \end{pmatrix} \begin{pmatrix} \delta^l_n & 0 \\ -b_{ln}(x_s) & \delta^n_l \end{pmatrix}.$$

$$b^{(s,V)}_{mn}(x) \equiv b_{mn}(x) + 2\,\partial_{[m}\zeta^{(s,V)}_{n]}(x)\,. \quad \begin{cases} \zeta^{(s,V)}_m(x) \equiv \int_0^s \mathrm{d}s'\,\hat{\mathrm{v}}^{(s',V)}_m(x)\,. \\ \hat{\mathrm{v}}^{(s,V)}_m(x) \equiv \dfrac{\partial x^n_s}{\partial x^m}\,\hat{\mathrm{v}}_n(x_s)\,. \end{cases}$$

Recalling the trivial coord. gauge sym., $\tilde{x}_m \sim \tilde{x}_m + \partial_m f\,,$
we introduce a diffeo. in the dual directions: $\tilde{x}_m \to \tilde{x}_m + \zeta^{(s,V)}_m(x)$.

⇨ **Composition law is explicitly shown*!***

§3. Finite transformations in SL(5) EFT

S-J. Rey, YS, arXiv:1510.06735.

N. Chaemjumrus, C.M. Hull, arXiv:1512.03837

SL(5) + SO(5,5) + E_6 EFT

Review: Exceptional Field Theory (1/3)

M-theory on n-torus : U-duality group : E_n.

Example $M_{11} = \mathbb{R}^{1,6} \times T^4$. U-duality group : SL(5)

x^μ x^i

7-dim. 4-dim.

$$(G_{MN},\ C_{MNP}) \longrightarrow (G_{ij},\ C_{ijk})$$

[Berman, Perry, '10]

$$\mathcal{M}_{MN} \equiv \begin{pmatrix} G_{ij} + \frac{1}{2} C_{ikl}\, C^{kl}{}_j & \frac{1}{\sqrt{2}} C_i{}^{j_1 j_2} \\ \frac{1}{\sqrt{2}} C^{i_1 i_2}{}_j & G^{i_1 i_2, j_1 j_2} \end{pmatrix} .$$

$$x^i \longrightarrow x^M = (x^i,\ y_{ij})$$ SL(5)-Exceptional space

4-dim. 4 + 6 = 10-dim.

$$\begin{aligned} \mathcal{L} = & \frac{1}{12} \mathcal{M}^{MN} \partial_M \mathcal{M}^{KL} \partial_N \mathcal{M}_{KL} - \frac{1}{2} \mathcal{M}^{MN} \partial_N \mathcal{M}^{KL} \partial_L \mathcal{M}_{MK} \\ & + \frac{1}{12} \mathcal{M}^{MN} (\mathcal{M}^{KL} \partial_M \mathcal{M}_{KL}) (\mathcal{M}^{RS} \partial_N \mathcal{M}_{RS}) \\ & + \frac{1}{4} \mathcal{M}^{MN} \mathcal{M}^{PQ} (\mathcal{M}^{RS} \partial_P \mathcal{M}_{RS}) (\partial_M \mathcal{M}_{NQ}) . \end{aligned}$$

Review: Exceptional Field Theory (2/3)

$(x^i,\ y_{ij})$ ⇨

※ SL(5)-manifest coordinates:

$$x^{ab} = x^{[ab]} \quad (a, b = 1, \dots, 5)\,.$$

$$\left(x^{i5} = x^i = -x^{5i}\,,\quad x^{ij} = \tfrac{1}{2}\,\epsilon^{ijkl}\,y_{kl}\right).$$

Consistency of the theory (section condition)

[Berman, Perry, '10]

$$\frac{\partial}{\partial y_{ij}} * = 0\,.$$

$$\boxed{\frac{\partial}{\partial x^{[ab}} * \frac{\partial}{\partial x^{cd]}} * = 0\,.}$$

$\boxed{y_{12},\ y_{13},\ y_{23}}$

$\cancel{y_{14},\ y_{24},\ y_{34}}$

$\cancel{x^i}$

7-dim.+4-dim.=11-diml theory

$$\mathcal{M}_{MN} \equiv \begin{pmatrix} G_{ij} + \frac{1}{2}\,C_{ikl}\,C^{kl}{}_j & \frac{1}{\sqrt{2}}\,C_i{}^{j_1 j_2} \\ \frac{1}{\sqrt{2}}\,C^{i_1 i_2}{}_j & G^{i_1 i_2, j_1 j_2} \end{pmatrix}.$$

$$\mathcal{L}_{\text{EFT}} = R(G) - \tfrac{1}{2}\,|F^{(4)}|^2\,.$$

$$M_{10} = \mathbb{R}^{1,6} \times \tilde{T}^3\,.$$

[Blair, Malek, J-H Park, '14]

$$\mathcal{M}_{MN} = \left(\begin{matrix} G_{ij},\ B_{ij},\ \phi, \\ C^{(0)},\ C^{(2)}\,. \end{matrix} \right)$$

7-dim.+3-dim.=10-diml theory

$$\mathcal{L}_{\text{EFT}} = \mathcal{L}_{\text{type IIB SUGRA}}\,.$$

EFT unifies the 11-diml SUGRA and type IIB SUGRA

Review: Exceptional Field Theory (3/3)

SL(5)-EFT action

$$\mathcal{L} = \frac{1}{12}\mathcal{M}^{MN}\,\partial_M\mathcal{M}^{KL}\,\partial_N\mathcal{M}_{KL} - \frac{1}{2}\mathcal{M}^{MN}\,\partial_N\mathcal{M}^{KL}\,\partial_L\mathcal{M}_{MK} + \frac{1}{12}\mathcal{M}^{MN}\left(\mathcal{M}^{KL}\,\partial_M\mathcal{M}_{KL}\right)\left(\mathcal{M}^{RS}\,\partial_N\mathcal{M}_{RS}\right) + \frac{1}{4}\mathcal{M}^{MN}\,\mathcal{M}^{PQ}\left(\mathcal{M}^{RS}\,\partial_P\mathcal{M}_{RS}\right)\left(\partial_M\mathcal{M}_{NQ}\right).$$

Gauge symmetry ... **Generalized Lie derivative**

$$\hat{\pounds}_V W^A = \underbrace{V^B\,\partial_B W^A - W^B\,\partial_B V^A}_{\text{Lie derivative}} + \varepsilon^{eAB}\,\varepsilon_{eCD}\,\partial_B V^C\,W^D\,.$$

10-dim $A = [a_1 a_2]$

$\varepsilon^{e a_1 a_2 b_1 b_2}\quad (\varepsilon^{12345} = 1)$

$$\delta_V\mathcal{M}_{MN} = \hat{\pounds}_V\mathcal{M}_{MN}$$

$$\mathcal{M}_{MN} \equiv \begin{pmatrix} G_{ij} + \frac{1}{2}C_{ikl}\,C^{kl}{}_j & \frac{1}{\sqrt{2}}C_i{}^{j_1 j_2} \\ \frac{1}{\sqrt{2}}C^{i_1 i_2}{}_j & G^{i_1 i_2, j_1 j_2} \end{pmatrix}.$$

$$\frac{\partial}{\partial y_{ij}} * = 0\,.$$

$$\begin{cases} \delta_V G_{ij}(x) = \pounds_v G_{ij}(x)\,, \\ \delta_V C_{ijk}(x) = \pounds_v C_{ijk}(x) + 3\,\partial_{[i}\tilde{v}_{jk]}(x)\,. \end{cases}$$

Diffeo + gauge transf. of 3-form pot.

There was no proposal for the Finite transf. law.

Finite transf. law in SL(5) EFT

Coordinates : $x^M = (x^i,\ y_{ij}) \qquad (\leftrightarrow x^A = x^{[ab]})$

Gen. vector : $W^M(x) \equiv \begin{pmatrix} w^i(x) \\ \frac{1}{\sqrt{2}}\,\tilde{w}_{i_1 i_2}(x) \end{pmatrix}. \qquad \delta_V W^M = \hat{\pounds}_V W^M\,.$

Untwisted vector :

$$\hat{W}^M(x) \equiv \begin{pmatrix} w^i(x) \\ \frac{1}{\sqrt{2}}\,\hat{w}_{i_1 i_2}(x) \end{pmatrix} \equiv \begin{pmatrix} \delta^i_j & 0 \\ \frac{1}{\sqrt{2}}\,C_{i_1 i_2 j}(x) & \delta^{j_1 j_2}_{i_1 i_2} \end{pmatrix} \begin{pmatrix} w^j(x) \\ \frac{1}{\sqrt{2}}\,\tilde{w}_{j_1 j_2}(x) \end{pmatrix}.$$

$$\Rightarrow \left\{ \begin{array}{ll} \delta_V w^i(x) = \pounds_v w^i(x)\,, & \longleftarrow \textbf{vector} \\ \delta_V \hat{w}_{ij}(x) = \pounds_v \hat{w}_{ij}(x)\,. & \longleftarrow \textbf{2-form !} \end{array} \right.$$

Finite version $\Rightarrow$

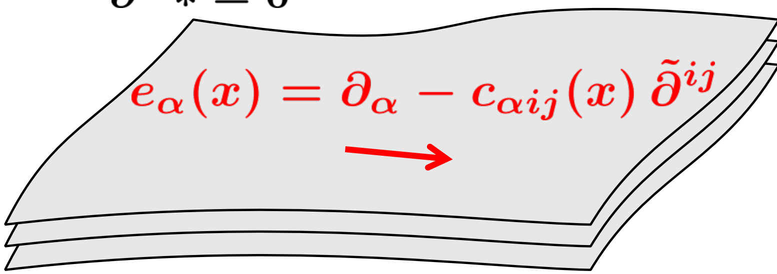
$$\hat{W}^M_{(s,V)}(x) = \begin{pmatrix} \frac{\partial x^i}{\partial x^j_s} & 0 \\ 0 & \frac{\partial x_s^{[j_1}}{\partial x^{[i_1}} \frac{\partial x_s^{j_2]}}{\partial x^{i_2]}} \end{pmatrix} \begin{pmatrix} w^j(x_s) \\ \frac{1}{\sqrt{2}}\,\hat{w}_{j_1 j_2}(x_s) \end{pmatrix}.$$

Our task : to obtain a finite transf. for $C_{ijk}(x)$.

Finite transf. law for $c_{\alpha ij}$

$$\frac{\partial}{\partial y_{ij}}$$

$$\tilde{\partial}^{ij} * = 0$$



4-dim subspace in 10-diml Exceptional space

$$c_{\alpha ij} = c_{ij\alpha} \longleftarrow \epsilon_{eAB}\, V^A_{(1)}\, V^B_{(2)} = 0\,.$$

Gauge fixing : $e^i{}_\alpha = \delta^i_\alpha$ $\Rightarrow$ $\delta_V c_{ijk}(x) = \pounds_v c_{ijk}(x) + 3\, \partial_{[i}\tilde{v}_{jk]}(x)\,.$

Involutive property $\Rightarrow$ $\partial_{[i}c_{jkl]}(x) = 0\,.$

Diff. eqs.

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}\, x^i = v^i(x)\,, \\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}s}\, c_{ijk}(x) = \pounds_v c_{ijk}(x) + 3\, \partial_{[i}\tilde{v}_{jk]}(x)\,. \end{cases}$$

Solution

$$\begin{cases} x^i_s = \mathrm{e}^{\,s\,v}\, x^i\,, \\ c^{(s,V)}_{ijk}(x) = c_{ijk}(x) + 3\, \partial_{[i}\zeta^{(s,V)}_{jk]}(x)\,. \end{cases}$$

$$\begin{cases} \zeta^{(s,V)}_{ij}(x) \equiv \displaystyle\int_0^s \mathrm{d}s'\, \hat{\mathbf{v}}^{(s',V)}_{ij}(x)\,. \\[2ex] \hat{\mathbf{v}}^{(s,V)}_{ij}(x) \equiv \dfrac{\partial x^k_s}{\partial x^i}\, \dfrac{\partial x^l_s}{\partial x^j}\, \hat{\mathbf{v}}_{kl}(x_s)\,. \\ \qquad (\hat{\mathbf{v}}_{ij} \equiv \tilde{v}_{ij} - c_{ijk}\, v^k)\,. \end{cases}$$

Results

Similar to the case of DFT:

$$W^M_{(s,V)}(x) = \mathcal{S}^M{}_N \, W^N(x_s) \, ,$$

$$\mathcal{S}^M{}_N \equiv \begin{pmatrix} \delta^i_k & 0 \\ -\frac{1}{\sqrt{2}} \, c^{(s,V)}_{i_1 i_2 k}(x) & \delta^{k_1 k_2}_{i_1 i_2} \end{pmatrix} \begin{pmatrix} \frac{\partial x^k}{\partial x^l_s} & 0 \\ 0 & \frac{\partial x_s^{[l_1}}{\partial x^{[k_1}} \frac{\partial x_s^{l_2]}}{\partial x^{k_2]}} \end{pmatrix} \begin{pmatrix} \delta^l_j & 0 \\ \frac{1}{\sqrt{2}} \, c_{l_1 l_2 j}(x_s) & \delta^{j_1 j_2}_{l_1 l_2} \end{pmatrix} .$$

Future works:

M-theory on n-torus : U-duality group : E_n .

n=4 → n= 5, 6, 7, (8?), ... **[Chaemjumrus, Hull, '15]**

generalization is straightforward.

Finite transformation in non-geometric BG in EFT.

c.f. [K. Lee, S-J. Rey, YS, work in progress]

Summary

- In **Hohm-Zwiebach's proposal** for the finite transfs., there was an **issue in the composition**.
- We proposed a **new transformation law**, which **satisfies the composition law** as usual in GR.
- We introduced a **foliation by *d*-dim'l null subspace**, and proposed a **patching condition** between open sets with different foliations. ***(Dirac manifold)***
- We obtained a fin. transf. law in non-geom. BG, and studied a **patching condition for a T-fold**. (5^2_2-brane) **(skipped today)**
- We applied our procedure to **SL(5) EFT**, and obtained a finite transf. Law.