Supplements for the lecture note

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Summary of Axioms

General axioms

(A1) Reflexivity. $X \overset{A}{\sim} X$.

(A2) Transitivity. $X \prec Y$ & $Y \prec Z$ implies $X \prec Z$.

(A3) Consistency. If $X \prec X'$ and $Y \prec Y'$, then $(X, Y) \prec (X', Y')$.

(A4) Scaling invariance. If $X \prec Y$, then $tX \prec tY$ for all $t > 0$.

(A5) Splitting and recombination. For $0 < t < 1$, $X \overset{A}{\sim} (tX, (1-t)X)$.

(A6) Stability. If $(X, \varepsilon Z_0) \prec (Y, \varepsilon Z_1)$ holds for a sequence of ε's tending to zero and some states Z_0, Z_1, then $X \prec Y$.

(A7) Convex combination. Assume X and Y are states in the same state space, Γ, that has a convex structure. If $t \in [0, 1]$ then
$(tX, (1-t)Y) \prec tX + (1-t)Y$.

Axioms for simple systems

(S1) Irreversibility. For each $X \in \Gamma$ there is a point $Y \in \Gamma$ such that $X \prec\prec Y$. (Note: This axiom is implied by T4, and hence it is not really independent.)

(S2) Lipschitz tangent planes. For each $X \in \Gamma$ the forward sector $A_X = \{Y \in \Gamma : X \prec Y\}$ has a *unique* support plane at X (i.e., A_X has a *tangent plane* at X). The slope of the tangent plane is assumed to be a *locally Lipschitz continuous* function of X.

(S3) Connectedness of the boundary. The boundary ∂A_X of a forward sector is connected.

Axioms for thermal equilibrium

(T1) Thermal contact. For any two simple systems with state spaces Γ_1 and Γ_2, there is another simple system, the *thermal join* of Γ_1 and Γ_2, with state space

$$\Delta_{12} = \{(U, V_1, V_2) | U = U_1 + U_2\}$$

with $(U_1, V_1) \in \Gamma_1, (U_2, V_2) \in \Gamma_2$. Moreover,

$$((U, V_1),\ (U_2, V_2)) \prec (U_1 + U_2, V_1, V_2)$$

(T2) Thermal splitting. For any point (U, V_1, V_2) in Δ_{12} there is at least one pair of states, $(U_1, V_1) \in \Gamma_1$, $(U_2, V_2)) \in \Gamma_2$, with $U = U_1 + U_2$, such that

$$(U, V_1, V_2) \overset{A}{\sim} ((U_1, V_1), (U_2, V_2)).$$

In particular, if (U, V) is a state of a simple system Γ and $\lambda \in [0, 1]$ then

$$(U, (1-\lambda)V, \lambda V) \overset{A}{\sim} (((1-\lambda)U, (1-\lambda)V), (\lambda U, \lambda V)).$$

If $(U, V_1, V_2) \overset{A}{\sim} ((U_1, V_1), (U_2, V_2))$ we write $(U_1, V_1) \overset{T}{\sim} (U_2, V_2)$.

(T3) Zeroth law. $X \overset{T}{\sim} Y$ & $Y \overset{T}{\sim} Z$ implies $X \overset{T}{\sim} Z$.

(T4) Transversality. If Γ is the state space of a simple system and if $X \in \Gamma$, then there exist states $X_0 \overset{T}{\sim} X_1$ with $X_0 \prec\prec X \prec\prec X_1$.

(T5) Universal temperature range. If Γ_1 and Γ_2 are state spaces of simple systems then, for every $X \in \Gamma_1$ and every V in the projection of Γ_2 onto the space of its work coordinates, there is a $Y \in \Gamma_2$ with work coordinates V such that $X \overset{T}{\sim} Y$.

Axiom for mixtures and reactions

(M) Absence of sinks. If Γ is connected to Γ' then Γ' is connected to Γ, i.e., $\Gamma \prec \Gamma'$ implies $\Gamma' \prec \Gamma$.

Transitivity of $\overset{A}{\sim}$

If $X \overset{A}{\sim} Y$ and $Y \overset{A}{\sim} Z$, then $X \overset{A}{\sim} Z$.

Pf) Since $X \prec Y$ and $Y \prec Z$, A2 gives $X \prec Z$. Likewise, using $Z \prec Y$ and $Y \prec X$, we arrive at $Z \prec X$. This completes the proof.

CP in multiple scaled copies

The proof resorts to mathematical induction for fixed but arbitrary N. First we observe that if $M = N+1$ with $t_i = t'_i$ ($i < N$) and $t_N = t'_N + t'_M$, then splitting $\Gamma^{(t_N)}$ into $\Gamma^{(t'_N)} \times \Gamma^{(t'_M)}$ shows that Y and Y' are comparable. Now we will prove the general cases.

Step 1. If $M = 1$, we can split Y' into N copies. Hence Y and Y' are comparable.

Step 2. Now assume CP is valid for M and consider Y' with $M+1$ multiple scaled copies. Set $t''_i = t'_i$ for $i = 1, \dots, M-1$ and $t''_M = t'_M + t''_{M+1}$. Then any state in a scaled state space Γ'' with two-primed parameters is comparable with that in a scaled state space Γ' with one-primed parameters as shown above. Since Y and $Y'' \in \Gamma''$ are comparable by assumption and comparability is an equivalence relation, Y and Y' are comparable.

Hence, by mathematical induction, the proof is complete.

Extension of A5

A5 says $X \overset{A}{\sim} (tX, (1-t)X)$ for $0 < t < 1$. By A4, we get

$$\frac{1}{t}X \overset{A}{\sim} \left(X, \frac{1-t}{t}X\right).$$

By definition of the generalized ordering, we write

$$X \overset{A}{\sim} \left(\frac{1}{t}X, \frac{t-1}{t}X\right).$$

If we define either $a = 1/t$ ($a > 1$) or $a = (t-1)/t$ ($a < 0$), we can write

$$X \overset{A}{\sim} (aX, (1-a)X)$$

for all $a \in R$, where we have used $(X, Y) \overset{A}{\sim} (Y, X)$. When $a = 0$ or 1, the above relation is trivially met.

Lemma 2.1

(i) If $X_0 \prec X$, $X \in \mathscr{S}_0$. If $X \prec\prec X_0$, we claim that $(1+\alpha)X_0 \prec (\alpha X_1, X)$ for some $\alpha > 0$, which in turn gives $\lambda = -\alpha$. Otherwise, $(\alpha X_1, X) \prec (1+\alpha)X_0$ for all $\alpha > 0$ by the CP. Thus, by A4 and A5, $(X_1, X/\alpha) \prec (X_0, X_0/\alpha)$ for all $\alpha > 0$. Thus by A6, $X_1 \prec X_0$ in contrast to the assumption.

Strip and canonical entropy

Let $\lambda = S_\Gamma(X)$. If $X_0 \prec X$, $X_0 \prec ((1-\lambda)X_0, \lambda X_1)$ by Lemma 2.3. By the cancellation law, we get $\lambda X_0 \prec \lambda X_1$. Hence $\lambda \geq 0$. Likewise, if $X \prec X_1$, $((1-\lambda)X_0, \lambda X_1) \prec X_1$ and, in turn, $(1-\lambda)X_0 \prec (1-\lambda)X_1$. Thus, $\lambda \leq 1$.

Entropy for states outside the strip

Let $\lambda_1 = S_\Gamma(X_1|X_0, X)$ for $X_1 \prec X$. Then

$$X_1 \overset{A}{\sim} ((1-\lambda_1)X_0, \lambda_1 X) \Rightarrow X \overset{A}{\sim} \left(\frac{\lambda_1 - 1}{\lambda_1}X_0, \frac{1}{\lambda_1}X_1\right),$$

which gives $S_\Gamma(X|X_0, X_1) = 1/\lambda_1 = S_\Gamma(X_1|X_0, X)^{-1}$. Let $\lambda_2 = S_\Gamma(X_0|X, X_1)$ for $X \prec X_0$. Then,

$$X_0 \overset{A}{\sim} ((1-\lambda_2)X, \lambda_2 X_1) \Rightarrow X \overset{A}{\sim} \left(\frac{1}{1-\lambda_2}X_0, \frac{\lambda_2}{\lambda_2 - 1}X_1\right),$$

which gives

$$S_\Gamma(X|X_0, X_1) = -\frac{\lambda_2}{1-\lambda_2} = -\frac{S_\Gamma(X_0|X, X_1)}{1 - S_\Gamma(X_0|X, X_1)}.$$

Consistent entropy scale

Let $X \in \Gamma$ and $Y \in \Gamma'$ with $(X, Y) \in \Gamma \times \Gamma'$. Let $\lambda_1 = S(X)$, $\lambda_2 = S(Y)$ and $\lambda_{12} = S(X, Y)$. Then $(X, \lambda_1 Z_0) \overset{A}{\sim} (X_\Gamma, \lambda_1 Z_1)$ and $(Y, \lambda_2 Z_0) \overset{A}{\sim} (X_{\Gamma'}, \lambda_1 Z_1)$. By A3 and A5,

$$(X, Y, (\lambda_1 + \lambda_2)Z_0) \overset{A}{\sim} (X_\Gamma, X_{\Gamma'}, (\lambda_1 + \lambda_2)Z_1).$$

Since $((X, Y), \lambda_{12} Z_0) \overset{A}{\sim} (X_{\Gamma \times \Gamma'}, \lambda_{12} Z_1)$ by definition, $X_{\Gamma \times \Gamma'} = (X_\Gamma, X_{\Gamma'})$, and supremum is uniquely defined, we get

$$S(X, Y) = \lambda_{12} = \lambda_1 + \lambda_2 = S(X) + S(Y).$$

By A4, $t(X, \lambda_1 Z_0) \overset{A}{\sim} t(X_\Gamma, \lambda_1 Z_1)$ or $(tX, t\lambda_1 Z_0) \overset{A}{\sim} (tX_\Gamma, \lambda_1 Z_1)$. Since $tX_\Gamma = X_{t\Gamma}$, we get $S(tX) = tS(X)$.

Since $X \prec Y$ iff $(X, Z_0) \prec (Y, Z_0)$, S is an entropy function and $S(X) = a_\Gamma S_\Gamma(X) + B_\Gamma$.

Thm 2.7

To prove Thm 2.7, we use $1-\lambda = t(1-\lambda_1)+(1-t)(1-\lambda_2)$, where $\lambda = t\lambda_1 + (1-t)\lambda_2$. Using A3-A5 and A7, we get

$$\begin{aligned}&(\lambda X_0, (1-\lambda)X_1) \overset{A}{\sim}\\ &(t(\lambda_1 X_0, (1-\lambda_1)X_1), (1-t)(\lambda_1 X_0, (1-\lambda_1)X_1))\\ &\prec (tX, (1-t)Y) \prec tX + (1-t)Y,\end{aligned}$$

which gives $tX + (1-t)Y \in \mathscr{S}_\lambda$.

Gauge and Hamiltonian

Consider a (classical) Hamiltonian $H(p, q; t)$ with explicit time dependence. Since the equations of motion are

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q},$$

they are invariant by gauge transformation $H \mapsto H' := H + g(t)$, where $g(t)$ is an arbitrary function of time. In other words, H' gives the exactly same equations of motion as derived from H.

Let us assume that $g(0) = 0$ which gives the same energy of the system at $t = 0$ irrespective of whether we use H or H' for our Hamiltonian. However, the energy at $t > 0$ depends on which Hamiltonian we use although the (microscopic) dynamics are identical! In this context, it is meaningless to compare energy of two states solely from Hamiltonian!

In quantum mechanics, the time-dependent Schrödinger equation is invariant under the following gauge transformtion

$$|\psi\rangle \mapsto |\psi'\rangle := \exp\left(\frac{1}{i\hbar}\int_0^t g(t')dt'\right)|\psi\rangle.$$

That is, if $|\psi\rangle$ is the solution of

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H(t)|\psi\rangle,$$

$|\psi'\rangle$ defined above is the solution of

$$i\hbar\frac{\partial}{\partial t}|\psi'\rangle = H'(t)|\psi\rangle.$$

Notice that (time-independent) observables are not affected by the (global) gauge transformation.

Equivalence of CP and S1

1. S1 implies CP.

Assume **CP** is false. Then there is X and its neighborhood N_X such that $X \overset{A}{\sim} Z \ \forall_{Z\in N_X}$. By definition, $N_X \subset A_X$. Choose an aribrary $Y \in A_X$. There is a point $Z \in N_X$ such that a set of all convex combination of Y and Z contains X. Then by the mathematical logic in the lecture note, we can conclude that **S1** is false.

2. CP implies S1, once all forward sectors have non-empty interior.

Assume **S1** is false. Then there is X_0 whose forward sector A_{X_0} only has states adiabatically equivalent to X_0. By assumption, A_{X_0} should have an interior point X which has a neighborhood N_X contained in A_{X_0}. Thus, the logic in the lecture note shows that **CP** is false.

Lipschitz continous but not everywhere differentiable

Let $f(x) = |x|$. Since $|f(x)-f(y)| = ||x|-|y|| \leq |x-y|$ for any x and y, $f(x)$ is Lipschitz continuous, but $f'(0)$ is not defined.

Properties of concave functions

A function f defined on an open interval (a, b) is called a concave function if

$$\lambda f(x) + (1-\lambda)f(y) \leq f(\lambda x + (1-\lambda)y)$$

for all $a < x < y < b$ and $0 \leq \lambda \leq 1$.

Since for $a < s < u < t < b$

$$f(u) = f\left(\frac{t-u}{t-s}s + \frac{u-s}{t-s}t\right) \geq \frac{t-u}{t-s}f(s) + \frac{u-s}{t-s}f(t),$$

we get

$$\begin{aligned}f(u) - f(s) &\geq \frac{u-s}{t-s}\left(f(t) - f(s)\right),\\ \frac{f(u)-f(s)}{u-s} &\geq \frac{f(t)-f(s)}{t-s},\end{aligned} \tag{1}$$

and

$$f(u) \geq \frac{t-u}{t-s} f(s) + \frac{u-s}{t-s}(f(t)-f(u)) + \frac{u-s}{t-s} f(u),$$
$$0 \geq \frac{t-u}{t-s}(f(s)-f(u)) + \frac{u-s}{t-s}(f(t)-f(u)),$$
$$\frac{f(u)-f(s)}{u-s} \geq \frac{f(t)-f(u)}{t-u} \tag{2}$$

If $a < s < u < v < t < b$, we get

$$\frac{f(u)-f(s)}{u-s} \geq \frac{f(t)-f(u)}{t-u} \geq \frac{f(t)-f(v)}{t-v}, \tag{3}$$

where Eqs. (1) and (2) have been used.

Let $F_x(h) = [f(x+h) - f(x)]/h$ $(0 < h < b-x)$. If we set $s = x$, $t = x + h'$, and $u = x + h$ $(h < h')$ in Eq. (1), we get $F_s(h) \geq F_s(h')$, that is, $F_s(h)$ is an decreasing function of h. If we set $v = x$, $t = x + h$ in Eq. (3), we find that $F_x(h)$ is bounded from above. Thus, $\lim_{h->0^+} F_x(h)$ is well-defined for all $a < x < b$.

Likewise, let $G_x(h) = [f(x) - f(x-h)]/h$ $(0 < h < x-a)$. If we set $t = x$, $v = x-h$, $u = x-h'$ $(h < h')$ in Eq. (3), we find $G_x(h)$ is an increasing function of h. If we set $u = x$ and $s = x-h$ in Eq. (3), we conclude that $G_x(h)$ is bounded from below. Thus, $\lim_{h->0} G_x(h)$ is also well-defined. To summarize, any concave function has well-defined *one side derivative* at all points.

One can easily construct a concave function with different one side derivative. An example of a concave function with $f'_+(1) \neq f'_-(1)$ is

$$f(x) = \begin{cases} 1-x, & 0 < x < 1, \\ 2(1-x), & 1 \leq x < 2. \end{cases}$$

If we denote the derivative from above [below] by $f'_+(x)$ [$f'_-(x)$], Eq. (2) shows that $f'_-(x) \geq f'_+(x)$ for all x. Also by setting $s = x$, $t = y$ in Eq. (3) we can conclude that $f'_+(x) \geq f'_-(y)$ if $x < y$.

S is a concave function of U for fixed V

This theorem can be derived from **T1, T2**, and **A5** without resorting to **A7**. The proof is as follows. Set $t \in [0,1]$. By **T1**,

$$(((1-t)U, (1-t)V), (tU', tV)) \prec$$
$$((1-t)U + tU', (1-t)V, tV).$$

By **T2** with $U'' := (1-t)U + tU'$, the latter is $\overset{A}{\sim}$

$$(((1-t)U'', (1-t)V), (tU'', tV)) \overset{A}{\sim} (U'', V),$$

where **A5** has been used. Since S is additive and non decreasing under $\prec$, we get

$$(1-t)S(U,V) + tS(U',V) \leq S((1-t)U + tU', V),$$