Lecture notes on non-Fermi liquid physics near quantum criticality

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a. How do quasiparticles disappear, approaching a quantum critical point?
a. Can we find ω/T scaling in dynamic correlation functions beyond the Hertz-Moriya-Millis theory?

2. Mott quantum criticality

a. How do quasiparticles disappear, approaching a Mott quantum critical point?
b. How does a Fermi surface disappear, approaching a Mott quantum critical point?
c. Can we obtain localized magnetic moments in the Fermi-surface problem?

III. LANDAU'S FERMI-LIQUID THEORY

A. Effective field theory

1. Introduction to the Path-integral representation

$$H = -t \sum_{ij} (c_i^\dagger c_j + H.c.) = \sum_k \epsilon_k c_k^\dagger c_k, \qquad \epsilon_k = -2t \sum_j \cos k_j \tag{1}$$

$$Z = \mathrm{tr} e^{-\beta(H-\mu N)} \tag{2}$$

a. Fock-basis representation

$$Z = \sum_{\{n_k\}=0}^{1} \exp\Big\{ -\beta \sum_k (\epsilon_k - \mu) n_k \Big\} = \Pi_k \Big\{ 1 + e^{-\beta(\epsilon_k - \mu)} \Big\} \tag{3}$$

$$F = -\frac{1}{\beta} \ln Z = -\frac{1}{\beta} \sum_k \ln \Big\{ 1 + e^{-\beta(\epsilon_k - \mu)} \Big\} \tag{4}$$

b. Coherent or Wave-function basis representation

$$\begin{aligned} Z &= \int Dc_i \exp\Big[-\int_0^\beta d\tau \Big\{ \sum_i c_i^\dagger (\partial_\tau - \mu) c_i - t \sum_{ij} (c_i^\dagger c_j + H.c.) \Big\} \Big] \\ &= \int Dc_k \exp\Big[-\sum_{i\omega} \sum_k c_k^\dagger (-i\omega - \mu + \epsilon_k) c_k \Big] = \mathcal{C} \Pi_{i\omega} \Pi_k (-i\omega - \mu + \epsilon_k) \end{aligned} \tag{5}$$

$$i\omega = i\frac{(2n+1)\pi}{\beta} \tag{6}$$

$$P = \int D\boldsymbol{c}^{\dagger} D\boldsymbol{c} \exp\Big\{ - \boldsymbol{c}^{\dagger}\boldsymbol{M}\boldsymbol{c}\Big\} = \mathcal{C}\det\boldsymbol{M} \tag{7}$$

$$F = -\frac{1}{\beta}\sum_{i\omega}\sum_{k} \ln(-i\omega - \mu + \epsilon_k) = -\frac{1}{\beta}\sum_{k}\ln\Big\{1 + e^{-\beta(\epsilon_k - \mu)}\Big\} \tag{8}$$

2. Effective field theory: Spinless fermions

$$Z = \int Dc_{\boldsymbol{p}} \exp\Big[- \int_0^{\beta} d\tau \Big\{ \sum_{\boldsymbol{p}} c_{\boldsymbol{p}}^{\dagger}(\partial_\tau - \mu + \epsilon_{\boldsymbol{p}})c_{\boldsymbol{p}} + \sum_{\boldsymbol{q}} V_{\boldsymbol{q}} \sum_{\boldsymbol{p}}\sum_{\boldsymbol{p}'} c_{\boldsymbol{p}+\boldsymbol{q}}^{\dagger} c_{\boldsymbol{p}'-\boldsymbol{q}}^{\dagger} c_{\boldsymbol{p}'} c_{\boldsymbol{p}} \Big\}\Big] \tag{9}$$

$$Z = \exp\Big(- \beta F_{FL}\Big) \tag{10}$$

B. Self-consistent mean-field theory and Boltzmann transport theory

1. Free-energy functional in terms of an order parameter

$$\text{Order parameter} = \delta n(\boldsymbol{p};\boldsymbol{r},t) = n(\boldsymbol{p};\boldsymbol{r},t) - n_{eq}(\boldsymbol{p}), \qquad n_{eq}(\boldsymbol{p}) = \frac{1}{\exp\Big\{\beta\Big(\varepsilon_{qp}(\boldsymbol{p}) - \mu\Big)\Big\} + 1} \tag{11}$$

$$E_{FL}[\delta n(\boldsymbol{p};\boldsymbol{r},t)] = \sum_{\boldsymbol{p}} \Big(\varepsilon_{qp}(\boldsymbol{p}) - \mu\Big)\delta n(\boldsymbol{p};\boldsymbol{r},t) + \sum_{\boldsymbol{p}}\sum_{\boldsymbol{p}'} F_{\boldsymbol{p}\boldsymbol{p}'}\delta n(\boldsymbol{p};\boldsymbol{r},t)\delta n(\boldsymbol{p}';\boldsymbol{r},t) \tag{12}$$

$$S[n(\boldsymbol{p};\boldsymbol{r},t)] = k_B \sum_{\boldsymbol{p}} \Big\{ n(\boldsymbol{p};\boldsymbol{r},t)\ln n(\boldsymbol{p};\boldsymbol{r},t) + [1 - n(\boldsymbol{p};\boldsymbol{r},t)]\ln[1 - n(\boldsymbol{p};\boldsymbol{r},t)]\Big\} \tag{13}$$

$$\begin{aligned} &F_{FL}[\delta n(\boldsymbol{p};\boldsymbol{r},t)] = E_{FL}[\delta n(\boldsymbol{p};\boldsymbol{r},t)] - TS[n(\boldsymbol{p};\boldsymbol{r},t)] \\ &= \sum_{\boldsymbol{p}} \Big(\varepsilon_{qp}(\boldsymbol{p}) - \mu\Big)\delta n(\boldsymbol{p};\boldsymbol{r},t) + \sum_{\boldsymbol{p}}\sum_{\boldsymbol{p}'} F_{\boldsymbol{p}\boldsymbol{p}'}\delta n(\boldsymbol{p};\boldsymbol{r},t)\delta n(\boldsymbol{p}';\boldsymbol{r},t) \\ &-k_B T \sum_{\boldsymbol{p}} \Big\{ n(\boldsymbol{p};\boldsymbol{r},t)\ln n(\boldsymbol{p};\boldsymbol{r},t) + [1 - n(\boldsymbol{p};\boldsymbol{r},t)]\ln[1 - n(\boldsymbol{p};\boldsymbol{r},t)]\Big\} \end{aligned} \tag{14}$$

$$\frac{\partial F_{FL}[\delta n(\boldsymbol{p};\boldsymbol{r},t)]}{\partial \delta n(\boldsymbol{p};\boldsymbol{r},t)} = 0 \longrightarrow n(\boldsymbol{p};\boldsymbol{r},t) = \frac{1}{\exp\Big\{\beta\Big(\varepsilon_{qp}(\boldsymbol{p}) - \mu + \sum_{\boldsymbol{p}'} F_{\boldsymbol{p}\boldsymbol{p}'}\delta n(\boldsymbol{p}';\boldsymbol{r},t)\Big)\Big\} + 1} \tag{15}$$

2. Physical quantities

3. Boltzmann transport theory

$$\frac{dn(\boldsymbol{p};\boldsymbol{r},t)}{dt} = I_{coll}[\delta n(\boldsymbol{p};\boldsymbol{r},t)] \tag{16}$$

$$\partial_t n(\boldsymbol{p};\boldsymbol{r},t) + \dot{\boldsymbol{r}}\cdot\boldsymbol{\nabla_r} n(\boldsymbol{p};\boldsymbol{r},t) + \dot{\boldsymbol{p}}\cdot\boldsymbol{\nabla_p} n(\boldsymbol{p};\boldsymbol{r},t) = -\frac{n(\boldsymbol{p};\boldsymbol{r},t)-n_{eq}(\boldsymbol{p})}{\tau} \tag{17}$$

$$\begin{aligned}\dot{\boldsymbol{r}} &= \boldsymbol{\nabla_p}\Big(\varepsilon_{qp}(\boldsymbol{p}) - \mu + \sum_{\boldsymbol{p}'} F_{\boldsymbol{pp}'}\delta n(\boldsymbol{p}';\boldsymbol{r},t)\Big),\\ \dot{\boldsymbol{p}} &= -\boldsymbol{\nabla_r}\Big(\varepsilon_{qp}(\boldsymbol{p}) - \mu + \sum_{\boldsymbol{p}'} F_{\boldsymbol{pp}'}\delta n(\boldsymbol{p}';\boldsymbol{r},t)\Big) + e\Big(\boldsymbol{E} + \frac{1}{c}\dot{\boldsymbol{r}}\times\boldsymbol{B}\Big)\end{aligned} \tag{18}$$

$$\dot{\boldsymbol{r}} = \boldsymbol{\nabla_p}\varepsilon_{qp}(\boldsymbol{p}) + \sum_{\boldsymbol{p}'}(\boldsymbol{\nabla_p}F_{\boldsymbol{pp}'})\delta n(\boldsymbol{p}';\boldsymbol{r},t) \approx \boldsymbol{\nabla_p}\varepsilon_{qp}(\boldsymbol{p}) \tag{19}$$

$$\dot{\boldsymbol{p}} = -\sum_{\boldsymbol{p}'} F_{\boldsymbol{pp}'}\boldsymbol{\nabla_r}\delta n(\boldsymbol{p}';\boldsymbol{r},t) + e\boldsymbol{E} + \frac{e}{c}\boldsymbol{\nabla_p}\varepsilon_{qp}(\boldsymbol{p})\times\boldsymbol{B} \tag{20}$$

$$\begin{aligned}&\partial_t n(\boldsymbol{p};\boldsymbol{r},t) + [\boldsymbol{\nabla_p}\varepsilon_{qp}(\boldsymbol{p})]\cdot\boldsymbol{\nabla_r} n(\boldsymbol{p};\boldsymbol{r},t) + \Big(-\partial_{\varepsilon_{qp}} n_{eq}[\varepsilon_{qp}(\boldsymbol{p})]\Big)\sum_{\boldsymbol{p}'} F_{\boldsymbol{pp}'}[\boldsymbol{\nabla_p}\varepsilon_{qp}(\boldsymbol{p})]\cdot\boldsymbol{\nabla_r}\delta n(\boldsymbol{p}';\boldsymbol{r},t)\\ &+\Big(e\boldsymbol{E} + \frac{e}{c}[\boldsymbol{\nabla_p}\varepsilon_{qp}(\boldsymbol{p})]\times\boldsymbol{B}\Big)\cdot\boldsymbol{\nabla_p} n(\boldsymbol{p};\boldsymbol{r},t) = -\frac{n(\boldsymbol{p};\boldsymbol{r},t)-n_{eq}(\boldsymbol{p})}{\tau}\end{aligned} \tag{21}$$

Dynamical dielectric constant

$$\langle\delta\rho(\boldsymbol{q},\nu)\rangle = \chi(\boldsymbol{q},\nu)\delta\phi(\boldsymbol{q},\nu) \tag{22}$$

$$\langle\delta\rho(\boldsymbol{q},\nu)\rangle \equiv \sum_{\boldsymbol{p}}\delta n(\boldsymbol{p};\boldsymbol{q},\nu) \longrightarrow \chi(\boldsymbol{q},\nu) = \frac{\sum_{\boldsymbol{p}}\delta n(\boldsymbol{p};\boldsymbol{q},\nu)}{\delta\phi(\boldsymbol{q},\nu)} \tag{23}$$

$$\begin{aligned}&\Big(-i\nu + \frac{1}{\tau} + [\boldsymbol{\nabla_p}\varepsilon_{qp}(\boldsymbol{p})]\cdot(i\boldsymbol{q})\Big)\delta n(\boldsymbol{p};\boldsymbol{q},\nu) + \Big(-\partial_{\varepsilon_{qp}} n_{eq}[\varepsilon_{qp}(\boldsymbol{p})]\Big)\sum_{\boldsymbol{p}'} F_{\boldsymbol{pp}'}[\boldsymbol{\nabla_p}\varepsilon_{qp}(\boldsymbol{p})]\cdot(i\boldsymbol{q})\delta n(\boldsymbol{p}';\boldsymbol{q},\nu)\\ &= -e\Big(-\partial_{\varepsilon_{qp}} n_{eq}[\varepsilon_{qp}(\boldsymbol{p})]\Big)(i\boldsymbol{q})\cdot[\boldsymbol{\nabla_p}\varepsilon_{qp}(\boldsymbol{p})]\delta\phi(\boldsymbol{q},\nu)\end{aligned} \tag{24}$$

a. Fermi gas

$$\frac{\delta n(\boldsymbol{p};\boldsymbol{q},\nu)}{\delta\phi(\boldsymbol{q},\nu)} = \frac{e\Big(-\partial_{\varepsilon_{qp}} n_{eq}[\varepsilon_{qp}(\boldsymbol{p})]\Big)[\boldsymbol{\nabla_p}\varepsilon_{qp}(\boldsymbol{p})]\cdot(i\boldsymbol{q})}{i\nu - \frac{1}{\tau} - [\boldsymbol{\nabla_p}\varepsilon_{qp}(\boldsymbol{p})]\cdot(i\boldsymbol{q})} \tag{25}$$

$$\chi_{FG}(\boldsymbol{q},\nu) = \sum_{\boldsymbol{p}}\frac{e\Big(-\partial_{\varepsilon_{qp}} n_{eq}[\varepsilon_{qp}(\boldsymbol{p})]\Big)[\boldsymbol{\nabla_p}\varepsilon_{qp}(\boldsymbol{p})]\cdot(i\boldsymbol{q})}{i\nu - \frac{1}{\tau} - [\boldsymbol{\nabla_p}\varepsilon_{qp}(\boldsymbol{p})]\cdot(i\boldsymbol{q})} \tag{26}$$

b. Landau's Fermi liquid: RPA (random phase approximation) result

C. Justification of Landau's Fermi-liquid theory I: Renormalization group analysis

Ref. R. Shankar, Rev. Mod. Phys. **66**, 129 (1994)

1. Effective field theory

$$Z = \int Dc_{\boldsymbol{p}} \exp\Big[-\int_0^\beta d\tau \Big\{ \sum_{\boldsymbol{p}} c^\dagger_{\boldsymbol{p}} (\partial_\tau - \mu + \epsilon_{\boldsymbol{p}}) c_{\boldsymbol{p}} + \sum_{\boldsymbol{q}} V_{\boldsymbol{q}} \sum_{\boldsymbol{p}} \sum_{\boldsymbol{p}'} c^\dagger_{\boldsymbol{p}+\boldsymbol{q}} c^\dagger_{\boldsymbol{p}'-\boldsymbol{q}} c_{\boldsymbol{p}'} c_{\boldsymbol{p}} \Big\}\Big] \tag{27}$$

$$\epsilon_{\boldsymbol{p}} - \mu \approx \boldsymbol{v}_F \cdot (\boldsymbol{p} - \boldsymbol{p}_F) \equiv v_F p \tag{28}$$

$$\begin{aligned} Z = &\int Dc_{\boldsymbol{p}} \exp\Big[-\int_0^\beta d\tau \Big\{ N_F \int \frac{d\Omega_d}{S_d} \int_{-\Lambda}^{\Lambda} dp c^\dagger_{\boldsymbol{p}} (\partial_\tau - v_F p) c_{\boldsymbol{p}} \\ &+ N_F^4 \int \frac{d\Omega_d}{S_d} \int_{-\Lambda}^{\Lambda} dp \int \frac{d\Omega'_d}{S_d} \int_{-\Lambda}^{\Lambda} dp' \int \frac{d\omega_d}{S_d} \int_{-\Lambda}^{\Lambda} dk \int \frac{d\omega'_d}{S_d} \int_{-\Lambda}^{\Lambda} dk' V(\boldsymbol{k}, \boldsymbol{k}'; \boldsymbol{p}', \boldsymbol{p}) c^\dagger_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}'} c_{\boldsymbol{p}'} c_{\boldsymbol{p}} \delta^{(d)}(\boldsymbol{p} + \boldsymbol{p}' - \boldsymbol{k} - \boldsymbol{k}') \Big\}\Big] \end{aligned} \tag{29}$$

$$\boldsymbol{p} + \boldsymbol{p}' = \boldsymbol{k} + \boldsymbol{k}' \tag{30}$$

Solutions in $d = 2$
a. $\boldsymbol{p} = \boldsymbol{k}$ and $\boldsymbol{p}' = \boldsymbol{k}'$ $\longrightarrow$ Forward scattering channel
b. $\boldsymbol{p} = \boldsymbol{k}'$ and $\boldsymbol{p}' = \boldsymbol{k}$ $\longrightarrow$ Backward (exchange) scattering channel (= Forward scattering channel in the case of spinless fermions)
c. $\boldsymbol{p} = -\boldsymbol{p}'$ and $\boldsymbol{k} = -\boldsymbol{k}'$ $\longrightarrow$ BCS (pairing) scattering channel

$$\begin{aligned} W = &\int Dc_{\boldsymbol{p}} \exp\Big[-\Big\{ N_F \int \frac{d\Omega_d}{S_d} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\Lambda}^{\Lambda} dp c^\dagger_{\boldsymbol{p}} (-i\omega - v_F p) c_{\boldsymbol{p}} \\ &+ N_F^3 \int \frac{d\Omega_d}{S_d} \int \frac{d\Omega'_d}{S_d} \int \frac{d\omega_d}{S_d} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int_{-\Lambda}^{\Lambda} dp \int_{-\Lambda}^{\Lambda} dp' \int_{-\Lambda}^{\Lambda} dq F(\boldsymbol{q}) c^\dagger_{\boldsymbol{p}+\boldsymbol{q}} c^\dagger_{\boldsymbol{p}'-\boldsymbol{q}} c_{\boldsymbol{p}'} c_{\boldsymbol{p}} \\ &+ N_F^3 \int \frac{d\Omega_d}{S_d} \int \frac{d\Omega'_d}{S_d} \int \frac{d\omega_d}{S_d} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int_{-\Lambda}^{\Lambda} dp \int_{-\Lambda}^{\Lambda} dp' \int_{-\Lambda}^{\Lambda} dq V(\boldsymbol{q}) c^\dagger_{\boldsymbol{p}'} c^\dagger_{-\boldsymbol{p}'+\boldsymbol{q}} c_{-\boldsymbol{p}+\boldsymbol{q}} c_{\boldsymbol{p}} \Big\}\Big] \end{aligned} \tag{31}$$

2. Scaling analysis

$$p = \frac{p_r}{r}, \qquad \omega = \frac{\omega_r}{r} \tag{32}$$

$$c(p_r/r, \omega_r/r) = r^{\Delta_c} c(p_r, \omega_r), \qquad F(q_r/r) = r^{\Delta_F} F(q_r), \qquad V(q_r/r) = r^{\Delta_V} V(q_r) \tag{33}$$

$$W = r^{2L^d \Delta_c} \int Dc(p_r, \omega_r) \exp \Big[- \Big\{ N_F \int \frac{d\Omega_d}{S_d} \int_{-\infty}^{\infty} \frac{d\omega_r}{2\pi} \int_{-\Lambda/r}^{\Lambda/r} dp_r r^{-3+2\Delta_c} c^\dagger(p_r, \omega_r)(-i\omega_r - v_F p_r) c(p_r, \omega_r)$$
$$+ N_F^3 \int \frac{d\Omega_d}{S_d} \int \frac{d\Omega_d'}{S_d} \int \frac{d\omega_d}{S_d} \int_{-\infty}^{\infty} \frac{d\omega_r}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_r'}{2\pi} \int_{-\infty}^{\infty} \frac{d\Omega_r}{2\pi} \int_{-\Lambda/r}^{\Lambda/r} dp_r \int_{-\Lambda/r}^{\Lambda/r} dp_r' \int_{-\Lambda/r}^{\Lambda/r} dq_r$$
$$r^{-6+4\Delta_c+\Delta_F} F_r(q_r) c^\dagger(p_r + q_r, \omega_r + \Omega_r) c^\dagger(p_r' - q_r, \omega_r' - \Omega_r) c(p_r', \omega_r') c(p_r, \omega_r)$$
$$+ N_F^3 \int \frac{d\Omega_d}{S_d} \int \frac{d\Omega_d'}{S_d} \int \frac{d\omega_d}{S_d} \int_{-\infty}^{\infty} \frac{d\omega_r}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_r'}{2\pi} \int_{-\infty}^{\infty} \frac{d\Omega_r}{2\pi} \int_{-\Lambda/r}^{\Lambda/r} dp_r \int_{-\Lambda/r}^{\Lambda/r} dp_r' \int_{-\Lambda/r}^{\Lambda/r} dq_r$$
$$r^{-6+4\Delta_c+\Delta_V} V(q_r) c^\dagger(p_r', \omega_r') c^\dagger(-p_r' + q_r, -\omega_r' + \Omega_r) c(-p_r + q_r, -\omega_r + \Omega_r) c(p_r, \omega_r) \Big\} \Big] \tag{34}$$

$$\begin{aligned} r^{-3+2\Delta_c} &= r^0 \longrightarrow \Delta_c = \frac{3}{2}, \\ r^{-6+4\Delta_c+\Delta_F} &= r^0 \longrightarrow \Delta_F = 0, \\ r^{-6+4\Delta_c+\Delta_V} &= r^0 \longrightarrow \Delta_V = 0 \end{aligned} \tag{35}$$

$$c(p_r/r, \omega_r/r) = r^{3/2} c(p_r, \omega_r), \qquad F(q_r/r) = F(q_r), \qquad V(q_r/r) = V(q_r) \tag{36}$$

3. Introduction of quantum corrections up to the one-loop order

$$W = \int Dc(p, \omega) \exp \Big[- \Big\{ N_F \int \frac{d\Omega_d}{S_d} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\Lambda}^{\Lambda} dp c^\dagger(p, \omega)(-i\omega - v_F p) c(p, \omega) + N_F^3 \int \frac{d\Omega_d}{S_d} \int \frac{d\Omega_d'}{S_d} \int \frac{d\omega_d}{S_d}$$
$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int_{-\Lambda}^{\Lambda} dp \int_{-\Lambda}^{\Lambda} dp' \int_{-\Lambda}^{\Lambda} dq F(q) c^\dagger(p + q, \omega + \Omega) c^\dagger(p' - q, \omega' - \Omega) c(p', \omega') c(p, \omega)$$
$$+ N_F^3 \int \frac{d\Omega_d}{S_d} \int \frac{d\Omega_d'}{S_d} \int \frac{d\omega_d}{S_d} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int_{-\Lambda}^{\Lambda} dp \int_{-\Lambda}^{\Lambda} dp' \int_{-\Lambda}^{\Lambda} dq V(q) c^\dagger(p', \omega') c^\dagger(-p' + q, -\omega' + \Omega)$$
$$c(-p + q, -\omega + \Omega) c(p, \omega) \Big\} \Big] \tag{37}$$

$$c(p, \omega) = c(p_h, \omega) \theta(|\Lambda| > |p_h| > |\Lambda|/r) + c(p_l, \omega) \theta(\Lambda/r > |p_l|) \tag{38}$$

$$W = \int Dc(p_l,\omega)Dc(p_h,\omega)\exp\Big[-\Big\{N_F\int\frac{d\Omega_d}{S_d}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{|\Lambda|/r}^{|\Lambda|}dp_h c^\dagger(p_h,\omega)(-i\omega - v_F p_h)c(p_h,\omega)$$
$$+N_F^3\int\frac{d\Omega_d}{S_d}\int\frac{d\Omega'_d}{S_d}\int\frac{d\omega_d}{S_d}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\int_{|\Lambda|/r}^{|\Lambda|}dp_h\int_{|\Lambda|/r}^{|\Lambda|}dp'_h\int_{-\Lambda}^{\Lambda}dqF(q)c^\dagger(p_h+q,\omega+\Omega)c^\dagger(p'_h-q,\omega'-\Omega)$$
$$c(p'_h,\omega')c(p_h,\omega) + N_F^3\int\frac{d\Omega_d}{S_d}\int\frac{d\Omega'_d}{S_d}\int\frac{d\omega_d}{S_d}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\int_{|\Lambda|/r}^{|\Lambda|}dp_h\int_{|\Lambda|/r}^{|\Lambda|}dp'_h\int_{-\Lambda}^{\Lambda}dqV(q)c^\dagger(p'_h,\omega')$$
$$c^\dagger(-p'_h+q,-\omega'+\Omega)c(-p_h+q,-\omega+\Omega)c(p_h,\omega)$$
$$+N_F\int\frac{d\Omega_d}{S_d}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\Lambda/r}^{\Lambda/r}dp_l c^\dagger(p_l,\omega)(-i\omega - v_F p_l)c(p_l,\omega)$$
$$+N_F^3\int\frac{d\Omega_d}{S_d}\int\frac{d\Omega'_d}{S_d}\int\frac{d\omega_d}{S_d}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\int_{-\Lambda/r}^{\Lambda/r}dp_l\int_{-\Lambda/r}^{\Lambda/r}dp'_l\int_{-\Lambda}^{\Lambda}dqF(q)c^\dagger(p_l+q,\omega+\Omega)c^\dagger(p'_l-q,\omega'-\Omega)$$
$$c(p'_l,\omega')c(p_l,\omega) + N_F^3\int\frac{d\Omega_d}{S_d}\int\frac{d\Omega'_d}{S_d}\int\frac{d\omega_d}{S_d}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\int_{-\Lambda/r}^{\Lambda/r}dp_l\int_{-\Lambda/r}^{\Lambda/r}dp'_l\int_{-\Lambda}^{\Lambda}dqV(q)c^\dagger(p'_l,\omega')$$
$$c^\dagger(-p'_l+q,-\omega'+\Omega)c(-p_l+q,-\omega+\Omega)c(p_l,\omega)$$
$$+N_F^3\int\frac{d\Omega_d}{S_d}\int\frac{d\Omega'_d}{S_d}\int\frac{d\omega_d}{S_d}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\int_{-\Lambda/r}^{\Lambda/r}dp_l\int_{|\Lambda|/r}^{|\Lambda|}dp'_h\int_{-\Lambda}^{\Lambda}dqF(q)c^\dagger(p_l+q,\omega+\Omega)c^\dagger(p'_h-q,\omega'-\Omega)$$
$$c(p'_h,\omega')c(p_l,\omega) + N_F^3\int\frac{d\Omega_d}{S_d}\int\frac{d\Omega'_d}{S_d}\int\frac{d\omega_d}{S_d}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\int_{-\Lambda/r}^{\Lambda/r}dp_l\int_{|\Lambda|/r}^{|\Lambda|}dp'_h\int_{-\Lambda}^{\Lambda}dqV(q)c^\dagger(p'_h,\omega')$$
$$c^\dagger(-p'_h+q,-\omega'+\Omega)c(-p_l+q,-\omega+\Omega)c(p_l,\omega)\Big\}\Big] \quad (39)$$

$$W \approx \int Dc(p_l,\omega)Dc(p_h,\omega)\exp\Big[-\Big\{N_F\int\frac{d\Omega_d}{S_d}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{|\Lambda|/r}^{|\Lambda|}dp_h c^\dagger(p_h,\omega)(-i\omega - v_F p_h)c(p_h,\omega)$$
$$+N_F\int\frac{d\Omega_d}{S_d}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\Lambda/r}^{\Lambda/r}dp_l c^\dagger(p_l,\omega)(-i\omega - v_F p_l)c(p_l,\omega)$$
$$+N_F^3\int\frac{d\Omega_d}{S_d}\int\frac{d\Omega'_d}{S_d}\int\frac{d\omega_d}{S_d}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\int_{-\Lambda/r}^{\Lambda/r}dp_l\int_{-\Lambda/r}^{\Lambda/r}dp'_l\int_{-\Lambda}^{\Lambda}dqF(q)c^\dagger(p_l+q,\omega+\Omega)c^\dagger(p'_l-q,\omega'-\Omega)$$
$$c(p'_l,\omega')c(p_l,\omega) + N_F^3\int\frac{d\Omega_d}{S_d}\int\frac{d\Omega'_d}{S_d}\int\frac{d\omega_d}{S_d}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\int_{-\Lambda/r}^{\Lambda/r}dp_l\int_{-\Lambda/r}^{\Lambda/r}dp'_l\int_{-\Lambda}^{\Lambda}dqV(q)c^\dagger(p'_l,\omega')$$
$$c^\dagger(-p'_l+q,-\omega'+\Omega)c(-p_l+q,-\omega+\Omega)c(p_l,\omega)$$
$$+N_F^3\int\frac{d\Omega_d}{S_d}\int\frac{d\Omega'_d}{S_d}\int\frac{d\omega_d}{S_d}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\int_{-\Lambda/r}^{\Lambda/r}dp_l\int_{|\Lambda|/r}^{|\Lambda|}dp'_h\int_{-\Lambda}^{\Lambda}dqF(q)c^\dagger(p_l+q,\omega+\Omega)c^\dagger(p'_h-q,\omega'-\Omega)$$
$$c(p'_h,\omega')c(p_l,\omega) + N_F^3\int\frac{d\Omega_d}{S_d}\int\frac{d\Omega'_d}{S_d}\int\frac{d\omega_d}{S_d}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\int_{-\Lambda/r}^{\Lambda/r}dp_l\int_{|\Lambda|/r}^{|\Lambda|}dp'_h\int_{-\Lambda}^{\Lambda}dqV(q)c^\dagger(p'_h,\omega')$$
$$c^\dagger(-p'_h+q,-\omega'+\Omega)c(-p_l+q,-\omega+\Omega)c(p_l,\omega)\Big\}\Big] \quad (40)$$

$$S_{int}$$
$$= N_F^3\int\frac{d\Omega_d}{S_d}\int\frac{d\Omega'_d}{S_d}\int\frac{d\omega_d}{S_d}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\int_{-\Lambda/r}^{\Lambda/r}dp_l\int_{|\Lambda|/r}^{|\Lambda|}dp'_h\int_{-\Lambda}^{\Lambda}dqF(q)c^\dagger(p_l+q,\omega+\Omega)c^\dagger(p'_h-q,\omega'-\Omega)$$
$$c(p'_h,\omega')c(p_l,\omega) + N_F^3\int\frac{d\Omega_d}{S_d}\int\frac{d\Omega'_d}{S_d}\int\frac{d\omega_d}{S_d}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\int_{-\Lambda/r}^{\Lambda/r}dp_l\int_{|\Lambda|/r}^{|\Lambda|}dp'_h\int_{-\Lambda}^{\Lambda}dqV(q)c^\dagger(p'_h,\omega')$$
$$c^\dagger(-p'_h+q,-\omega'+\Omega)c(-p_l+q,-\omega+\Omega)c(p_l,\omega) \quad (41)$$

$$\mathcal{S}^{(2)} = -\frac{1}{2}\Big(\Big\langle S_{int}^2\Big\rangle_h - \Big\langle S_{int}\Big\rangle_h^2\Big) \tag{42}$$

a. Renormalization of forward scattering channel
a-1. Zero sound (ZS) channel

$$\mathcal{S}_F^{ZS} = -\frac{N_F^5}{2}\Big(\int\frac{d\Omega_d}{S_d}\int\frac{d\Omega_d'}{S_d}\int\frac{d\omega_d}{S_d}\Big)^2\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega_c'}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega_c}{2\pi}$$
$$\int_{-\Lambda/r}^{\Lambda/r}dp_l\int_{|\Lambda|/r}^{|\Lambda|}dp_h'\int_{-\Lambda}^{\Lambda}dq\int_{-\Lambda/r}^{\Lambda/r}dp_{lc}\int_{|\Lambda|/r}^{|\Lambda|}dp_{hc}'F(q)F(-q)\Big\langle c^\dagger(p_h'-q,\omega'-\Omega)c(p_h',\omega')c^\dagger(p_{hc}'+q,\omega_c'+\Omega)c(p_{hc}',\omega_c')\Big\rangle_c$$
$$c^\dagger(p_l+q,\omega+\Omega)c^\dagger(p_{lc}-q,\omega_c-\Omega)c(p_{lc},\omega_c)c(p_l,\omega) \tag{43}$$

$$\begin{aligned}&\Big\langle c^\dagger(p_h'-q,\omega'-\Omega)c(p_h',\omega')c^\dagger(p_{hc}'+q,\omega_c'+\Omega)c(p_{hc}',\omega_c')\Big\rangle_c\\&=-\Big\langle c^\dagger(p_h'-q,\omega'-\Omega)c^\dagger(p_{hc}'+q,\omega_c'+\Omega)c(p_{hc}',\omega_c')c(p_h',\omega')\Big\rangle_c\\&=-\Big\langle c^\dagger(p_h'-q,\omega'-\Omega)c^\dagger(p_h',\omega')c(p_h'-q,\omega'-\Omega)c(p_h',\omega')\Big\rangle_c\\&=G(p_h',\omega')G(p_h'-q,\omega'-\Omega)\end{aligned} \tag{44}$$

$$\mathcal{S}_F^{ZS} = -\frac{N_F^3}{2}\Big(\int\frac{d\Omega_d}{S_d}\int\frac{d\Omega_d'}{S_d}\int\frac{d\omega_d}{S_d}\Big)^2\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega_c}{2\pi}\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\int_{-\Lambda/r}^{\Lambda/r}dp_l\int_{-\Lambda/r}^{\Lambda/r}dp_{lc}\int_{-\Lambda}^{\Lambda}dq$$
$$\Big(F(q)F(-q)\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int_{|\Lambda|/r}^{|\Lambda|}dp_h'G(p_h',\omega')G(p_h'-q,\omega'-\Omega)\Big)c^\dagger(p_l+q,\omega+\Omega)c^\dagger(p_{lc}-q,\omega_c-\Omega)c(p_{lc},\omega_c)c(p_l,\omega) \tag{45}$$

a-2. Zero sound' (ZS') channel (= Zero sound (ZS) channel for spinless fermions)
a-3. BCS (pairing) channel
b. Renormalization of BCS scattering channel
b-1. Zero sound (ZS) channel
b-2. Zero sound' (ZS') channel (= Zero sound (ZS) channel for spinless fermions)
b-3. BCS (pairing) channel

$$\mathcal{S}_{BCS}^{BCS} = -\frac{N_F^5}{2}\Big(\int\frac{d\Omega_d}{S_d}\int\frac{d\Omega_d'}{S_d}\int\frac{d\omega_d}{S_d}\Big)^2\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega_c}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega_c'}{2\pi}$$
$$\int_{-\Lambda/r}^{\Lambda/r}dp_l\int_{|\Lambda|/r}^{|\Lambda|}dp_h'\int_{-\Lambda}^{\Lambda}dq\int_{-\Lambda/r}^{\Lambda/r}dp_{lc}\int_{|\Lambda|/r}^{|\Lambda|}dp_{hc}'[V(q)]^2\Big\langle c^\dagger(p_h',\omega')c^\dagger(-p_h'+q,-\omega'+\Omega)c(-p_{hc}'+q,-\omega_c'+\Omega)c(p_{hc}',\omega_c')\Big\rangle_c$$
$$c^\dagger(p_{lc},\omega_c)c^\dagger(-p_{lc}+q,-\omega_c+\Omega)c(-p_l+q,-\omega+\Omega)c(p_l,\omega) \tag{46}$$

$$\begin{aligned}&\Big\langle c^\dagger(p_h',\omega')c^\dagger(-p_h'+q,-\omega'+\Omega)c(-p_{hc}'+q,-\omega_c'+\Omega)c(p_{hc}',\omega_c')\Big\rangle_c\\&=\Big\langle c^\dagger(p_h',\omega')c^\dagger(-p_h'+q,-\omega'+\Omega)c(p_h',\omega')c(-p_h'+q,-\omega'+\Omega)\Big\rangle_c=-G(p_h',\omega')G(-p_h'+q,-\omega'+\Omega)\end{aligned} \tag{47}$$

$$\mathcal{S}_{BCS}^{BCS} = \frac{N_F^3}{2}\Big(\int\frac{d\Omega_d}{S_d}\int\frac{d\Omega_d'}{S_d}\int\frac{d\omega_d}{S_d}\Big)^2\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega_c}{2\pi}\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\int_{-\Lambda/r}^{\Lambda/r}dp_l\int_{-\Lambda/r}^{\Lambda/r}dp_{lc}\int_{-\Lambda}^{\Lambda}dq$$
$$[V(q)]^2\Big(\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int_{|\Lambda|/r}^{|\Lambda|}dp_h'G(p_h',\omega')G(-p_h'+q,-\omega'+\Omega)\Big)c^\dagger(p_{lc},\omega_c)c^\dagger(-p_{lc}+q,-\omega_c+\Omega)c(-p_l+q,-\omega+\Omega)c(p_l,\omega) \tag{48}$$

4. Renormalization group equations

$$F(r) = F - \frac{N_F}{2} F^2(r) \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{|\Lambda|/r}^{|\Lambda|} dp'_h G(p'_h, \omega') G(p'_h - q, \omega' - \Omega) + \mathrm{BCS}_F \longrightarrow \frac{dF(r)}{d\ln r} = 0 \tag{49}$$

$$V(r) = V + \frac{N_F}{2} V^2(r) \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{|\Lambda|/r}^{|\Lambda|} dp'_h G(p'_h, \omega') G(-p'_h + q, -\omega' + \Omega) + \mathrm{ZS}_{BCS} \longrightarrow \frac{dV(r)}{d\ln r} = -cN_F V^2(r) \tag{50}$$

$$V(T) = \frac{V}{1 - cN_F V \ln(D/T)} \longrightarrow T_c = D \exp\Big(- \frac{1}{cN_F V} \Big) \tag{51}$$

5. Main message

D. Justification of Landau's Fermi-liquid theory II: Self-consistent RPA (random phase approximation) theory

$$Z = \int Dc_{\boldsymbol{p}} \exp \Big[- \int_0^\beta d\tau \Big\{ \sum_{\boldsymbol{p}} c_{\boldsymbol{p}}^\dagger (\partial_\tau - \mu + \epsilon_{\boldsymbol{p}}) c_{\boldsymbol{p}} + \sum_{\boldsymbol{q}} V_{\boldsymbol{q}} \sum_{\boldsymbol{p}} \sum_{\boldsymbol{p}'} c_{\boldsymbol{p}+\boldsymbol{q}}^\dagger c_{\boldsymbol{p}'-\boldsymbol{q}}^\dagger c_{\boldsymbol{p}'} c_{\boldsymbol{p}} \Big\} \Big] \tag{52}$$

1. Collective excitations for forward scattering channel and pairing interaction channel

$$Z = \int Dc_{\boldsymbol{p}} D\phi_{\boldsymbol{q}} D\Delta_{\boldsymbol{q}} \exp \Big[- \int_0^\beta d\tau \Big\{ \sum_{\boldsymbol{p}} c_{\boldsymbol{p}}^\dagger (\partial_\tau - \mu + \epsilon_{\boldsymbol{p}}) c_{\boldsymbol{p}} - i \sum_{\boldsymbol{q}} \sum_{\boldsymbol{p}} \phi_{\boldsymbol{q}} c_{\boldsymbol{p}+\boldsymbol{q}}^\dagger c_{\boldsymbol{p}} + \sum_{\boldsymbol{q}} \frac{1}{4V_{\boldsymbol{q}}} \phi_{\boldsymbol{q}} \phi_{-\boldsymbol{q}}$$
$$- \sum_{\boldsymbol{q}} \sum_{\boldsymbol{p}} \Delta_{\boldsymbol{q}}^\dagger c_{-\boldsymbol{p}+\boldsymbol{q}} c_{\boldsymbol{p}} - \sum_{\boldsymbol{q}} \sum_{\boldsymbol{p}} \Delta_{\boldsymbol{q}} c_{\boldsymbol{p}+\boldsymbol{q}}^\dagger c_{-\boldsymbol{p}}^\dagger - \sum_{\boldsymbol{q}} \frac{1}{V_{\boldsymbol{q}}} \Delta_{\boldsymbol{q}}^\dagger \Delta_{\boldsymbol{q}} \Big\} \Big] \tag{53}$$

2. Forward scattering channel only

$$Z_{FL} = \int Dc_{\boldsymbol{p}} D\phi_{\boldsymbol{q}} \exp \Big[- \int_0^\beta d\tau \Big\{ \sum_{\boldsymbol{p}} c_{\boldsymbol{p}}^\dagger (\partial_\tau - \mu + \epsilon_{\boldsymbol{p}}) c_{\boldsymbol{p}} - i \sum_{\boldsymbol{q}} \sum_{\boldsymbol{p}} \phi_{\boldsymbol{q}} c_{\boldsymbol{p}+\boldsymbol{q}}^\dagger c_{\boldsymbol{p}} + \sum_{\boldsymbol{q}} \frac{1}{4V_{\boldsymbol{q}}} \phi_{\boldsymbol{q}} \phi_{-\boldsymbol{q}} \Big\} \Big] \tag{54}$$

3. Cumulant expansion

$$S_{int} = \int_0^\beta d\tau i \sum_{\boldsymbol{q}} \sum_{\boldsymbol{p}} \phi_{\boldsymbol{q}} c_{\boldsymbol{p}+\boldsymbol{q}}^\dagger c_{\boldsymbol{p}} \tag{55}$$

$$\langle e^{-S_{int}} \rangle_0 \approx e^{-\langle S_{int} \rangle_0 + \frac{1}{2} \Big(\langle S_{int}^2 \rangle_0 - \langle S_{int} \rangle_0^2 \Big)},$$
$$\langle S_{int} \rangle_0 = \frac{1}{Z_0} \int Dc_{\boldsymbol{p}} S_{int} e^{-S_0}, \qquad S_0 = \int_0^\beta d\tau \sum_{\boldsymbol{p}} c_{\boldsymbol{p}}^\dagger (\partial_\tau - \mu + \epsilon_{\boldsymbol{p}}) c_{\boldsymbol{p}}, \qquad Z_0 = \int Dc_{\boldsymbol{p}} e^{-S_0} \tag{56}$$

$$\mathcal{S}^{(2)} = -\frac{1}{2}\Big(\langle S_{int}^2\rangle_0 - \langle S_{int}\rangle_0^2\Big)$$
$$= \frac{1}{2}\sum_{i\omega}\frac{1}{\beta}\sum_{i\Omega}\frac{1}{\beta}\sum_{i\omega'}\frac{1}{\beta}\sum_{i\Omega'}\sum_{\boldsymbol{q}}\sum_{\boldsymbol{p}}\sum_{\boldsymbol{q}'}\sum_{\boldsymbol{p}'}\Big\langle \phi_{\boldsymbol{q}}\phi_{\boldsymbol{q}'}c^\dagger_{\boldsymbol{p}+\boldsymbol{q}}c_{\boldsymbol{p}}c^\dagger_{\boldsymbol{p}'+\boldsymbol{q}'}c_{\boldsymbol{p}'}\Big\rangle_c \tag{57}$$

$$\mathcal{S}^{(2)} = \frac{1}{2}\sum_{i\omega}\frac{1}{\beta}\sum_{i\Omega}\frac{1}{\beta}\sum_{i\omega'}\frac{1}{\beta}\sum_{i\Omega'}\sum_{\boldsymbol{q}}\sum_{\boldsymbol{p}}\sum_{\boldsymbol{q}'}\sum_{\boldsymbol{p}'}\Big\{\Big\langle\phi_{\boldsymbol{q}}\phi_{\boldsymbol{q}'}\Big\rangle_c c^\dagger_{\boldsymbol{p}+\boldsymbol{q}}c_{\boldsymbol{p}}c^\dagger_{\boldsymbol{p}'+\boldsymbol{q}'}c_{\boldsymbol{p}'} + \phi_{\boldsymbol{q}}\phi_{\boldsymbol{q}'}\Big\langle c^\dagger_{\boldsymbol{p}+\boldsymbol{q}}c_{\boldsymbol{p}}c^\dagger_{\boldsymbol{p}'+\boldsymbol{q}'}c_{\boldsymbol{p}'}\Big\rangle_c$$
$$-\Big\langle\phi_{\boldsymbol{q}}\phi_{\boldsymbol{q}'}\Big\rangle\Big\langle c^\dagger_{\boldsymbol{p}+\boldsymbol{q}}c_{\boldsymbol{p}}c^\dagger_{\boldsymbol{p}'+\boldsymbol{q}'}c_{\boldsymbol{p}'}\Big\rangle_c\Big\} \equiv \frac{1}{2}\sum_{i\Omega}\sum_{\boldsymbol{q}}\Big\{D(\boldsymbol{q},i\Omega)\frac{1}{\beta}\sum_{i\omega}\frac{1}{\beta}\sum_{i\omega'}\sum_{\boldsymbol{p}}\sum_{\boldsymbol{p}'}c^\dagger_{\boldsymbol{p}+\boldsymbol{q}}c_{\boldsymbol{p}}c^\dagger_{\boldsymbol{p}'-\boldsymbol{q}}c_{\boldsymbol{p}'}$$
$$+\Pi(\boldsymbol{q},i\Omega)\phi_{\boldsymbol{q}}\phi_{-\boldsymbol{q}} - \Pi(\boldsymbol{q},i\Omega)D(\boldsymbol{q},i\Omega)\Big\} \tag{58}$$

$$\frac{1}{\beta}\sum_{i\omega}\frac{1}{\beta}\sum_{i\omega'}\sum_{\boldsymbol{p}}\sum_{\boldsymbol{p}'}c^\dagger_{\boldsymbol{p}+\boldsymbol{q}}c_{\boldsymbol{p}}c^\dagger_{\boldsymbol{p}'-\boldsymbol{q}}c_{\boldsymbol{p}'} = \frac{1}{\beta}\sum_{i\omega}\frac{1}{\beta}\sum_{i\omega'}\sum_{\boldsymbol{p}}\sum_{\boldsymbol{p}'}\Big\{c^\dagger_{\boldsymbol{p}+\boldsymbol{q}}\Big\langle c_{\boldsymbol{p}}c^\dagger_{\boldsymbol{p}'-\boldsymbol{q}}\Big\rangle_c c_{\boldsymbol{p}'} + c^\dagger_{\boldsymbol{p}'-\boldsymbol{q}}\Big\langle c_{\boldsymbol{p}'}c^\dagger_{\boldsymbol{p}+\boldsymbol{q}}\Big\rangle_c c_{\boldsymbol{p}}$$
$$+\Big\langle c_{\boldsymbol{p}}c^\dagger_{\boldsymbol{p}'-\boldsymbol{q}}\Big\rangle_c\Big\langle c_{\boldsymbol{p}'}c^\dagger_{\boldsymbol{p}+\boldsymbol{q}}\Big\rangle_c\Big\} = \frac{1}{\beta}\sum_{i\omega}\sum_{\boldsymbol{p}}\Big\{-2G(\boldsymbol{p}+\boldsymbol{q},i\omega+i\Omega)c^\dagger_{\boldsymbol{p}}c_{\boldsymbol{p}} + G(\boldsymbol{p}+\boldsymbol{q},i\omega+i\Omega)G(\boldsymbol{p},i\omega)\Big\} \tag{59}$$

$$\mathcal{S}^{(2)} = \sum_{i\omega}\sum_{\boldsymbol{p}}\Big(-\frac{1}{\beta}\sum_{i\Omega}\sum_{\boldsymbol{q}}D(\boldsymbol{q},i\Omega)G(\boldsymbol{p}+\boldsymbol{q},i\omega+i\Omega)\Big)c^\dagger_{\boldsymbol{p}}c_{\boldsymbol{p}} + \frac{1}{2}\sum_{i\omega}\sum_{\boldsymbol{p}}\frac{1}{\beta}\sum_{i\Omega}\sum_{\boldsymbol{q}}G(\boldsymbol{p}+\boldsymbol{q},i\omega+i\Omega)D(\boldsymbol{q},i\Omega)G(\boldsymbol{p},i\omega)$$
$$+\frac{1}{2}\sum_{i\Omega}\sum_{\boldsymbol{q}}\Pi(\boldsymbol{q},i\Omega)\phi_{\boldsymbol{q}}\phi_{-\boldsymbol{q}} - \frac{1}{2}\sum_{i\Omega}\sum_{\boldsymbol{q}}\Pi(\boldsymbol{q},i\Omega)D(\boldsymbol{q},i\Omega) \tag{60}$$

$$Z_{FL} = \int Dc_{\boldsymbol{p}}D\phi_{\boldsymbol{q}}\exp\Big[-\sum_{i\omega}\sum_{\boldsymbol{p}}c^\dagger_{\boldsymbol{p}}\Big(-i\omega-\mu+\epsilon_{\boldsymbol{p}}-\frac{1}{\beta}\sum_{i\Omega}\sum_{\boldsymbol{q}}D(\boldsymbol{q},i\Omega)G(\boldsymbol{p}+\boldsymbol{q},i\omega+i\Omega)\Big)c_{\boldsymbol{p}}$$
$$-\frac{1}{2}\sum_{i\Omega}\sum_{\boldsymbol{q}}\Big\{\phi_{\boldsymbol{q}}\Big(\frac{1}{2V_{\boldsymbol{q}}}+\Pi(\boldsymbol{q},i\Omega)\Big)\phi_{-\boldsymbol{q}} - \Pi(\boldsymbol{q},i\Omega)D(\boldsymbol{q},i\Omega)\Big\}$$
$$-\frac{1}{2}\sum_{i\omega}\sum_{\boldsymbol{p}}\frac{1}{\beta}\sum_{i\Omega}\sum_{\boldsymbol{q}}G(\boldsymbol{p}+\boldsymbol{q},i\omega+i\Omega)D(\boldsymbol{q},i\Omega)G(\boldsymbol{p},i\omega)\Big] \tag{61}$$

$$F_{FL}[G(\boldsymbol{p},i\omega),D(\boldsymbol{q},i\Omega),\Pi(\boldsymbol{q},i\Omega)]$$
$$= -\frac{1}{\beta}\sum_{i\omega}\sum_{\boldsymbol{p}}\ln\Big(-i\omega-\mu+\epsilon_{\boldsymbol{p}}-\frac{1}{\beta}\sum_{i\Omega}\sum_{\boldsymbol{q}}D(\boldsymbol{q},i\Omega)G(\boldsymbol{p}+\boldsymbol{q},i\omega+i\Omega)\Big)$$
$$+\frac{1}{2\beta}\sum_{i\Omega}\sum_{\boldsymbol{q}}\Big\{\ln\Big(\frac{1}{2V_{\boldsymbol{q}}}+\Pi(\boldsymbol{q},i\Omega)\Big) - \Pi(\boldsymbol{q},i\Omega)D(\boldsymbol{q},i\Omega)\Big\}$$
$$+\frac{1}{2\beta}\sum_{i\omega}\sum_{\boldsymbol{p}}\frac{1}{\beta}\sum_{i\Omega}\sum_{\boldsymbol{q}}G(\boldsymbol{p}+\boldsymbol{q},i\omega+i\Omega)D(\boldsymbol{q},i\Omega)G(\boldsymbol{p},i\omega) \tag{62}$$

$$\frac{\delta F_{FL}[G(\boldsymbol{p},i\omega),D(\boldsymbol{q},i\Omega),\Pi(\boldsymbol{q},i\Omega)]}{\delta G(\boldsymbol{p},i\omega)} = 0,$$
$$\frac{\delta F_{FL}[G(\boldsymbol{p},i\omega),D(\boldsymbol{q},i\Omega),\Pi(\boldsymbol{q},i\Omega)]}{\delta D(\boldsymbol{q},i\Omega)} = 0,$$
$$\frac{\delta F_{FL}[G(\boldsymbol{p},i\omega),D(\boldsymbol{q},i\Omega),\Pi(\boldsymbol{q},i\Omega)]}{\delta \Pi(\boldsymbol{q},i\Omega)} = 0 \tag{63}$$

$$\Sigma(\boldsymbol{p}, i\omega) \equiv -\frac{1}{\beta} \sum_{i\Omega} \sum_{\boldsymbol{q}} D(\boldsymbol{q}, i\Omega) G(\boldsymbol{p}+\boldsymbol{q}, i\omega + i\Omega),$$
$$\Pi(\boldsymbol{q}, i\Omega) = \frac{1}{2\beta} \sum_{i\omega} \sum_{\boldsymbol{p}} G(\boldsymbol{p}+\boldsymbol{q}, i\omega + i\Omega) G(\boldsymbol{p}, i\omega) \tag{64}$$

$$G(\boldsymbol{p}, i\omega) = \frac{1}{i\omega + \mu + \epsilon_{\boldsymbol{p}} - \Sigma(\boldsymbol{p}, i\omega)},$$
$$D(\boldsymbol{q}, i\Omega) = \frac{1}{\frac{1}{2V_{\boldsymbol{q}}} + \Pi(\boldsymbol{q}, i\Omega)} \tag{65}$$

4. Fermi-liquid ansatz

$$\Sigma(\boldsymbol{p}, i\omega \to \omega + i\delta) \approx \Sigma(\boldsymbol{p}_F, \omega = 0) + \omega \frac{\partial}{\partial \omega} \Re\Sigma(\boldsymbol{p}_F, \omega)\Big|_{\omega=0} + (\boldsymbol{p} - \boldsymbol{p}_F) \cdot \boldsymbol{\nabla}_{\boldsymbol{p}} \Re\Sigma(\boldsymbol{p}, \omega = 0)\Big|_{\boldsymbol{p}=\boldsymbol{p}_F}$$
$$= \omega \frac{\partial}{\partial \omega} \Re\Sigma(\boldsymbol{p}_F, \omega)\Big|_{\omega=0} + (\boldsymbol{p} - \boldsymbol{p}_F) \cdot \boldsymbol{\nabla}_{\boldsymbol{p}} \Re\Sigma(\boldsymbol{p}, \omega = 0)\Big|_{\boldsymbol{p}=\boldsymbol{p}_F} \tag{66}$$

$$\epsilon_{\boldsymbol{p}} - \mu = \boldsymbol{v}_F \cdot (\boldsymbol{p} - \boldsymbol{p}_F) \tag{67}$$

$$G(\boldsymbol{p}, \omega + i\delta) = \frac{1}{\Big(1 - \frac{\partial}{\partial \omega} \Re\Sigma(\boldsymbol{p}_F, \omega)\Big|_{\omega=0}\Big)\omega - \Big(\boldsymbol{v}_F + \boldsymbol{\nabla}_{\boldsymbol{p}} \Re\Sigma(\boldsymbol{p}, \omega = 0)\Big|_{\boldsymbol{p}=\boldsymbol{p}_F}\Big) \cdot (\boldsymbol{p} - \boldsymbol{p}_F)}$$
$$-i\pi\delta\Big\{\Big(1 - \frac{\partial}{\partial \omega} \Re\Sigma(\boldsymbol{p}_F, \omega)\Big|_{\omega=0}\Big)\omega - \Big(\boldsymbol{v}_F + \boldsymbol{\nabla}_{\boldsymbol{p}} \Re\Sigma(\boldsymbol{p}, \omega = 0)\Big|_{\boldsymbol{p}=\boldsymbol{p}_F}\Big) \cdot (\boldsymbol{p} - \boldsymbol{p}_F)\Big\} \tag{68}$$

$$Z(\omega) = \Big(1 - \frac{\partial}{\partial \omega} \Re\Sigma(\boldsymbol{p}_F, \omega)\Big|_{\omega=0}\Big)^{-1},$$
$$Z(\boldsymbol{p}) = \Big(1 + \frac{\boldsymbol{v}_F}{|\boldsymbol{v}_F|^2} \cdot \boldsymbol{\nabla}_{\boldsymbol{p}} \Re\Sigma(\boldsymbol{p}, \omega = 0)\Big|_{\boldsymbol{p}=\boldsymbol{p}_F}\Big), \qquad \boldsymbol{v}_F^R = Z(\boldsymbol{p}) \boldsymbol{v}_F \tag{69}$$

$$G(\boldsymbol{p}, \omega + i\delta) = \frac{Z(\omega)}{\omega - Z(\omega) Z(\boldsymbol{p}) \boldsymbol{v}_F \cdot (\boldsymbol{p} - \boldsymbol{p}_F)} - i\pi\delta\Big(Z^{-1}(\omega)\omega - Z(\boldsymbol{p}) \boldsymbol{v}_F \cdot (\boldsymbol{p} - \boldsymbol{p}_F)\Big) \tag{70}$$

$$\Pi(\boldsymbol{q},\Omega+i\delta)=\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\sum_{\boldsymbol{p}}\Big\{\frac{Z(\omega+\Omega)}{\omega+\Omega-Z(\omega+\Omega)Z(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{v}_F\cdot(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}_F)}$$
$$-i\pi\delta\Big(Z^{-1}(\omega+\Omega)(\omega+\Omega)-Z(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{v}_F\cdot(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}_F)\Big)\Big\}\Big\{\frac{Z(\omega)}{\omega-Z(\omega)Z(\boldsymbol{p})\boldsymbol{v}_F\cdot(\boldsymbol{p}-\boldsymbol{p}_F)}$$
$$-i\pi\delta\Big(Z^{-1}(\omega)\omega-Z(\boldsymbol{p})\boldsymbol{v}_F\cdot(\boldsymbol{p}-\boldsymbol{p}_F)\Big)\Big\}$$
$$\approx\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\sum_{\boldsymbol{p}}\Big\{\frac{Z_\omega}{\omega+\Omega-Z_\omega Z_p\boldsymbol{v}_F\cdot(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}_F)}-i\pi\delta\Big(Z_\omega^{-1}(\omega+\Omega)-Z_p\boldsymbol{v}_F\cdot(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}_F)\Big)\Big\}$$
$$\Big\{\frac{Z_\omega}{\omega-Z_\omega Z_p\boldsymbol{v}_F\cdot(\boldsymbol{p}-\boldsymbol{p}_F)}-i\pi\delta\Big(Z_\omega^{-1}\omega-Z_p\boldsymbol{v}_F\cdot(\boldsymbol{p}-\boldsymbol{p}_F)\Big)\Big\}$$
$$=\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\sum_{\boldsymbol{p}}\frac{Z_\omega}{\omega+\Omega-Z_\omega Z_p\boldsymbol{v}_F\cdot(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}_F)+i\delta}\frac{Z_\omega}{\omega-Z_\omega Z_p\boldsymbol{v}_F\cdot(\boldsymbol{p}-\boldsymbol{p}_F)+i\delta}\tag{71}$$

$$\Pi(\boldsymbol{q},\Omega+i\delta)=\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\sum_{\boldsymbol{p}}\Big\{\frac{1}{\omega-Z_\omega Z_p\boldsymbol{v}_F\cdot(\boldsymbol{p}-\boldsymbol{p}_F)+i\delta}-\frac{1}{\omega+\Omega-Z_\omega Z_p\boldsymbol{v}_F\cdot(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}_F)+i\delta}\Big\}\frac{Z_\omega}{\Omega-Z_\omega Z_p\boldsymbol{v}_F\cdot\boldsymbol{q}}$$
$$=\frac{1}{4\pi}\sum_{\boldsymbol{p}}\frac{Z_\omega Z_\omega Z_p\boldsymbol{v}_F\cdot\boldsymbol{q}}{\Omega-Z_\omega Z_p\boldsymbol{v}_F\cdot\boldsymbol{q}}=Z_\omega\frac{N_F}{2}\int_{-1}^{1}d\cos\theta\Big(-1+\frac{\Omega}{\Omega-Z_\omega Z_p v_F q\cos\theta}\Big)$$
$$=Z_\omega\frac{N_F}{2}\Big\{-2+\frac{\Omega}{Z_\omega Z_p v_F q}\ln\Big(\frac{\Omega+Z_\omega Z_p v_F q}{\Omega-Z_\omega Z_p v_F q}\Big)\Big\}=-Z_\omega N_F+\frac{N_F}{2}\frac{\Omega}{Z_p v_F q}\ln\Big(\frac{\Omega+Z_\omega Z_p v_F q}{\Omega-Z_\omega Z_p v_F q}\Big)\tag{72}$$

$$\Sigma(\boldsymbol{p}_F,\omega=0)+\Big(1-Z_\omega^{-1}\Big)\omega+\Big(Z_p-1\Big)\boldsymbol{v}_F\cdot(\boldsymbol{p}-\boldsymbol{p}_F)$$
$$=-\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\sum_{\boldsymbol{q}}D(\boldsymbol{q},\Omega+i\delta)\Big\{\frac{Z(\omega+\Omega)}{\omega+\Omega-Z(\omega+\Omega)Z(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{v}_F\cdot(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}_F)}$$
$$-i\pi\delta\Big(Z^{-1}(\omega+\Omega)(\omega+\Omega)-Z(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{v}_F\cdot(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}_F)\Big)\Big\}$$
$$\approx-\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\sum_{\boldsymbol{q}}D(\boldsymbol{q},\Omega+i\delta)\Big\{\frac{Z_\omega}{\omega+\Omega-Z_\omega Z_p\boldsymbol{v}_F\cdot(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}_F)}-i\pi\delta\Big(Z_\omega^{-1}(\omega+\Omega)-Z_p\boldsymbol{v}_F\cdot(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}_F)\Big)\Big\}$$
$$=-\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\sum_{\boldsymbol{q}}\frac{1}{\frac{1}{2V_{\boldsymbol{q}}}-Z_\omega N_F+\frac{N_F}{2}\frac{\Omega}{Z_p v_F q}\ln\Big(\frac{\Omega+Z_\omega Z_p v_F q}{\Omega-Z_\omega Z_p v_F q}\Big)}\frac{Z_\omega}{\omega+\Omega-Z_\omega Z_p\boldsymbol{v}_F\cdot(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}_F)+i\delta}\tag{73}$$

$$\Sigma(\boldsymbol{p}_F,\omega=0)+\Big(1-Z_\omega^{-1}\Big)\omega+\Big(Z_p-1\Big)\boldsymbol{v}_F\cdot(\boldsymbol{p}-\boldsymbol{p}_F)=-\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\sum_{\boldsymbol{q}}\frac{1}{\frac{1}{2V_{\boldsymbol{q}}}-Z_\omega N_F+\frac{N_F}{2}\frac{\Omega}{Z_p v_F q}\ln\Big(\frac{\Omega+Z_\omega Z_p v_F q}{\Omega-Z_\omega Z_p v_F q}\Big)}$$
$$\Big\{\frac{Z_\omega}{\Omega-Z_\omega Z_p\boldsymbol{v}_F\cdot\boldsymbol{q}+i\delta}-\omega\frac{Z_\omega}{[\Omega-Z_\omega Z_p\boldsymbol{v}_F\boldsymbol{q}+i\delta]^2}+Z_\omega Z_p\boldsymbol{v}_F\cdot(\boldsymbol{p}-\boldsymbol{p}_F)\frac{Z_\omega}{[\Omega-Z_\omega Z_p\boldsymbol{v}_F\cdot\boldsymbol{q}+i\delta]^2}\Big\}\tag{74}$$

$$\Sigma(\boldsymbol{p}_F,\omega=0)=-\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\sum_{\boldsymbol{q}}\frac{1}{\frac{1}{2V_{\boldsymbol{q}}}-Z_\omega N_F+\frac{N_F}{2}\frac{\Omega}{Z_p v_F q}\ln\Big(\frac{\Omega+Z_\omega Z_p v_F q}{\Omega-Z_\omega Z_p v_F q}\Big)}\frac{Z_\omega}{\Omega-Z_\omega Z_p\boldsymbol{v}_F\cdot\boldsymbol{q}+i\delta}\tag{75}$$

$$1-Z_\omega^{-1}=\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\sum_{\boldsymbol{q}}\frac{1}{\frac{1}{2V_{\boldsymbol{q}}}-Z_\omega N_F+\frac{N_F}{2}\frac{\Omega}{Z_p v_F q}\ln\Big(\frac{\Omega+Z_\omega Z_p v_F q}{\Omega-Z_\omega Z_p v_F q}\Big)}\frac{Z_\omega}{[\Omega-Z_\omega Z_p\boldsymbol{v}_F\boldsymbol{q}+i\delta]^2}\tag{76}$$

$$Z_p - 1 = -Z_\omega Z_p \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \sum_{\boldsymbol{q}} \frac{1}{\frac{1}{2V_{\boldsymbol{q}}} - Z_\omega N_F + \frac{N_F}{2} \frac{\Omega}{Z_p v_F q} \ln\left(\frac{\Omega + Z_\omega Z_p v_F q}{\Omega - Z_\omega Z_p v_F q}\right)} \frac{Z_\omega}{[\Omega - Z_\omega Z_p \boldsymbol{v}_F \cdot \boldsymbol{q} + i\delta]^2} \tag{77}$$

$$1 - Z_\omega^{-1} = \frac{1 - Z_p}{Z_\omega Z_p} \longrightarrow Z_\omega Z_p = 1 \tag{78}$$

$$Z_\omega \neq 0, \qquad Z_p \neq 0 \tag{79}$$

6. Main message

E. Problem: Landau's Fermi-liquid theory for a Weyl metal phase ?

IV. SYMMETRY BREAKING IN LANDAU'S FERMI-LIQUID STATE

A. Symmetry breaking pattern of order parameter

Ref. C. Nayak, Quantum Condensed Matter Physics - Lecture Notes

1. Particle-particle channel

a. Spin-singlet channel

$$\langle \psi_\alpha(\boldsymbol{k}, t) \psi_\beta(-\boldsymbol{k}, t) \rangle = \Delta_s(\boldsymbol{k}) \epsilon_{\alpha\beta} = \Delta_s(k) \Big(\sum_{m=-l}^{l} d_m Y_{lm}(\hat{\boldsymbol{k}}) \Big) \epsilon_{\alpha\beta} \tag{80}$$

$l = 0,\ 2,\ 4$: s-wave, d-wave, g-wave
Symmetry breaking pattern: $U(1) \times O(3) \times SU(2) \times Z_2^T \longrightarrow Z_2 \times U(1) \times Z_l \times SU(2) \times Z_2^T$
$U(1) \rightarrow Z_2$: The pairing ground state is invariant under $\psi_\alpha(\boldsymbol{k}, t) \longrightarrow -\psi_\alpha(\boldsymbol{k}, t)$.
$O(3) \rightarrow U(1) \times Z_l$: $U(1)$ is the subgroup of rotations about the direction of the angular momentum vector of the pair, and Z_l is the discrete set of rotations which leave $Y_{lm}(\hat{\boldsymbol{k}})$ invariant.
b. Spin-triplet channel

$$\langle \psi_\alpha(\boldsymbol{k}, t) \psi_\beta(-\boldsymbol{k}, t) \rangle = \boldsymbol{\Delta}_t(\boldsymbol{k}) \cdot \boldsymbol{\sigma}_{\alpha\gamma} \epsilon_{\gamma\beta} = \Delta_t(k) \Big(\sum_{m=-l}^{l} Y_{lm}(\hat{\boldsymbol{k}}) \boldsymbol{d}_m \Big) \cdot \boldsymbol{\sigma}_{\alpha\gamma} \epsilon_{\gamma\beta} \tag{81}$$

$$\langle \psi_\alpha(\boldsymbol{k}, t) \psi_\beta(-\boldsymbol{k}, t) \rangle = \Delta_t(k) \Big(\sum_{m=-1}^{1} Y_{1m}(\hat{\boldsymbol{k}}) \boldsymbol{d}_m \Big) \cdot \boldsymbol{\sigma}_{\alpha\gamma} \epsilon_{\gamma\beta} \tag{82}$$

b-1. ^{3}He A phase

$$\mathbf{d}_0 = \mathbf{d}_- = 0, \qquad \mathbf{d}_+ = \mathbf{d} \tag{83}$$

$$\langle \psi_\alpha(\boldsymbol{k}, t) \psi_\beta(-\boldsymbol{k}, t) \rangle = \Delta_t(k) (k_x + i k_y) \boldsymbol{d} \cdot \boldsymbol{\sigma}_{\alpha\gamma} \epsilon_{\gamma\beta} \tag{84}$$

Symmetry breaking pattern: $U(1)\times O(3)\times SU(2)\times Z_2^T \longrightarrow Z_2\times U(1)\times U(1)$ The former $U(1)$ corresponds to spin rotations about the $\boldsymbol{d}-$direction, and the latter $U(1)$, rotations about the $\hat{\boldsymbol{z}}-$axis combined with gauge transformations (the former yields a phase factor which is cancelled by the latter).

b-2. ^{3}He B phase

$$\mathbf{d}_0=\hat{\boldsymbol{z}},\qquad \mathbf{d}_\pm=\frac{\hat{\boldsymbol{x}}\pm i\hat{\boldsymbol{y}}}{\sqrt{2}} \tag{85}$$

$$\langle\psi_\alpha(\boldsymbol{k},t)\psi_\beta(-\boldsymbol{k},t)\rangle=\Delta_t(k)\hat{\boldsymbol{k}}\cdot\boldsymbol{\sigma}_{\alpha\gamma}\epsilon_{\gamma\beta} \tag{86}$$

$l=1$, $s=1$, and $J=0$

Symmetry breaking pattern: $U(1)\times O(3)\times SU(2)\times Z_2^T \longrightarrow Z_2\times SO(3)\times Z_2^T$ $SO(3)$ is the group of simultaneous rotations of both space and spin.

c. Odd-gap pairing

If the pairing order parameter is an odd function of $t-t'$, so that the correlation function acutally vanishes at $t=t'$, then the order is called odd-gap superconductivity.

d. Examples on 2d square lattice

The symmetry group of the 2d square lattice is D_4 with 8 elements: 4 rotations and 4 reflections.

s-wave: $\langle\psi_\alpha(\boldsymbol{k},t)\psi_\beta(-\boldsymbol{k},t)\rangle=\Delta_s\epsilon_{\alpha\beta}$

^{3}He A phase analogue: $\langle\psi_\alpha(\boldsymbol{k},t)\psi_\beta(-\boldsymbol{k},t)\rangle=\Delta_p(\sin k_xa+i\sin k_ya)\boldsymbol{d}\cdot\boldsymbol{\sigma}_{\alpha\gamma}\epsilon_{\gamma\beta}$

p_x wave: $\langle\psi_\alpha(\boldsymbol{k},t)\psi_\beta(-\boldsymbol{k},t)\rangle=\Delta_p(\sin k_xa)\boldsymbol{d}\cdot\boldsymbol{\sigma}_{\alpha\gamma}\epsilon_{\gamma\beta}$

$d_{x^2-y^2}$ wave: $\langle\psi_\alpha(\boldsymbol{k},t)\psi_\beta(-\boldsymbol{k},t)\rangle=\Delta_d(\cos k_xa-\cos k_ya)\epsilon_{\alpha\beta}$

$d_{x^2-y^2}+id_{xy}$ wave: $\langle\psi_\alpha(\boldsymbol{k},t)\psi_\beta(-\boldsymbol{k},t)\rangle=\Delta_d(\cos k_xa-\cos k_ya+i\sin k_xa\sin k_ya)\epsilon_{\alpha\beta}$

2. Particle-hole channel I: Finite-momentum ordering

a. Spin-singlet channel

$$\langle\psi^\dagger_\alpha(\boldsymbol{k}+\boldsymbol{Q},t)\psi_\beta(\boldsymbol{k},t)\rangle=\Phi_{\boldsymbol{Q}}(\boldsymbol{k})\delta_{\alpha\beta} \tag{87}$$

$\Phi_{\boldsymbol{Q}}(\boldsymbol{k})=\Phi_{\boldsymbol{Q}}$: Charge density wave (CDW)

b. Spin-triplet channel

$$\langle\psi^\dagger_\alpha(\boldsymbol{k}+\boldsymbol{Q},t)\psi_\beta(\boldsymbol{k},t)\rangle=\boldsymbol{\Phi}_{\boldsymbol{Q}}(\boldsymbol{k})\cdot\boldsymbol{\sigma}_{\alpha\beta} \tag{88}$$

$\boldsymbol{\Phi}_{\boldsymbol{Q}}(\boldsymbol{k})=\boldsymbol{\Phi}_{\boldsymbol{Q}}$: Spin density wave (SDW)

c. Higher angular momentum

$$\Phi_{\boldsymbol{Q}}(\boldsymbol{k})=\Phi_{\boldsymbol{Q}}f(\boldsymbol{k}),\qquad \boldsymbol{\Phi}_{\boldsymbol{Q}}(\boldsymbol{k})=\boldsymbol{\Phi}_{\boldsymbol{Q}}g(\boldsymbol{k}) \tag{89}$$

d. Commensurate ordering: $\boldsymbol{Q}=(\pi/a,0$ or $\boldsymbol{Q}=(\pi/a,\pi/a)$

$$\begin{aligned}
&\langle\psi^\dagger_\alpha(\boldsymbol{k}+\boldsymbol{Q},t)\psi_\beta(\boldsymbol{k},t)\rangle=\Phi_{\boldsymbol{Q}}f(\boldsymbol{k})\delta_{\alpha\beta},\\
&\langle\psi^\dagger_\beta(\boldsymbol{k},t)\psi_\alpha(\boldsymbol{k}+\boldsymbol{Q},t)\rangle=\Phi^*_{\boldsymbol{Q}}f(\boldsymbol{k})^*\delta_{\alpha\beta},\\
&\langle\psi^\dagger_\beta(\boldsymbol{k}+2\boldsymbol{Q},t)\psi_\alpha(\boldsymbol{k}+\boldsymbol{Q},t)\rangle=\Phi^*_{\boldsymbol{Q}}f(\boldsymbol{k})^*\delta_{\alpha\beta},\\
&\Phi_{\boldsymbol{Q}}f(\boldsymbol{k}+\boldsymbol{Q})=\Phi^*_{\boldsymbol{Q}}f^*(\boldsymbol{k})\longrightarrow\frac{f(\boldsymbol{k}+\boldsymbol{Q})}{f^*(\boldsymbol{k})}=\frac{\Phi^*_{\boldsymbol{Q}}}{\Phi_{\boldsymbol{Q}}}
\end{aligned} \tag{90}$$

If $f(\boldsymbol{k}+\boldsymbol{Q})=-f^*(\boldsymbol{k})$, $\Phi_{\boldsymbol{Q}}$ must be imaginary. If $f(\boldsymbol{k}+\boldsymbol{Q})=f^*(\boldsymbol{k})$, $\Phi_{\boldsymbol{Q}}$ must be real.

e. Examples in the 2d square lattice

Commensurate singlet p_x density-wave ordering: $\langle\psi^\dagger_\alpha(\boldsymbol{k}+\boldsymbol{Q},t)\psi_\beta(\boldsymbol{k},t)\rangle=\Phi_{\boldsymbol{Q}}\sin k_xa\delta_{\alpha\beta}$

$\boldsymbol{Q}=(\pi/a,0)$: $\langle\psi^\dagger_\alpha(\boldsymbol{k}+\boldsymbol{Q},t)\psi_\beta(\boldsymbol{k},t)\rangle\longrightarrow\langle\psi^\dagger_\alpha(\boldsymbol{x},t)\psi_\beta(\boldsymbol{x}+a\hat{\boldsymbol{x}},t)-\psi^\dagger_\alpha(\boldsymbol{x},t)\psi_\beta(\boldsymbol{x}-a\hat{\boldsymbol{x}},t)\rangle=...+|\Phi_{\boldsymbol{Q}}|e^{i\boldsymbol{Q}\cdot\boldsymbol{x}}\delta_{\alpha\beta}$

$\boldsymbol{Q}=(0,\pi/a)$: $\langle\psi^\dagger_\alpha(\boldsymbol{k}+\boldsymbol{Q},t)\psi_\beta(\boldsymbol{k},t)\rangle \longrightarrow \langle\psi^\dagger_\alpha(\boldsymbol{x},t)\psi_\beta(\boldsymbol{x}+a\hat{\boldsymbol{x}},t)-\psi^\dagger_\alpha(\boldsymbol{x},t)\psi_\beta(\boldsymbol{x}-a\hat{\boldsymbol{x}},t)\rangle = ... - i|\Phi_{\boldsymbol{Q}}|e^{i\boldsymbol{Q}\cdot\boldsymbol{x}}\delta_{\alpha\beta}$
Although this states looks to break time reversal symmetry, the combination of T and translation by an odd number of lattice spacings remains unbroken. This p density-wave state is called Peierls state.

Commensurate singlet p_x+ip_y density-wave ordering: $\langle\psi^\dagger_\alpha(\boldsymbol{k}+\boldsymbol{Q},t)\psi_\beta(\boldsymbol{k},t)\rangle = \Phi_{\boldsymbol{Q}}(\sin k_xa+i\sin k_ya)\delta_{\alpha\beta}$ This state does not break time reversal symmetry, either.

Commensurate singlet $d_{x^2-y^2}$ density-wave ordering: $\langle\psi^\dagger_\alpha(\boldsymbol{k}+\boldsymbol{Q},t)\psi_\beta(\boldsymbol{k},t)\rangle = \Phi_{\boldsymbol{Q}}(\cos k_xa-\cos k_ya)\delta_{\alpha\beta}$

$\boldsymbol{Q}=(\pi/a,\pi/a)$: $\langle\psi^\dagger_\alpha(\boldsymbol{k}+\boldsymbol{Q},t)\psi_\beta(\boldsymbol{k},t)\rangle \longrightarrow \langle\psi^\dagger_\alpha(\boldsymbol{x},t)\psi_\beta(\boldsymbol{x}+a\hat{\boldsymbol{x}},t)+\psi^\dagger_\alpha(\boldsymbol{x},t)\psi_\beta(\boldsymbol{x}-a\hat{\boldsymbol{x}},t)\rangle - \langle\psi^\dagger_\alpha(\boldsymbol{x},t)\psi_\beta(\boldsymbol{x}+a\hat{\boldsymbol{y}},t)-\psi^\dagger_\alpha(\boldsymbol{x},t)\psi_\beta(\boldsymbol{x}-a\hat{\boldsymbol{y}},t)\rangle = ... + \frac{i}{2}|\Phi_{\boldsymbol{Q}}|e^{i\boldsymbol{Q}\cdot\boldsymbol{x}}\delta_{\alpha\beta}$

This looks break T as well as translational and rotational invariance. The combination of time-reversal and a translation by one-lattice spacing is preserved by this ordering. The commensurate $\boldsymbol{Q}=(\pi/a,\pi/a)$ singlet $d_{x^2-y^2}$ density-wave state is called the staggered-flux phase, proposed to describe the normal state of high T_c cuprates.

3. *Particle-hole channel II: Uniform ordering*

a. Spin-singlet channel

$$\langle\psi^\dagger_\alpha(\boldsymbol{k},t)\psi_\beta(\boldsymbol{k},t)\rangle = \Phi f(\boldsymbol{k})\delta_{\alpha\beta} = \Phi\Big(\sum_{m=-l}^{l} d_m Y_{lm}(\hat{\boldsymbol{k}})\Big)\delta_{\alpha\beta} \tag{91}$$

b. Spin-triplet channel

$$\langle\psi^\dagger_\alpha(\boldsymbol{k},t)\psi_\beta(\boldsymbol{k},t)\rangle = m\boldsymbol{g}(\boldsymbol{k})\cdot\boldsymbol{\sigma}_{\alpha\beta} = m\Big(\sum_{m=-l}^{l} Y_{lm}(\hat{\boldsymbol{k}})\boldsymbol{d}_m\Big)\cdot\boldsymbol{\sigma}_{\alpha\beta} \tag{92}$$

4. *Effective theory*

Ref. Phys. Rev. B **80**, 064410 (2009)
a. Effective Hamiltonian

$$H = -J_1\sum_{\langle jl\rangle}\boldsymbol{S}_j\cdot\boldsymbol{S}_l + J_2\sum_{\langle\langle jl\rangle\rangle}\boldsymbol{S}_j\cdot\boldsymbol{S}_l \tag{93}$$

b. SU(2) projective representation (Fermion parton construction)

$$\boldsymbol{S}_j = \frac{1}{2}f^\dagger_{j\alpha}\boldsymbol{\sigma}_{\alpha\beta}f_{j\beta} \tag{94}$$

c. Matrix Nambu spinor for $SU_s(2)\times SU_c(2)$

$$\Psi_j = \begin{pmatrix} f_{j\uparrow} & f_{j\downarrow} \\ f^\dagger_{j\downarrow} & -f^\dagger_{j\uparrow}\end{pmatrix} \tag{95}$$

d. Effective theory

$$Z = \int D\Psi_j D\boldsymbol{a}_\tau \exp\Big[-\int_0^\beta d\tau\Big\{\frac{1}{2}\sum_j \mathrm{tr}[\Psi^\dagger_j(\partial_\tau\boldsymbol{I}+i\boldsymbol{a}_\tau\cdot\boldsymbol{\sigma})\Psi_j]+H\Big\}\Big] \tag{96}$$

$$Z = \int D\Psi_j D\boldsymbol{a}_\tau D\boldsymbol{U}^{sin}_{jl} D\boldsymbol{U}^{tri}_{jl,\mu} \exp\Big[- \int_0^\beta d\tau \Big\{ \frac{1}{2} \sum_j \mathrm{tr}[\Psi_j^\dagger (\partial_\tau \boldsymbol{I} + i\boldsymbol{a}_\tau \cdot \boldsymbol{\sigma})\Psi_j]$$
$$- \frac{J_2}{4} \sum_{\langle\langle jl \rangle\rangle} \mathrm{tr}[\Psi_j^\dagger \boldsymbol{U}^{sin}_{jl} \Psi_l] + \frac{J_2}{4} \sum_{\langle\langle jl \rangle\rangle} (|\chi_{jl}|^2 + |\Delta_{jl}|^2)$$
$$- \frac{J_1}{4} \sum_{\langle jl \rangle} \mathrm{tr}[\Psi_j^\dagger \boldsymbol{U}^{tri}_{jl,\mu} \Psi_l \boldsymbol{\sigma}_\mu^T] + \frac{J_1}{4} \sum_{\langle jl \rangle} (|\boldsymbol{E}_{jl}|^2 + |\boldsymbol{D}_{jl}|^2) \Big\} \Big] \quad (97)$$

e. Order-parameter matrix field

$$\boldsymbol{U}^{sin}_{jl} = \begin{pmatrix} \chi^*_{jl} & \Delta_{jl} \\ \Delta^*_{jl} & -\chi_{jl} \end{pmatrix}, \qquad \boldsymbol{U}^{tri}_{jl,\mu} = \begin{pmatrix} \boldsymbol{E}^*_{jl,\mu} & \boldsymbol{D}_{jl,\mu} \\ -\boldsymbol{D}^*_{jl,\mu} & \boldsymbol{E}_{jl,\mu} \end{pmatrix} \quad (98)$$

f. How to perform the decomposition (Hubbard-Stratonovich transformation)

$$4\boldsymbol{S}_j \cdot \boldsymbol{S}_l = -(f^\dagger_{j\alpha} f_{l\alpha})(f^\dagger_{l\beta} f_{j\beta}) - (f^\dagger_{j\alpha} [\boldsymbol{\sigma}_2]_{\alpha\beta} f^\dagger_{l\beta})(f_{l\gamma} [\boldsymbol{\sigma}_2]_{\gamma\delta} f_{j\delta}),$$
$$-4\boldsymbol{S}_j \cdot \boldsymbol{S}_l = -(f^\dagger_{j\alpha} [\boldsymbol{\sigma}_\mu]_{\alpha\beta} f_{l\beta})(f^\dagger_{l\gamma} [\boldsymbol{\sigma}_\mu]_{\gamma\delta} f_{j\delta}) - (f^\dagger_{j\alpha} [\boldsymbol{\sigma}_2 \boldsymbol{\sigma}_\mu]_{\alpha\beta} f^\dagger_{l\beta})(f_{l\gamma} [\boldsymbol{\sigma}_\mu \boldsymbol{\sigma}_2]_{\gamma\delta} f_{j\delta}) \quad (99)$$

$$\chi^*_{jl} = \langle f^\dagger_{j\alpha} f_{l\alpha} \rangle, \qquad \Delta_{jl} = \langle f_{l\gamma} [\boldsymbol{\sigma}_2]_{\gamma\delta} f_{j\delta} \rangle,$$
$$\boldsymbol{E}^*_{jl,\mu} = \langle f^\dagger_{j\alpha} [\boldsymbol{\sigma}_\mu]_{\alpha\beta} f_{l\beta} \rangle, \qquad \boldsymbol{D}_{jl,\mu} = \langle f_{l\gamma} [\boldsymbol{\sigma}_\mu \boldsymbol{\sigma}_2]_{\gamma\delta} f_{j\delta} \rangle \quad (100)$$

g. Spin quadrupolar ordering

$$-2Q_{jl,\mu\nu} = \mathrm{tr}[\boldsymbol{U}^{tri}_{lj,\mu} \boldsymbol{U}^{tri}_{jl,\nu}] - \frac{\delta_{\mu\nu}}{3} \mathrm{tr}[\boldsymbol{U}^{tri}_{lj,\lambda} \boldsymbol{U}^{tri}_{jl,\lambda}], \qquad -2iP_{jl,\lambda} = \mathrm{tr}[\boldsymbol{U}^{sin}_{lj} \boldsymbol{U}^{tri}_{jl,\lambda}] \quad (101)$$

$$K_{jl,\mu\nu} = \langle \boldsymbol{S}_{j\mu} \boldsymbol{S}_{l\nu} \rangle - \frac{\delta_{\mu\nu}}{3} \langle \boldsymbol{S}_j \cdot \boldsymbol{S}_l \rangle, \qquad Pjl,\lambda = \epsilon_{\lambda\mu\nu} K_{jl,\mu\nu},$$
$$Q_{jl,\mu\nu} = \frac{1}{2}(K_{jl,\mu\nu} + K_{jl,\nu\mu}), \qquad Q_{\mu\nu}(\boldsymbol{r}) = d_\mu(\boldsymbol{r}) d_\nu(\boldsymbol{r}) - \frac{1}{3}\delta_{\mu\nu} |\boldsymbol{d}(\boldsymbol{r})|^2 \quad (102)$$

B. Self-consistent RPA theory: Nambu-basis formalism vs. original-basis formalism

1. BCS theory

a. Effective theory

$$Z = \int Dc_{\boldsymbol{p}\sigma} \exp\Big[\int_0^\beta d\tau \Big\{ \sum_{\boldsymbol{p}} c^\dagger_{\boldsymbol{p}\sigma} (\partial_\tau - \mu + \epsilon_{\boldsymbol{p}}) c_{\boldsymbol{p}\sigma} + \sum_{\boldsymbol{q}} V_{\boldsymbol{q}} \sum_{\boldsymbol{p}} \sum_{\boldsymbol{p}'} c^\dagger_{\boldsymbol{p}+\boldsymbol{q}\sigma} c^\dagger_{\boldsymbol{p}'-\boldsymbol{q}\sigma'} c_{\boldsymbol{p}'\sigma'} c_{\boldsymbol{p}\sigma} \Big\} \Big] \quad (103)$$

$$Z = \int Dc_{\boldsymbol{p}\sigma} D\phi_{\boldsymbol{q}} D\Delta_{\boldsymbol{q}} \exp\Big[- \int_0^\beta d\tau \Big\{ \sum_{\boldsymbol{p}} c^\dagger_{\boldsymbol{p}\sigma} (\partial_\tau - \mu + \epsilon_{\boldsymbol{p}}) c_{\boldsymbol{p}\sigma} - i \sum_{\boldsymbol{q}} \sum_{\boldsymbol{p}} \phi_{\boldsymbol{q}} c^\dagger_{\boldsymbol{p}+\boldsymbol{q}\sigma} c_{\boldsymbol{p}\sigma} + \sum_{\boldsymbol{q}} \frac{1}{4V_{\boldsymbol{q}}} \phi_{\boldsymbol{q}} \phi_{-\boldsymbol{q}}$$
$$- \sum_{\boldsymbol{q}} \sum_{\boldsymbol{p}} \Delta^\dagger_{\boldsymbol{q}} c_{-\boldsymbol{p}+\boldsymbol{q}\downarrow} c_{\boldsymbol{p}\uparrow} - \sum_{\boldsymbol{q}} \sum_{\boldsymbol{p}} \Delta_{\boldsymbol{q}} c^\dagger_{\boldsymbol{p}+\boldsymbol{q}\uparrow} c^\dagger_{-\boldsymbol{p}\downarrow} - \sum_{\boldsymbol{q}} \frac{1}{V_{\boldsymbol{q}}} \Delta^\dagger_{\boldsymbol{q}} \Delta_{\boldsymbol{q}} \Big\} \Big] \quad (104)$$

$$V_{\boldsymbol{q}} \longrightarrow -V_{\boldsymbol{q}} \quad (105)$$

$$Z = \int Dc_{\boldsymbol{p}\sigma} D\Delta_{\boldsymbol{q}} \exp\Big[- \int_0^\beta d\tau \Big\{ \sum_{\boldsymbol{p}} c^\dagger_{\boldsymbol{p}\sigma}(\partial_\tau - \mu + \epsilon_{\boldsymbol{p}}) c_{\boldsymbol{p}\sigma}$$
$$- \sum_{\boldsymbol{q}} \sum_{\boldsymbol{p}} \Delta^\dagger_{\boldsymbol{q}} c_{-\boldsymbol{p}+\boldsymbol{q}\downarrow} c_{\boldsymbol{p}\uparrow} - \sum_{\boldsymbol{q}} \sum_{\boldsymbol{p}} \Delta_{\boldsymbol{q}} c^\dagger_{\boldsymbol{p}+\boldsymbol{q}\uparrow} c^\dagger_{-\boldsymbol{p}\downarrow} + \sum_{\boldsymbol{q}} \frac{1}{V_{\boldsymbol{q}}} \Delta^\dagger_{\boldsymbol{q}} \Delta_{\boldsymbol{q}} \Big\} \Big] \tag{106}$$

b. Nambu formulation

$$\psi_{\boldsymbol{p}} = \begin{pmatrix} c_{\boldsymbol{p}\uparrow} \\ c^\dagger_{-\boldsymbol{p}\downarrow} \end{pmatrix} \tag{107}$$

$$Z = \int D\psi_{\boldsymbol{p}} \exp\Big[- \sum_{i\omega} \sum_{\boldsymbol{p}} \Big\{ \psi^\dagger_{\boldsymbol{p}} \Big(- i\omega \boldsymbol{I} + [\epsilon_{\boldsymbol{p}} - \mu] \boldsymbol{\tau}_z + \Delta \boldsymbol{\tau}_x \Big) \psi_{\boldsymbol{p}} + \frac{\Delta^2}{V} \Big\} \Big] \tag{108}$$

$$F(\Delta) = -\frac{1}{\beta} \sum_{i\omega} \sum_{\boldsymbol{p}} \mathrm{tr} \ln \Big(- i\omega \boldsymbol{I} + [\epsilon_{\boldsymbol{p}} - \mu] \boldsymbol{\tau}_z + \Delta \boldsymbol{\tau}_x \Big) + L^d \frac{\Delta^2}{V} \tag{109}$$

$$F(\Delta) = -\frac{1}{\beta} \sum_{\boldsymbol{p}} \ln \Big\{ 1 + \exp\Big(- \beta \sqrt{(\epsilon_{\boldsymbol{p}} - \mu)^2 + \Delta^2} \Big) \Big\} - \frac{1}{\beta} \sum_{\boldsymbol{p}} \ln \Big\{ 1 + \exp\Big(\beta \sqrt{(\epsilon_{\boldsymbol{p}} - \mu)^2 + \Delta^2} \Big) \Big\} + L^d \frac{\Delta^2}{V}$$
$$= -\frac{1}{\beta} \sum_{\boldsymbol{p}} \ln \Big\{ 2 \cosh \Big(\frac{\beta}{2} \sqrt{(\epsilon_{\boldsymbol{p}} - \mu)^2 + \Delta^2} \Big) \Big\} + L^d \frac{\Delta^2}{V} \tag{110}$$

$$\frac{\partial F(\Delta)}{\partial \Delta} = 0 \longrightarrow \frac{1}{V} = \frac{1}{L^d} \sum_{\boldsymbol{p}} \frac{1}{\sqrt{(\epsilon_{\boldsymbol{p}} - \mu)^2 + \Delta^2}} \tanh \Big(\frac{\beta}{2} \sqrt{(\epsilon_{\boldsymbol{p}} - \mu)^2 + \Delta^2} \Big) \tag{111}$$

$$\boldsymbol{G}(\boldsymbol{p}, i\omega) = -\Big\langle \psi(\boldsymbol{p}, i\omega) \psi^\dagger(\boldsymbol{p}, i\omega) \Big\rangle = \begin{pmatrix} -\langle c_\uparrow(\boldsymbol{p}, i\omega) c^\dagger_\uparrow(\boldsymbol{p}, i\omega) \rangle & -\langle c_\uparrow(\boldsymbol{p}, i\omega) c_\downarrow(-\boldsymbol{p}, -i\omega) \rangle \\ -\langle c^\dagger_\downarrow(-\boldsymbol{p}, -i\omega) c_\uparrow(\boldsymbol{p}, i\omega) \rangle & -\langle c^\dagger_\downarrow(-\boldsymbol{p}, -i\omega) c_\downarrow(-\boldsymbol{p}, -i\omega) \rangle \end{pmatrix}$$
$$= \begin{pmatrix} G_{\uparrow\uparrow}(\boldsymbol{p}, i\omega) & F_{\uparrow\downarrow}(\boldsymbol{p}, i\omega) \\ \{F_{\uparrow\downarrow}(\boldsymbol{p}, i\omega)\}^\dagger & -G_{\downarrow\downarrow}(-\boldsymbol{p}, -i\omega) \end{pmatrix} = \frac{1}{(i\omega)^2 - (\epsilon_{\boldsymbol{p}} - \mu)^2 - \Delta^2} \begin{pmatrix} i\omega - (\epsilon_{\boldsymbol{p}} - \mu) & -\Delta \\ -\Delta & i\omega - (\epsilon_{\boldsymbol{p}} - \mu) \end{pmatrix} \tag{112}$$

$$G_{\uparrow\uparrow}(\boldsymbol{p}, i\omega) = \frac{i\omega + (\epsilon_{\boldsymbol{p}} - \mu)}{(i\omega)^2 - (\epsilon_{\boldsymbol{p}} - \mu)^2 - \Delta^2} = \frac{u^2_{\boldsymbol{p}}}{i\omega - \sqrt{(\epsilon_{\boldsymbol{p}} - \mu)^2 + \Delta^2}} + \frac{v^2_{\boldsymbol{p}}}{i\omega + \sqrt{(\epsilon_{\boldsymbol{p}} - \mu)^2 + \Delta^2}} \tag{113}$$

$$u^2_{\boldsymbol{p}} = \frac{1}{2} \Big(1 + \frac{(\epsilon_{\boldsymbol{p}} - \mu)}{\sqrt{(\epsilon_{\boldsymbol{p}} - \mu)^2 + \Delta^2}} \Big), \qquad v^2_{\boldsymbol{p}} = \frac{1}{2} \Big(1 - \frac{(\epsilon_{\boldsymbol{p}} - \mu)}{\sqrt{(\epsilon_{\boldsymbol{p}} - \mu)^2 + \Delta^2}} \Big) \tag{114}$$

$$A_{\uparrow\uparrow}(\boldsymbol{p}, \omega + i\delta) = u^2_{\boldsymbol{p}} \delta(\omega - E_{\boldsymbol{p}}) + v^2_{\boldsymbol{p}} \delta(\omega + E_{\boldsymbol{p}}), \qquad E_{\boldsymbol{p}} = \sqrt{(\epsilon_{\boldsymbol{p}} - \mu)^2 + \Delta^2} \tag{115}$$

c. Self-consistent RPA theory

$$Z = \int Dc_{\boldsymbol{p}\sigma} D\Delta_{\boldsymbol{q}} \exp\Big[- \int_0^\beta d\tau \Big\{ \sum_{\boldsymbol{p}} c^\dagger_{\boldsymbol{p}\sigma}(\partial_\tau - \mu + \epsilon_{\boldsymbol{p}}) c_{\boldsymbol{p}\sigma}$$
$$- \sum_{\boldsymbol{q}} \sum_{\boldsymbol{p}} \Delta^\dagger_{\boldsymbol{q}} c_{-\boldsymbol{p}+\boldsymbol{q}\downarrow} c_{\boldsymbol{p}\uparrow} - \sum_{\boldsymbol{q}} \sum_{\boldsymbol{p}} \Delta_{\boldsymbol{q}} c^\dagger_{\boldsymbol{p}+\boldsymbol{q}\uparrow} c^\dagger_{-\boldsymbol{p}\downarrow} + \sum_{\boldsymbol{q}} \frac{1}{V_{\boldsymbol{q}}} \Delta^\dagger_{\boldsymbol{q}} \Delta_{\boldsymbol{q}} \Big\} \Big] \tag{116}$$

$$S_{int} = -\int_0^\beta d\tau \sum_{\boldsymbol{q}} \sum_{\boldsymbol{p}} \left(\Delta_{\boldsymbol{q}}^\dagger c_{-\boldsymbol{p}+\boldsymbol{q}\downarrow} c_{\boldsymbol{p}\uparrow} + \Delta_{\boldsymbol{q}} c_{\boldsymbol{p}+\boldsymbol{q}\uparrow}^\dagger c_{-\boldsymbol{p}\downarrow}^\dagger \right) \tag{117}$$

$$S_{eff}^{(2)} = -\frac{1}{2}\left(\langle S_{int}^2 \rangle - \langle S_{int} \rangle^2 \right) \tag{118}$$

$$\begin{aligned} S_{eff}^{(2)} &= -\int_0^\beta d\tau \int_0^\beta d\tau' \sum_{\boldsymbol{q}} \sum_{\boldsymbol{p}} \sum_{\boldsymbol{q}'} \sum_{\boldsymbol{p}'} \Delta_{\boldsymbol{q}'}^\dagger \Delta_{\boldsymbol{q}} c_{\boldsymbol{p}+\boldsymbol{q}\uparrow}^\dagger c_{-\boldsymbol{p}\downarrow}^\dagger c_{-\boldsymbol{p}'+\boldsymbol{q}'\downarrow} c_{\boldsymbol{p}'\uparrow} \\ &= -\int_0^\beta d\tau \int_0^\beta d\tau' \sum_{\boldsymbol{q}} \sum_{\boldsymbol{p}} \sum_{\boldsymbol{p}'} \Big\{ D(\boldsymbol{q}, \tau'-\tau) c_{\boldsymbol{p}+\boldsymbol{q}\uparrow}^\dagger c_{-\boldsymbol{p}\downarrow}^\dagger c_{-\boldsymbol{p}'+\boldsymbol{q}\downarrow} c_{\boldsymbol{p}'\uparrow} + \Delta_{\boldsymbol{q}}^\dagger \Pi(\boldsymbol{q}, \tau-\tau') \Delta_{\boldsymbol{q}} - D(\boldsymbol{q}, \tau'-\tau) \Pi(\boldsymbol{q}, \tau-\tau') \Big\} \end{aligned} \tag{119}$$

$$\begin{aligned} &-\int_0^\beta d\tau \int_0^\beta d\tau' \sum_{\boldsymbol{q}} \sum_{\boldsymbol{p}} \sum_{\boldsymbol{p}'} D(\boldsymbol{q}, \tau'-\tau) c_{\boldsymbol{p}+\boldsymbol{q}\uparrow}^\dagger c_{-\boldsymbol{p}\downarrow}^\dagger c_{-\boldsymbol{p}'+\boldsymbol{q}\downarrow} c_{\boldsymbol{p}'\uparrow} \\ &= -\int_0^\beta d\tau \int_0^\beta d\tau' \sum_{\boldsymbol{q}} \sum_{\boldsymbol{p}} D(\boldsymbol{q}, \tau'-\tau) \Big(c_{\boldsymbol{p}\uparrow}^\dagger G_{\downarrow\downarrow}(-\boldsymbol{p}+\boldsymbol{q}, \tau-\tau') c_{\boldsymbol{p}\uparrow} + c_{\boldsymbol{p}\downarrow}^\dagger G_{\uparrow\uparrow}(-\boldsymbol{p}+\boldsymbol{q}, \tau'-\tau) c_{\boldsymbol{p}\downarrow} \\ &- G_{\downarrow\downarrow}(-\boldsymbol{p}, \tau-\tau') G_{\uparrow\uparrow}(\boldsymbol{p}+\boldsymbol{q}, \tau'-\tau) \Big) \end{aligned} \tag{120}$$

$$\begin{aligned} Z &= \int Dc_{\boldsymbol{p}\sigma} D\Delta_{\boldsymbol{q}} \exp\Big[-\Big\{ \sum_{i\omega} \sum_{\boldsymbol{p}} c_{\boldsymbol{p}\uparrow}^\dagger \Big(-i\omega - \mu + \epsilon_{\boldsymbol{p}} - \frac{1}{\beta} \sum_{i\Omega} \sum_{\boldsymbol{q}} D(\boldsymbol{q}, i\Omega) G_{\downarrow\downarrow}(-\boldsymbol{p}+\boldsymbol{q}, -i\omega+i\Omega) \Big) c_{\boldsymbol{p}\uparrow} \\ &+ \sum_{i\omega} \sum_{\boldsymbol{p}} c_{\boldsymbol{p}\uparrow}^\dagger \Big(-i\omega - \mu + \epsilon_{\boldsymbol{p}} - \frac{1}{\beta} \sum_{i\Omega} \sum_{\boldsymbol{q}} D(\boldsymbol{q}, i\Omega) G_{\uparrow\uparrow}(-\boldsymbol{p}+\boldsymbol{q}, -i\omega+i\Omega) \Big) c_{\boldsymbol{p}\uparrow} \\ &+ \sum_{i\Omega} \sum_{\boldsymbol{q}} \left(\Delta_{\boldsymbol{q}}^\dagger \Big(\frac{1}{V_{\boldsymbol{q}}} - \Pi(\boldsymbol{q}, i\Omega) \Big) \Delta_{\boldsymbol{q}} + \Pi(\boldsymbol{q}, i\Omega) D(\boldsymbol{q}, i\Omega) \right) + \sum_{i\omega} \frac{1}{\beta} \sum_{i\Omega} \sum_{\boldsymbol{p}} \sum_{\boldsymbol{q}} D(\boldsymbol{q}, i\Omega) G_{\downarrow\downarrow}(-\boldsymbol{p}, -i\omega) G_{\uparrow\uparrow}(\boldsymbol{p}+\boldsymbol{q}, i\omega+i\Omega) \Big\} \Big] \end{aligned} \tag{121}$$

$$\begin{aligned} F[D(\boldsymbol{q}, i\Omega), G_{\downarrow\downarrow}(\boldsymbol{p}, i\omega), G_{\uparrow\uparrow}(\boldsymbol{p}, i\omega)] &= -\frac{1}{\beta} \sum_{i\omega} \sum_{\boldsymbol{p}} \ln\Big(-i\omega - \mu + \epsilon_{\boldsymbol{p}} - \frac{1}{\beta} \sum_{i\Omega} \sum_{\boldsymbol{q}} D(\boldsymbol{q}, i\Omega) G_{\downarrow\downarrow}(-\boldsymbol{p}+\boldsymbol{q}, i\omega+i\Omega) \Big) \\ &- \frac{1}{\beta} \sum_{i\omega} \sum_{\boldsymbol{p}} \ln\Big(-i\omega - \mu + \epsilon_{\boldsymbol{p}} - \frac{1}{\beta} \sum_{i\Omega} \sum_{\boldsymbol{q}} D(\boldsymbol{q}, i\Omega) G_{\uparrow\uparrow}(-\boldsymbol{p}+\boldsymbol{q}, i\omega+i\Omega) \Big) \\ &+ \frac{1}{\beta} \sum_{i\Omega} \sum_{\boldsymbol{q}} \Big\{ \ln\Big(\frac{1}{V_{\boldsymbol{q}}} - \Pi(\boldsymbol{q}, i\Omega) \Big) + \Pi(\boldsymbol{q}, i\Omega) D(\boldsymbol{q}, i\Omega) \Big\} \\ &+ \frac{1}{\beta} \sum_{i\omega} \frac{1}{\beta} \sum_{i\Omega} \sum_{\boldsymbol{p}} \sum_{\boldsymbol{q}} D(\boldsymbol{q}, i\Omega) G_{\downarrow\downarrow}(-\boldsymbol{p}, -i\omega) G_{\uparrow\uparrow}(\boldsymbol{p}+\boldsymbol{q}, i\omega+i\Omega) \end{aligned} \tag{122}$$

$$\begin{aligned} G_{\uparrow\uparrow}(\boldsymbol{p}, i\omega) &= \frac{1}{i\omega + \mu - \epsilon_{\boldsymbol{p}} + \frac{1}{\beta} \sum_{i\Omega} \sum_{\boldsymbol{q}} D(\boldsymbol{q}, i\Omega) G_{\downarrow\downarrow}(-\boldsymbol{p}+\boldsymbol{q}, -i\omega+i\Omega)}, \\ G_{\downarrow\downarrow}(\boldsymbol{p}, i\omega) &= \frac{1}{i\omega + \mu - \epsilon_{\boldsymbol{p}} + \frac{1}{\beta} \sum_{i\Omega} \sum_{\boldsymbol{q}} D(\boldsymbol{q}, i\Omega) G_{\uparrow\uparrow}(-\boldsymbol{p}+\boldsymbol{q}, -i\omega+i\Omega)}, \\ \Pi(\boldsymbol{q}, i\Omega) &= -\frac{1}{\beta} \sum_{i\omega} \sum_{\boldsymbol{p}} G_{\downarrow\downarrow}(-\boldsymbol{p}, -i\omega) G_{\uparrow\uparrow}(\boldsymbol{p}+\boldsymbol{q}, i\omega+i\Omega), \\ D(\boldsymbol{q}, i\Omega) &= \frac{1}{\frac{1}{V_{\boldsymbol{q}}} - \Pi(\boldsymbol{q}, i\Omega)} \end{aligned} \tag{123}$$

$$G_{\uparrow\uparrow}(\boldsymbol{p},i\omega) \approx \frac{1}{i\omega+\mu-\epsilon_{\boldsymbol{p}}+D(0,0)G_{\downarrow\downarrow}(-\boldsymbol{p},-i\omega)},$$
$$G_{\downarrow\downarrow}(\boldsymbol{p},i\omega) \approx \frac{1}{i\omega+\mu-\epsilon_{\boldsymbol{p}}+D(\boldsymbol{0},0)G_{\uparrow\uparrow}(-\boldsymbol{p},-i\omega)},$$
$$\Pi(0,0) \approx -\frac{1}{\beta}\sum_{i\omega}\sum_{\boldsymbol{p}} G_{\downarrow\downarrow}(-\boldsymbol{p},-i\omega)G_{\uparrow\uparrow}(\boldsymbol{p},i\omega), \qquad D(0,0) = \frac{1}{\frac{1}{V_0}-\Pi(0,0)} \tag{124}$$

$$\Sigma_{\uparrow\uparrow}(\boldsymbol{p},i\omega) = D(0,0)G_{\downarrow\downarrow}(-\boldsymbol{p},-i\omega) \approx D(0,0)g_{\downarrow\downarrow}(-\boldsymbol{p},-i\omega) = -\frac{\Delta^2}{i\omega+(\epsilon_{\boldsymbol{p}}-\mu)},$$
$$\Sigma_{\downarrow\downarrow}(\boldsymbol{p},i\omega) = D(\boldsymbol{0},0)G_{\uparrow\uparrow}(-\boldsymbol{p},-i\omega) \approx D(\boldsymbol{0},0)g_{\uparrow\uparrow}(-\boldsymbol{p},-i\omega) = -\frac{\Delta^2}{i\omega+(\epsilon_{\boldsymbol{p}}-\mu)} \tag{125}$$

$$\Delta^2 = D(0,0) = \frac{1}{\frac{1}{V_0}-\Pi(0,0)} \tag{126}$$

$$\Pi(0,0) = -\frac{1}{\beta}\sum_{i\omega}\sum_{\boldsymbol{p}} G_{\downarrow\downarrow}(-\boldsymbol{p},-i\omega)G_{\uparrow\uparrow}(\boldsymbol{p},i\omega) \neq -\frac{1}{\beta}\sum_{i\omega}\sum_{\boldsymbol{p}} g_{\downarrow\downarrow}(-\boldsymbol{p},-i\omega)g_{\uparrow\uparrow}(\boldsymbol{p},i\omega)$$
$$= -\frac{1}{\beta}\sum_{i\omega}\sum_{\boldsymbol{p}} \frac{1}{-i\omega+\mu-\epsilon_{\boldsymbol{p}}-\frac{\Delta^2}{-i\omega+(\epsilon_{\boldsymbol{p}}-\mu)}}\frac{1}{i\omega+\mu-\epsilon_{\boldsymbol{p}}-\frac{\Delta^2}{i\omega+(\epsilon_{\boldsymbol{p}}-\mu)}} \tag{127}$$

$$\Delta^2 = \frac{1}{\frac{1}{V_0}+\frac{1}{\beta}\sum_{i\omega}\sum_{\boldsymbol{p}}\frac{1}{-i\omega+\mu-\epsilon_{\boldsymbol{p}}-\frac{\Delta^2}{-i\omega+(\epsilon_{\boldsymbol{p}}-\mu)}}\frac{1}{i\omega+\mu-\epsilon_{\boldsymbol{p}}-\frac{\Delta^2}{i\omega+(\epsilon_{\boldsymbol{p}}-\mu)}}} \tag{128}$$

C. Effective field theory in symmetry breaking

Ref. I. Vekhter and A. V. Chubukov, Phys. Rev. Lett. **93**, 016405 (2004)
a. Spin-fermion model for magnetic instability

$$Z = \int Dc_{\boldsymbol{k}\sigma}D\boldsymbol{S}_{\boldsymbol{q}} \exp\Big[-\int_0^\beta d\tau\Big\{\sum_{\boldsymbol{k}} c_{\boldsymbol{k}\sigma}^\dagger(\partial_\tau-\mu+\epsilon_{\boldsymbol{k}})c_{\boldsymbol{k}\sigma}$$
$$+g\sum_{\boldsymbol{k}}\frac{1}{L^d}\sum_{\boldsymbol{q}}\boldsymbol{S}_{\boldsymbol{q}}\cdot c_{\boldsymbol{k}+\boldsymbol{q}\alpha}^\dagger\boldsymbol{\sigma}_{\alpha\beta}c_{\boldsymbol{k}\beta}+\sum_{\boldsymbol{q}}\boldsymbol{S}_{\boldsymbol{q}}(-\partial_\tau^2+v_s^2\boldsymbol{q}^2+m_s^2)\boldsymbol{S}_{-\boldsymbol{q}}\Big\}\Big] \tag{129}$$

b. Self-consistent RPA theory $\longrightarrow$ Spontaneous symmetry breaking

$$\boldsymbol{S}_{\boldsymbol{q}} \approx m\hat{\boldsymbol{z}}\delta^{(d)}(\boldsymbol{q}-\boldsymbol{Q})+\delta\boldsymbol{S}_{\boldsymbol{q}} \tag{130}$$

$$Z = \int Dc_{\boldsymbol{k}\sigma}D\delta\boldsymbol{S}_{\boldsymbol{q}} \exp\Big[-\int_0^\beta d\tau\Big\{\sum_{\boldsymbol{k}}\Big(c_{\boldsymbol{k}\sigma}^\dagger(\partial_\tau-\mu+\epsilon_{\boldsymbol{k}})c_{\boldsymbol{k}\sigma}+gm\sigma c_{\boldsymbol{k}+\boldsymbol{Q}\sigma}^\dagger c_{\boldsymbol{k}\sigma}\Big)$$
$$+g\sum_{\boldsymbol{k}}\frac{1}{L^d}\sum_{\boldsymbol{q}}\delta\boldsymbol{S}_{\boldsymbol{q}}\cdot c_{\boldsymbol{k}+\boldsymbol{q}\alpha}^\dagger\boldsymbol{\sigma}_{\alpha\beta}c_{\boldsymbol{k}\beta}+L[m\hat{\boldsymbol{z}}\delta^{(d)}(\boldsymbol{q}-\boldsymbol{Q})+\delta\boldsymbol{S}_{\boldsymbol{q}}]\Big\}\Big] \tag{131}$$

c. Effective order-parameter field theory in symmetry breaking

$$L[\boldsymbol{S_x}] = \int d^d\boldsymbol{x}\Big\{\boldsymbol{S_x}(-\partial_\tau^2 - v_s^2\boldsymbol{\partial}^2 + m_s^2)\boldsymbol{S_x} + u[\boldsymbol{S_x}\cdot\boldsymbol{S_x}]^2 + v[\boldsymbol{S_x}\times\boldsymbol{S_x}]^2\Big\} \tag{132}$$

$$m_s^2 < 0 \tag{133}$$

Ref. L. H. Ryder, Quantum Field Theory (Cambridge)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_i\partial_\mu\phi_i - \frac{m^2}{2}\phi_i\phi_i - \lambda(\phi_i\phi_i)^2 \tag{134}$$

$$G:\ \phi_i \longrightarrow e^{iQ_k\alpha_k}\phi_i e^{-iQ_k\alpha_k} = (e^{-iT_k\alpha_k})_{ij}\phi_j = U_{ij}\phi_j = [U(g)\phi]_i \tag{135}$$

$$V[\phi_i\phi_i] = \frac{m^2}{2}\phi_i\phi_i + \lambda(\phi_i\phi_i)^2 \tag{136}$$

$$m^2 < 0 \longrightarrow \vec{\phi}_0 = a\hat{e}_3, \qquad a = \Big(-\frac{m^2}{4\lambda}\Big)^{1/2} \tag{137}$$

$$G:\ \phi_0' = U(g)\phi_0 \neq \phi_0 \tag{138}$$

$$H:\ \phi_0' = U(h)\phi_0 = \phi_0, \qquad U(h) = e^{iT_3\alpha_3} \tag{139}$$

$$V(\phi') = V(\phi), \qquad \phi' = U(g)\phi \tag{140}$$

$$\phi_3 = \chi + a \tag{141}$$

$$V(\phi_1,\phi_2,\chi) = 4a^2\lambda\chi^2 + 4a\lambda\chi(\phi_1^2+\phi_2^2+\chi^2) + \lambda(\phi_1^2+\phi_2^2+\chi^2)^2 - \lambda a^4 \tag{142}$$

$$m_\chi^2 = 8a^2\lambda, \qquad m_{\phi_1}^2 = m_{\phi_2}^2 = 0 \tag{143}$$

$$\mathrm{Dim}(G/H) = \text{The number of gapless transverse modes} \tag{144}$$

This counting rule works only in the case of Lorentz invariance. What happens in the case of Galilean invariance?
d. Effective field theory for an itinerant SDW (spin-density-wave) phase

$$H_f = \sum_{n=\pm}\sum_{\boldsymbol{k}} E_n(\boldsymbol{k}) a^\dagger_{\boldsymbol{k}n\sigma}a_{\boldsymbol{k}n\sigma},$$
$$E_\pm(\boldsymbol{k}) = \frac{1}{2}(\epsilon_{\boldsymbol{k}} - \mu + \epsilon_{\boldsymbol{k}+\boldsymbol{Q}} - \mu) \pm \sqrt{\Big(\frac{\epsilon_k - \epsilon_{\boldsymbol{k}+\boldsymbol{Q}}}{2}\Big) + m^2} \tag{145}$$

$$a^\dagger_{\boldsymbol{k}+\sigma} = u_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\sigma} - \sigma v_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}+\boldsymbol{Q}\sigma}, \qquad a^\dagger_{\boldsymbol{k}-\sigma} = \sigma v_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\sigma} + u_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}+\boldsymbol{Q}\sigma},$$
$$u^2_{\boldsymbol{k}} = \frac{1}{2}\Big(1 + \frac{\frac{\epsilon_k - \epsilon_{\boldsymbol{k}+\boldsymbol{Q}}}{2}}{\sqrt{\Big(\frac{\epsilon_k - \epsilon_{\boldsymbol{k}+\boldsymbol{Q}}}{2}\Big) + m^2}}\Big), \qquad v^2_{\boldsymbol{k}} = \frac{1}{2}\Big(1 - \frac{\frac{\epsilon_k - \epsilon_{\boldsymbol{k}+\boldsymbol{Q}}}{2}}{\sqrt{\Big(\frac{\epsilon_k - \epsilon_{\boldsymbol{k}+\boldsymbol{Q}}}{2}\Big) + m^2}}\Big) \tag{146}$$

$$H_{sf} = g \sum_{\boldsymbol{k}} \frac{1}{L^d} \sum_{\boldsymbol{q}} \Gamma^{\alpha\beta}_{nn'}(\boldsymbol{k}, \boldsymbol{k}+\boldsymbol{q}) a^\dagger_{\boldsymbol{k}n\alpha} a_{\boldsymbol{k}+\boldsymbol{q}n'\beta} \boldsymbol{\sigma}_{\alpha\beta} \cdot \delta \boldsymbol{S}_{-\boldsymbol{q}},$$
$$\Gamma^{\alpha\beta}_{nn'}(\boldsymbol{k}, \boldsymbol{k}+\boldsymbol{q}) = [u_{\boldsymbol{k}} u_{\boldsymbol{k}+\boldsymbol{q}} - \alpha\beta v_{\boldsymbol{k}} v_{\boldsymbol{k}+\boldsymbol{q}}][1 - \delta_{nn'}] - n\delta_{nn'}[\alpha v_{\boldsymbol{k}} u_{\boldsymbol{k}+\boldsymbol{q}} + \beta u_{\boldsymbol{k}} v_{\boldsymbol{k}+\boldsymbol{q}}] \tag{147}$$

The vertex for the Goldstone transverse mode is given by $\Gamma^{\alpha\beta,tr}_{nn'}(\boldsymbol{k}, \boldsymbol{k}+\boldsymbol{q}) \propto |\boldsymbol{q}|$ at small $\boldsymbol{q}$, as required by Adler's principle.

Degrees of freedom: Renormalized quasiparticles + 1 gapped longitudinal mode + 2 gapless transverse modes + a gradient-type coupling between such renormalized fermions and gapless transverse modes $\longrightarrow$ Landau's Fermi liquid state with symmetry breaking

D. Problem: Effective field theory of symmetry breaking with spin-orbit coupling or nematicity ?

V. HERTZ-MORIYA-MILLIS QUANTUM CRITICALITY

A. Hertz-Moriya-Millis theory and renormalization group analysis

1. *Effective field theory for an itinerant Ising ferromagnetic phase transition*

$$Z = \int Dc_{\boldsymbol{p}\sigma} D\phi_{\boldsymbol{q}} \exp\Big[- \int_0^\beta d\tau \Big\{ \sum_{\boldsymbol{p}} c^\dagger_{\boldsymbol{p}\sigma}(\partial_\tau - \mu + \epsilon_{\boldsymbol{p}}) c_{\boldsymbol{p}\sigma} + g \sum_{\boldsymbol{p}} \frac{1}{L^d} \sum_{\boldsymbol{q}} \phi_{\boldsymbol{q}} \sigma c^\dagger_{\boldsymbol{p}+\boldsymbol{q}\sigma} c_{\boldsymbol{p}\sigma}$$
$$+ \sum_{\boldsymbol{q}} \phi_{\boldsymbol{q}}(-\partial^2_\tau + v^2_\phi \boldsymbol{q}^2 + m^2_\phi)\phi_{-\boldsymbol{q}} + \frac{u_\phi}{2} \sum_{\boldsymbol{q}} \frac{1}{L^d} \sum_{\boldsymbol{q}'} \frac{1}{L^d} \sum_{\boldsymbol{l}} \phi_{\boldsymbol{q}+\boldsymbol{l}} \phi_{\boldsymbol{q}'-\boldsymbol{l}} \phi_{-\boldsymbol{q}'} \phi_{-\boldsymbol{q}} \Big\} \Big] \tag{148}$$

2. *Recalling the polarization bubble*

$$\Pi(\boldsymbol{q}, \Omega + i\delta) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_{\boldsymbol{p}} \Big\{ \frac{Z(\omega+\Omega)}{\omega + \Omega - Z(\omega+\Omega) Z(\boldsymbol{p}+\boldsymbol{q}) \boldsymbol{v}_F \cdot (\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}_F)}$$
$$- i\pi\delta\Big(Z^{-1}(\omega+\Omega)(\omega+\Omega) - Z(\boldsymbol{p}+\boldsymbol{q})\boldsymbol{v}_F \cdot (\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}_F)\Big)\Big\} \Big\{ \frac{Z(\omega)}{\omega - Z(\omega) Z(\boldsymbol{p}) \boldsymbol{v}_F \cdot (\boldsymbol{p}-\boldsymbol{p}_F)}$$
$$- i\pi\delta\Big(Z^{-1}(\omega)\omega - Z(\boldsymbol{p})\boldsymbol{v}_F \cdot (\boldsymbol{p}-\boldsymbol{p}_F)\Big)\Big\}$$
$$\approx \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_{\boldsymbol{p}} \Big\{ \frac{Z_\omega}{\omega + \Omega - Z_\omega Z_p \boldsymbol{v}_F \cdot (\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}_F)} - i\pi\delta\Big(Z^{-1}_\omega(\omega+\Omega) - Z_p \boldsymbol{v}_F \cdot (\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}_F)\Big)\Big\}$$
$$\Big\{ \frac{Z_\omega}{\omega - Z_\omega Z_p \boldsymbol{v}_F \cdot (\boldsymbol{p}-\boldsymbol{p}_F)} - i\pi\delta\Big(Z^{-1}_\omega \omega - Z_p \boldsymbol{v}_F \cdot (\boldsymbol{p}-\boldsymbol{p}_F)\Big)\Big\}$$
$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sum_{\boldsymbol{p}} \frac{Z_\omega}{\omega + \Omega - Z_\omega Z_p \boldsymbol{v}_F \cdot (\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}_F) + i\delta} \frac{Z_\omega}{\omega - Z_\omega Z_p \boldsymbol{v}_F \cdot (\boldsymbol{p}-\boldsymbol{p}_F) + i\delta} \tag{149}$$

$$\Pi(\boldsymbol{q},\Omega+i\delta)=\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\sum_{\boldsymbol{p}}\Big\{\frac{1}{\omega-Z_\omega Z_p\boldsymbol{v}_F\cdot(\boldsymbol{p}-\boldsymbol{p}_F)+i\delta}-\frac{1}{\omega+\Omega-Z_\omega Z_p\boldsymbol{v}_F\cdot(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}_F)+i\delta}\Big\}\frac{Z_\omega}{\Omega-Z_\omega Z_p\boldsymbol{v}_F\cdot\boldsymbol{q}}$$
$$=\frac{1}{4\pi}\sum_{\boldsymbol{p}}\frac{Z_\omega Z_\omega Z_p\boldsymbol{v}_F\cdot\boldsymbol{q}}{\Omega-Z_\omega Z_p\boldsymbol{v}_F\cdot\boldsymbol{q}}=Z_\omega\frac{N_F}{2}\int_{-1}^{1}d\cos\theta\Big(-1+\frac{\Omega}{\Omega-Z_\omega Z_p v_F q\cos\theta}\Big)$$
$$=Z_\omega\frac{N_F}{2}\Big\{-2+\frac{\Omega}{Z_\omega Z_p v_F q}\ln\Big(\frac{\Omega+Z_\omega Z_p v_F q}{\Omega-Z_\omega Z_p v_F q}\Big)\Big\}=-Z_\omega N_F+\frac{N_F}{2}\frac{\Omega}{Z_p v_F q}\ln\Big(\frac{\Omega+Z_\omega Z_p v_F q}{\Omega-Z_\omega Z_p v_F q}\Big)$$
$$\approx -Z_\omega N_F+\frac{N_F}{2}\frac{\Omega}{Z_p v_F q}i\mathrm{sgn}(\Omega)+v^2\boldsymbol{q}^2 \tag{150}$$

3. *Hertz-Moriya-Millis theory for an itinerant Ising ferromagnetic phase transition*

$$Z=\int D\phi_{\boldsymbol{q}}\exp\Big[-\Big\{\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\int\frac{d^d\boldsymbol{q}}{(2\pi)^d}\phi_{\boldsymbol{q}}\Big(\gamma\frac{|\Omega|}{|\boldsymbol{q}|}+v_\phi^2\boldsymbol{q}^2+m_\phi^2\Big)\phi_{-\boldsymbol{q}}$$
$$+\int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\Omega'}{2\pi}\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\int\frac{d^d\boldsymbol{q}}{(2\pi)^d}\int\frac{d^d\boldsymbol{q}'}{(2\pi)^d}\int\frac{d^d\boldsymbol{l}}{(2\pi)^d}\frac{u_\phi}{2}\phi_{\boldsymbol{q}+\boldsymbol{l}}\phi_{\boldsymbol{q}'-\boldsymbol{l}}\phi_{-\boldsymbol{q}'}\phi_{-\boldsymbol{q}}\Big\}\Big] \tag{151}$$

4. *Renormalization group analysis*

a. Tree level scaling

$$\Omega=\frac{\Omega_r}{r^z},\qquad \boldsymbol{q}=\frac{\boldsymbol{q}_r}{r},\qquad z=3 \tag{152}$$

$$\phi(\boldsymbol{q}_r/r,i\Omega_r/r^z)=r^{\Delta_\phi}\phi(\boldsymbol{q}_r,i\Omega_r),\qquad m_\phi^2=r^{\Delta_{m^2}}m_\phi^2,\qquad u_\phi=r^{\Delta_u}u_\phi \tag{153}$$

$$Z=r^{L^d\Delta_\phi}\int D\phi_r(\boldsymbol{q}_r,i\Omega_r)\exp\Big[-\Big\{\int_{-\infty_r}^{\infty_r}\frac{d\Omega_r}{2\pi}\int\frac{d^d\boldsymbol{q}_r}{(2\pi)^d}\phi(\boldsymbol{q}_r,i\Omega_r)\Big(\gamma\frac{|\Omega_r|}{|\boldsymbol{q}_r|^{z-2}}+v_\phi^2\boldsymbol{q}_r^2+m_\phi^2\Big)\phi(-\boldsymbol{q}_r,-i\Omega_r)$$
$$+\int_{-\infty_r}^{\infty_r}\frac{d\Omega_r}{2\pi}\int_{-\infty_r}^{\infty_r}\frac{d\Omega_r'}{2\pi}\int_{-\infty_r}^{\infty_r}\frac{d\nu_r}{2\pi}\int\frac{d^d\boldsymbol{q}_r}{(2\pi)^d}\int\frac{d^d\boldsymbol{q}_r'}{(2\pi)^d}\int\frac{d^d\boldsymbol{l}_r}{(2\pi)^d}$$
$$\frac{u_\phi}{2}\phi(\boldsymbol{q}_r+\boldsymbol{l}_r,i\Omega_r+i\nu_r)\phi'(\boldsymbol{q}_r'-\boldsymbol{l}_r,i\Omega_r'-i\nu_r)\phi(-\boldsymbol{q}_r',-i\Omega_r')\phi(-\boldsymbol{q}_r,-i\Omega_r)\Big\}\Big] \tag{154}$$

$$r^{-z}r^{-d}r^{-z}r^{z-2}r^{2\Delta_\phi}=r^0\longrightarrow\Delta_\phi=\frac{d+z+2}{2},$$
$$r^{-z}r^{-d}r^{\Delta_{m^2}}r^{2\Delta_\phi}=r^0\longrightarrow\Delta_{m^2}=-2,$$
$$r^{-3z}r^{-3d}r^{\Delta_u}r^{4\Delta_\phi}=r^0\longrightarrow\Delta_u=-[4-(d+z)] \tag{155}$$

$$\frac{dm_\phi^2}{d\ln r}=0\longrightarrow\frac{dm_\phi^2}{d\ln r}=2m_\phi^2,$$
$$\frac{du_\phi}{d\ln r}=0\longrightarrow\frac{du_\phi}{d\ln r}=[4-(d+z)]u_\phi \tag{156}$$

b. Introduction of quantum corrections

$$\phi_{\boldsymbol{q}} = \phi_{\boldsymbol{q}_h}\Theta(\Lambda > |\boldsymbol{q}_h| > \Lambda/r) + \phi_{\boldsymbol{q}_l}\Theta(\Lambda/r > |\boldsymbol{q}_l|) \tag{157}$$

$$\begin{aligned} Z = \int D\phi_{\boldsymbol{q}_l} D\phi_{\boldsymbol{q}_h} \exp\Big[- \Big\{ \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int_{\Lambda/r}^{\Lambda} \frac{d^d\boldsymbol{q}_h}{(2\pi)^d} \phi_{\boldsymbol{q}_h} \Big(\gamma \frac{|\Omega|}{|\boldsymbol{q}_h|} + v_\phi^2 \boldsymbol{q}_h^2 + m_\phi^2 \Big) \phi_{-\boldsymbol{q}_h} \\ + \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\Omega'}{2\pi} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \int_{\Lambda/r}^{\Lambda} \frac{d^d\boldsymbol{q}_h}{(2\pi)^d} \int_{\Lambda/r}^{\Lambda} \frac{d^d\boldsymbol{q}'_h}{(2\pi)^d} \int_{\Lambda/r}^{\Lambda} \frac{d^d\boldsymbol{l}_h}{(2\pi)^d} \int_{\Lambda/r}^{\Lambda} \frac{d^d\boldsymbol{l}'_h}{(2\pi)^d} \delta^{(d)}(\boldsymbol{l}_h + \boldsymbol{l}'_h - \boldsymbol{q}'_h - \boldsymbol{q}_h) \frac{u_\phi}{2} \phi_{\boldsymbol{l}_h} \phi_{\boldsymbol{l}'_h} \phi_{-\boldsymbol{q}'_h} \phi_{-\boldsymbol{q}_h} \\ + \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int^{\Lambda/r} \frac{d^d\boldsymbol{q}_l}{(2\pi)^d} \phi_{\boldsymbol{q}_l} \Big(\gamma \frac{|\Omega|}{|\boldsymbol{q}_l|} + v_\phi^2 \boldsymbol{q}_l^2 + m_\phi^2 \Big) \phi_{-\boldsymbol{q}_l} \\ + \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\Omega'}{2\pi} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \int^{\Lambda/r} \frac{d^d\boldsymbol{q}_l}{(2\pi)^d} \int^{\Lambda/r} \frac{d^d\boldsymbol{q}'_l}{(2\pi)^d} \int^{\Lambda/r} \frac{d^d\boldsymbol{l}_l}{(2\pi)^d} \int^{\Lambda/r} \frac{d^d\boldsymbol{l}'_l}{(2\pi)^d} \delta^{(d)}(\boldsymbol{l}_l + \boldsymbol{l}'_l - \boldsymbol{q}'_l - \boldsymbol{q}_l) \frac{u_\phi}{2} \phi_{\boldsymbol{l}_l} \phi_{\boldsymbol{l}'_l} \phi_{-\boldsymbol{q}'_l} \phi_{-\boldsymbol{q}_l} \\ + \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\Omega'}{2\pi} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \int^{\Lambda/r} \frac{d^d\boldsymbol{q}_l}{(2\pi)^d} \int^{\Lambda/r} \frac{d^d\boldsymbol{l}_l}{(2\pi)^d} \int_{\Lambda/r}^{\Lambda} \frac{d^d\boldsymbol{q}'_h}{(2\pi)^d} \int_{\Lambda/r}^{\Lambda} \frac{d^d\boldsymbol{l}'_h}{(2\pi)^d} \delta^{(d)}(\boldsymbol{l}_l + \boldsymbol{l}'_h - \boldsymbol{q}'_h - \boldsymbol{q}_l) \frac{u_\phi}{2} \phi_{\boldsymbol{l}_l} \phi_{\boldsymbol{l}'_h} \phi_{-\boldsymbol{q}'_h} \phi_{-\boldsymbol{q}_l} \Big\} \Big] \end{aligned} \tag{158}$$

$$\begin{aligned} S_{int}[\phi_{\boldsymbol{q}_h}, \phi_{\boldsymbol{q}_l}] = \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\Omega'}{2\pi} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \int^{\Lambda/r} \frac{d^d\boldsymbol{q}_l}{(2\pi)^d} \int^{\Lambda/r} \frac{d^d\boldsymbol{l}_l}{(2\pi)^d} \int_{\Lambda/r}^{\Lambda} \frac{d^d\boldsymbol{q}'_h}{(2\pi)^d} \int_{\Lambda/r}^{\Lambda} \frac{d^d\boldsymbol{l}'_h}{(2\pi)^d} \\ \delta^{(d)}(\boldsymbol{l}_l + \boldsymbol{l}'_h - \boldsymbol{q}'_h - \boldsymbol{q}_l) \frac{u_\phi}{2} \phi_{\boldsymbol{l}_l} \phi_{\boldsymbol{l}'_h} \phi_{-\boldsymbol{q}'_h} \phi_{-\boldsymbol{q}_l} \end{aligned} \tag{159}$$

$$\Big\langle \exp\Big(- S_{int}[\phi_{\boldsymbol{q}_h}, \phi_{\boldsymbol{q}_l}] \Big) \Big\rangle_h = \exp\Big\{ - \Big\langle S_{int}[\phi_{\boldsymbol{q}_h}, \phi_{\boldsymbol{q}_l}] \Big\rangle_h + \frac{1}{2} \Big(\Big\langle S_{int}^2[\phi_{\boldsymbol{q}_h}, \phi_{\boldsymbol{q}_l}] \Big\rangle_h - \Big\langle S_{int}[\phi_{\boldsymbol{q}_h}, \phi_{\boldsymbol{q}_l}] \Big\rangle_h^2 \Big) \Big\} \tag{160}$$

c. Mass renormalization in the one-loop level

$$\begin{aligned} S_{m^2}^{(1)}[\phi_{\boldsymbol{q}_l}] = \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\Omega'}{2\pi} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \int^{\Lambda/r} \frac{d^d\boldsymbol{q}_l}{(2\pi)^d} \int^{\Lambda/r} \frac{d^d\boldsymbol{l}_l}{(2\pi)^d} \int_{\Lambda/r}^{\Lambda} \frac{d^d\boldsymbol{q}'_h}{(2\pi)^d} \int_{\Lambda/r}^{\Lambda} \frac{d^d\boldsymbol{l}'_h}{(2\pi)^d} \\ \delta^{(d)}(\boldsymbol{l}_l + \boldsymbol{l}'_h - \boldsymbol{q}'_h - \boldsymbol{q}_l) \frac{u_\phi}{2} \phi_{\boldsymbol{l}_l} \Big\langle \phi_{\boldsymbol{l}'_h} \phi_{-\boldsymbol{q}'_h} \Big\rangle_h^c \phi_{-\boldsymbol{q}_l} \end{aligned} \tag{161}$$

$$S_{m^2}^{(1)}[\phi_{\boldsymbol{q}_l}] = \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int^{\Lambda/r} \frac{d^d\boldsymbol{q}_l}{(2\pi)^d} \delta m_\phi^2 \phi_{\boldsymbol{q}_l} \phi_{-\boldsymbol{q}_l} \tag{162}$$

$$m_\phi^2(\Lambda/r) = m_\phi^2(\Lambda) + \delta m_\phi^2 \tag{163}$$

$$\delta m_\phi^2 = \frac{u_\phi}{2} \int_{-\infty}^{\infty} \frac{d\Omega'}{2\pi} \int_{\Lambda/r}^{\Lambda} \frac{d^d\boldsymbol{q}'_h}{(2\pi)^d} G(\boldsymbol{q}'_h, i\Omega') = \frac{u_\phi}{2} \int_{-\infty}^{\infty} \frac{d\Omega'}{2\pi} \int_{\Lambda/r}^{\Lambda} \frac{d^d\boldsymbol{q}'_h}{(2\pi)^d} \frac{1}{\gamma \frac{|\Omega'|}{|\boldsymbol{q}'_h|^{z-2}} + v_\phi^2 \boldsymbol{q}'^2_h + m_\phi^2} \approx c_{m^2} u_\phi \ln r \tag{164}$$

$$\frac{dm_\phi^2}{d\ln r} = 2m_\phi^2 + c_{m^2} u_\phi, \qquad c_{m^2} > 0 \tag{165}$$

d. Vertex renormalization in the one-loop level

$$S_u^{(2)}[\phi_{\boldsymbol{q}_l}] = -\frac{1}{2} \Big(\Big\langle S_{int}^2[\phi_{\boldsymbol{q}_h}, \phi_{\boldsymbol{q}_l}] \Big\rangle_h - \Big\langle S_{int}[\phi_{\boldsymbol{q}_h}, \phi_{\boldsymbol{q}_l}] \Big\rangle_h^2 \Big) \tag{166}$$

$$S_u^{(2)}[\phi_{\boldsymbol{q}_l}] = \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\Omega'}{2\pi} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \int^{\Lambda/r} \frac{d^d\boldsymbol{q}_l}{(2\pi)^d} \int^{\Lambda/r} \frac{d^d\boldsymbol{q}'_l}{(2\pi)^d} \int^{\Lambda/r} \frac{d^d\boldsymbol{l}_l}{(2\pi)^d} \int^{\Lambda/r} \frac{d^d\boldsymbol{l}'_l}{(2\pi)^d}$$
$$\delta^{(d)}(\boldsymbol{l}_l + \boldsymbol{l}'_l - \boldsymbol{q}'_l - \boldsymbol{q}_l) \frac{\delta u_\phi}{2} \phi_{\boldsymbol{l}_l} \phi_{\boldsymbol{l}'_l} \phi_{-\boldsymbol{q}'_l} \phi_{-\boldsymbol{q}_l}$$
$$= \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\Omega'}{2\pi} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \int^{\Lambda/r} \frac{d^d\boldsymbol{q}_l}{(2\pi)^d} \int^{\Lambda/r} \frac{d^d\boldsymbol{q}'_l}{(2\pi)^d} \int^{\Lambda/r} \frac{d^d\boldsymbol{l}}{(2\pi)^d} \frac{\delta u_\phi}{2} \phi_{\boldsymbol{q}_l+\boldsymbol{l}} \phi_{\boldsymbol{q}'_l-\boldsymbol{l}} \phi_{-\boldsymbol{q}'_l} \phi_{-\boldsymbol{q}_l} \tag{167}$$

$$u_\phi(\Lambda/r) = u_\phi(\Lambda) + \delta u_\phi \tag{168}$$

$$\delta u_\phi = \delta u_\phi^{ZS} + \delta u_\phi^{ZS'} + \delta u_\phi^{BCS} \tag{169}$$

d-1. ZS (zero sound channel-ph)

$$\delta u_\phi^{ZS} \propto -u_\phi^2 \int_{-\infty}^{\infty} \frac{d\Omega'}{2\pi} \int_{\Lambda/r}^{\Lambda} \frac{d^d\boldsymbol{q}'_h}{(2\pi)^d} G(\boldsymbol{q}'_h + \boldsymbol{l}, i\Omega' + i\nu) G(\boldsymbol{q}'_h, i\Omega')$$
$$= -u_\phi^2 \int_{-\infty}^{\infty} \frac{d\Omega'}{2\pi} \int_{\Lambda/r}^{\Lambda} \frac{d^d\boldsymbol{q}'_h}{(2\pi)^d} \frac{1}{\gamma \frac{|\Omega'+\nu|}{|\boldsymbol{q}'_h+\boldsymbol{l}|^{z-2}} + v_\phi^2 |\boldsymbol{q}'_h + \boldsymbol{l}|^2 + m_\phi^2} \frac{1}{\gamma \frac{|\Omega'|}{|\boldsymbol{q}'_h|^{z-2}} + v_\phi^2 \boldsymbol{q}'^2_h + m_\phi^2}$$
$$\propto -u_\phi^2 \ln r \tag{170}$$

d-2. ZS' (zero sound' channel-ph)
d-3. BCS (pairing channel-pp)

$$\frac{du_\phi}{d\ln r} = [4 - (d+z)] u_\phi - c_u u_\phi^2 \tag{171}$$

e. Wave-function renormalization in the two-loop level

5. *Wilson's effective field theory*

$$Z = r^{L^d \Delta_\phi} \int D\phi_r(\boldsymbol{q}_r, i\Omega_r) \exp\Big[- \Big\{ \int_{-\infty_r}^{\infty_r} \frac{d\Omega_r}{2\pi} \int^{\Lambda} \frac{d^d\boldsymbol{q}_r}{(2\pi)^d} \phi_r(\boldsymbol{q}_r, i\Omega_r) \Big(\gamma_r \frac{|\Omega_r|}{|\boldsymbol{q}_r|^{z-2}} + v_{\phi r}^2 \boldsymbol{q}_r^2 + m_{\phi r}^2 \Big) \phi_r(-\boldsymbol{q}_r, -i\Omega_r)$$
$$+ \int_{-\infty_r}^{\infty_r} \frac{d\Omega_r}{2\pi} \int_{-\infty_r}^{\infty_r} \frac{d\Omega'_r}{2\pi} \int_{-\infty_r}^{\infty_r} \frac{d\nu_r}{2\pi} \int^{\Lambda} \frac{d^d\boldsymbol{q}_r}{(2\pi)^d} \int^{\Lambda} \frac{d^d\boldsymbol{q}'_r}{(2\pi)^d} \int^{\Lambda} \frac{d^d\boldsymbol{l}_r}{(2\pi)^d}$$
$$\frac{u_{\phi r}(i\nu_r)}{2} \phi_r(\boldsymbol{q}_r + \boldsymbol{l}_r, i\Omega_r + i\nu_r) \phi'_r(\boldsymbol{q}'_r - \boldsymbol{l}_r, i\Omega'_r - i\nu_r) \phi_r(-\boldsymbol{q}'_r, -i\Omega'_r) \phi_r(-\boldsymbol{q}_r, -i\Omega_r) \Big\} \Big] \tag{172}$$

B. Scaling theory

1. *Singular free energy*

$$F(m_\phi^2, T) = \frac{1}{\beta} \sum_{i\Omega} \sum_{\boldsymbol{q}} \ln \Big(\gamma \frac{|\Omega|}{|\boldsymbol{q}|^{z-2}} + v_\phi^2 \boldsymbol{q}^2 + m_\phi^2 \Big) = r^{-(d+z)} \frac{1}{\beta} \sum_{i\Omega_r} \sum_{\boldsymbol{q}_r} \ln \Big(\gamma_r \frac{|\Omega_r|}{|\boldsymbol{q}_r|} + v_{\phi r}^2 \boldsymbol{q}_r^2 + m_{\phi r}^2 \Big) + F_{const.}$$
$$= r^{-(d+z)} F(m_{\phi r}^2 r^{-\Delta_{m^2}}, T_r r^{-z}) \tag{173}$$

2. Scaling theory

$$F(\delta,T)=r^{-(d+z)}F(\delta r^{-\frac{1}{\nu}},Tr^{-z}) \tag{174}$$

3. Callan-Symanzik equation

$$\frac{dF(\delta,T)}{d\ln r}=\Big(\frac{\partial}{\partial\ln r}+\beta_\delta\frac{\partial}{\partial\ln\delta}+\beta_T\frac{\partial}{\partial\ln T}\Big)F(\delta,T)=0\Longleftrightarrow F(\delta,T)=r^{-(d+z)}F(\delta r^{-\frac{1}{\nu}},Tr^{-z}) \tag{175}$$

$$\frac{\partial F(\delta,T)}{\partial\ln r}=(d+z)F(\delta,T),\qquad \beta_\delta=\frac{\partial\ln\delta}{\partial\ln r}=\frac{1}{\nu},\qquad \beta_T=\frac{\partial\ln T}{\partial\ln r}=z \tag{176}$$

4. Thermodynamics

$$r=T^{1/z}\longrightarrow F(\delta,T)=T^{-\frac{d+z}{z}}F(\delta T^{-\frac{1}{\nu z}},1) \tag{177}$$

$$S=-\frac{\partial F(\delta,T)}{\partial T},\qquad \gamma_v=\frac{C_v}{T}=\frac{\partial^2F(\delta,T)}{\partial T^2} \tag{178}$$

$-\partial_\delta^2F(\delta,T)$: Thermal expansion coefficient
Gruneisen ratio at quantum criticality vs. at classical criticality

5. Dangerously irrelevant operators

Singular dependence for u in Free energy near quantum criticality

$$\frac{dF(\delta,T,u)}{d\ln r}=\Big(\frac{\partial}{\partial\ln r}+\beta_\delta\frac{\partial}{\partial\ln\delta}+\beta_T\frac{\partial}{\partial\ln T}+\beta_u\frac{\partial}{\partial\ln u}\Big)F(\delta,T,u)=0 \tag{179}$$

$$\frac{\partial F(\delta,T,u)}{\partial\ln r}=(d+z)F(\delta,T,u),\qquad \beta_\delta=\frac{\partial\ln\delta}{\partial\ln r}=\frac{1}{\nu},\qquad \beta_T=\frac{\partial\ln T}{\partial\ln r}=z,\qquad \beta_u=\frac{\partial\ln u}{\partial\ln r}=4-(d+z) \tag{180}$$

$$F(\delta,T,u)=r^{-(d+z)}F(\delta r^{-\frac{1}{\nu}},Tr^{-z},ur^{-[4-(d+z)]}) \tag{181}$$

$$r=T^{1/z}\longrightarrow F(\delta,T,u)=T^{-\frac{d+z}{z}}F(\delta T^{-\frac{1}{\nu z}},1,uT^{-\frac{4-(d+z)}{z}}) \tag{182}$$

The existence of a dangerously irrelevant operator gives rise to breakdown of ω/T scaling in dynamical susceptibility near quantum criticality.

$$\chi(\omega,\delta,T,u)=T^{-\frac{\Delta_\chi}{z}}\chi(\omega/T,\delta T^{-\frac{1}{\nu z}},1,uT^{-\frac{4-(d+z)}{z}}) \tag{183}$$

$$\delta=0\longrightarrow\chi(\omega,0,T,u)=T^{-\frac{\Delta_\chi}{z}}\chi(\omega/T,0,1,uT^{-\frac{4-(d+z)}{z}}) \tag{184}$$

C. Self-consistent treatment of both fermions and critical bosons

1. Self-consistent RPA theory

$$Z = \int Dc_{\boldsymbol{p}\sigma} D\phi_{\boldsymbol{q}} \exp\Big[- \int_0^\beta d\tau \Big\{ \sum_{\boldsymbol{p}} c_{\boldsymbol{p}\sigma}^\dagger (\partial_\tau - \mu + \epsilon_{\boldsymbol{p}}) c_{\boldsymbol{p}\sigma} + g \sum_{\boldsymbol{p}} \frac{1}{L^d} \sum_{\boldsymbol{q}} \phi_{\boldsymbol{q}} \sigma c_{\boldsymbol{p}+\boldsymbol{q}\sigma}^\dagger c_{\boldsymbol{p}\sigma}$$
$$+ \sum_{\boldsymbol{q}} \phi_{\boldsymbol{q}} (-\partial_\tau^2 + v_\phi^2 \boldsymbol{q}^2 + m_\phi^2) \phi_{-\boldsymbol{q}} + \frac{u_\phi}{2} \sum_{\boldsymbol{q}} \frac{1}{L^d} \sum_{\boldsymbol{q}'} \frac{1}{L^d} \sum_{\boldsymbol{l}} \phi_{\boldsymbol{q}+\boldsymbol{l}} \phi_{\boldsymbol{q}'-\boldsymbol{l}} \phi_{-\boldsymbol{q}'} \phi_{-\boldsymbol{q}} \Big\} \Big] \tag{185}$$

$$Z = \int Dc_{\boldsymbol{p}\sigma} D\phi_{\boldsymbol{q}} \exp\Big[- \sum_{i\omega} \sum_{\boldsymbol{p}} c_{\boldsymbol{p}\sigma}^\dagger \Big(- i\omega - \mu + \epsilon_{\boldsymbol{p}} - \frac{1}{\beta} \sum_{i\Omega} \sum_{\boldsymbol{q}} D(\boldsymbol{q}, i\Omega) G(\boldsymbol{p}+\boldsymbol{q}, i\omega + i\Omega) \Big) c_{\boldsymbol{p}\sigma}$$
$$- \sum_{i\Omega} \sum_{\boldsymbol{q}} \Big\{ \phi_{\boldsymbol{q}} \Big(\Omega^2 + v_\phi^2 \boldsymbol{q}^2 + m_\phi^2 + \Pi(\boldsymbol{q}, i\Omega) \Big) \phi_{-\boldsymbol{q}} - \frac{1}{2} \Pi(\boldsymbol{q}, i\Omega) D(\boldsymbol{q}, i\Omega) \Big\}$$
$$- \frac{1}{2} \sum_{i\omega} \sum_{\boldsymbol{p}} \frac{1}{\beta} \sum_{i\Omega} \sum_{\boldsymbol{q}} G(\boldsymbol{p}+\boldsymbol{q}, i\omega + i\Omega) D(\boldsymbol{q}, i\Omega) G(\boldsymbol{p}, i\omega)$$
$$- \int_0^\beta d\tau \Big\{ g \sum_{\boldsymbol{p}} \frac{1}{L^d} \sum_{\boldsymbol{q}} \phi_{\boldsymbol{q}} \sigma c_{\boldsymbol{p}+\boldsymbol{q}\sigma}^\dagger c_{\boldsymbol{p}\sigma} + \frac{u_\phi}{2} \sum_{\boldsymbol{q}} \frac{1}{L^d} \sum_{\boldsymbol{q}'} \frac{1}{L^d} \sum_{\boldsymbol{l}} \phi_{\boldsymbol{q}+\boldsymbol{l}} \phi_{\boldsymbol{q}'-\boldsymbol{l}} \phi_{-\boldsymbol{q}'} \phi_{-\boldsymbol{q}} \Big\} \Big] \tag{186}$$

$$Z_{eff} = \int Dc_{\boldsymbol{p}\sigma} D\phi_{\boldsymbol{q}} \exp\Big[- \sum_{i\omega} \sum_{\boldsymbol{p}} c_{\boldsymbol{p}\sigma}^\dagger \Big(- i\omega - \mu + \epsilon_{\boldsymbol{p}} - i c_f g^2 \mathrm{sgn}(\omega) |\omega|^{d/z} \Big) c_{\boldsymbol{p}\sigma}$$
$$- \sum_{i\Omega} \sum_{\boldsymbol{q}} \phi_{\boldsymbol{q}} \Big(\Omega^2 + v_\phi^2 \boldsymbol{q}^2 + m_\phi^2 + \gamma \frac{|\Omega|}{|\boldsymbol{q}|^{z-2}} \Big) \phi_{-\boldsymbol{q}}$$
$$- \int_0^\beta d\tau \Big\{ g \sum_{\boldsymbol{p}} \frac{1}{L^d} \sum_{\boldsymbol{q}} \phi_{\boldsymbol{q}} \sigma c_{\boldsymbol{p}+\boldsymbol{q}\sigma}^\dagger c_{\boldsymbol{p}\sigma} + \frac{u_\phi}{2} \sum_{\boldsymbol{q}} \frac{1}{L^d} \sum_{\boldsymbol{q}'} \frac{1}{L^d} \sum_{\boldsymbol{l}} \phi_{\boldsymbol{q}+\boldsymbol{l}} \phi_{\boldsymbol{q}'-\boldsymbol{l}} \phi_{-\boldsymbol{q}'} \phi_{-\boldsymbol{q}} \Big\} \Big] \tag{187}$$

2. Two dimensions

$$Z_{eff} = \int Dc_{\boldsymbol{p}\sigma} D\phi_{\boldsymbol{q}} \exp\Big[- \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^2\boldsymbol{p}}{(2\pi)^2} c_{\boldsymbol{p}\sigma}^\dagger \Big(- i\omega - \mu + \epsilon_{\boldsymbol{p}} - i c_f g^2 \mathrm{sgn}(\omega) |\omega|^{2/z} \Big) c_{\boldsymbol{p}\sigma}$$
$$- \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} \phi_{\boldsymbol{q}} \Big(\Omega^2 + v_\phi^2 \boldsymbol{q}^2 + m_\phi^2 + \gamma \frac{|\Omega|}{|\boldsymbol{q}|^{z-2}} \Big) \phi_{-\boldsymbol{q}} - \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^2\boldsymbol{p}}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} g \phi_{\boldsymbol{q}} \sigma c_{\boldsymbol{p}+\boldsymbol{q}\sigma}^\dagger c_{\boldsymbol{p}\sigma}$$
$$- \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\Omega'}{2\pi} \int \frac{d^2\boldsymbol{q}'}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \int \frac{d^2\boldsymbol{l}}{(2\pi)^2} \frac{u_\phi}{2} \phi_{\boldsymbol{q}+\boldsymbol{l}} \phi_{\boldsymbol{q}'-\boldsymbol{l}} \phi_{-\boldsymbol{q}'} \phi_{-\boldsymbol{q}} \Big] \tag{188}$$

$$Z_{eff} \approx \int Dc_{\boldsymbol{p}\sigma} D\phi_{\boldsymbol{q}} \exp\Big[- \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^2\boldsymbol{p}}{(2\pi)^2} c_{\boldsymbol{p}\sigma}^\dagger \Big(- i c_f g^2 \mathrm{sgn}(\omega) |\omega|^{2/z} + \boldsymbol{v}_F \cdot (\boldsymbol{p} - \boldsymbol{p}_F) + c_\perp [(\boldsymbol{v}_F/|\boldsymbol{v}_F|) \times (\boldsymbol{p} - \boldsymbol{p}_F)]^2 \Big) c_{\boldsymbol{p}\sigma}$$
$$- \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} \phi_{\boldsymbol{q}} \Big(\gamma \frac{|\Omega|}{|\boldsymbol{q}|^{z-2}} + v_\phi^2 \boldsymbol{q}^2 + m_\phi^2 \Big) \phi_{-\boldsymbol{q}} - \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^2\boldsymbol{p}}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} g \phi_{\boldsymbol{q}} \sigma c_{\boldsymbol{p}+\boldsymbol{q}\sigma}^\dagger c_{\boldsymbol{p}\sigma}$$
$$- \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int \frac{d^2\boldsymbol{q}}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\Omega'}{2\pi} \int \frac{d^2\boldsymbol{q}'}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \int \frac{d^2\boldsymbol{l}}{(2\pi)^2} \frac{u_\phi}{2} \phi_{\boldsymbol{q}+\boldsymbol{l}} \phi_{\boldsymbol{q}'-\boldsymbol{l}} \phi_{-\boldsymbol{q}'} \phi_{-\boldsymbol{q}} \Big] \tag{189}$$

3. Patch construction: To be proven below

$$Z_{eff} \approx \int Dc_{\boldsymbol{p}s\sigma} D\phi_{\boldsymbol{q}} \exp\Big[-\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} dp_{\parallel} \int_{-\infty}^{\infty} dp_{\perp} c^{\dagger}_{\boldsymbol{p}s\sigma} \Big(-ic_f g^2 \mathrm{sgn}(\omega)|\omega|^{2/z} + sv_F p_{\parallel} + \frac{p_{\perp}^2}{2m} \Big) c_{\boldsymbol{p}s\sigma}$$
$$- \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int_{0}^{\infty} dq_{\parallel} \int_{-\infty}^{\infty} dq_{\perp} \phi_{\boldsymbol{q}} \Big(\gamma \frac{|\Omega|}{|q_{\perp}^2 + q_{\parallel}^2|^{\frac{z-2}{2}}} + v_{\phi}^2 (q_{\perp}^2 + q_{\parallel}^2) \Big) \phi_{-\boldsymbol{q}}$$
$$- \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} dp_{\parallel} \int_{-\infty}^{\infty} dp_{\perp} \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int_{0}^{\infty} dq_{\parallel} \int_{-\infty}^{\infty} dq_{\perp} g \phi_{\boldsymbol{q}} \sigma c^{\dagger}_{\boldsymbol{p}+\boldsymbol{q}s\sigma} c_{\boldsymbol{p}s\sigma}$$
$$- \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int_{0}^{\infty} dq_{\parallel} \int_{-\infty}^{\infty} dq_{\perp} \int_{-\infty}^{\infty} \frac{d\Omega'}{2\pi} \int_{0}^{\infty} dq'_{\parallel} \int_{-\infty}^{\infty} dq'_{\perp} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \int_{0}^{\infty} dl_{\parallel} \int_{-\infty}^{\infty} dl_{\perp} \frac{u_{\phi}}{2} \phi_{\boldsymbol{q}+\boldsymbol{l}} \phi_{\boldsymbol{q}'-\boldsymbol{l}} \phi_{-\boldsymbol{q}'} \phi_{-\boldsymbol{q}} \Big] \quad (190)$$

4. Scaling analysis

$$\omega = \frac{\omega_r}{r^z}, \qquad p_{\parallel} = \frac{p_{\parallel r}}{r^2}, \qquad p_{\perp} = \frac{p_{\perp r}}{r} \quad (191)$$

$$c_{s\sigma}(p_{\parallel r}/r^2, p_{\perp r}/r, i\omega/r^z) = r^{\Delta_c} c_{s\sigma}(p_{\parallel r}, p_{\perp r}, i\omega) \quad (192)$$

$$r^{-z} r^{-2} r^{-1} r^{2\Delta_c} r^{-2} = r^0 \longrightarrow \Delta_c = \frac{z+5}{2} = 4 \quad (193)$$

$$\phi(q_{\parallel r}/r^2, q_{\perp r}/r, i\Omega/r^z) = r^{\Delta_{\phi}} \phi(q_{\parallel r}, q_{\perp r}, i\Omega) \quad (194)$$

$$r^{-z} r^{-2} r^{-1} r^{2\Delta_{\phi}} r^{-2} = r^0 \longrightarrow \Delta_{\phi} = \Delta_c = \frac{z+5}{2} = 4 \quad (195)$$

$$g = r^{\Delta_g} g \quad (196)$$

$$r^{-2z} r^{-4} r^{-2} r^{2\Delta_c} r^{\Delta_{\phi}} r^{\Delta_g} = r^0 \longrightarrow \Delta_g = 2z + 6 - \frac{3}{2}(z+5) = \frac{z-3}{2} = 0 \quad (197)$$

$$u_{\phi} = r^{\Delta_u} u_{\phi} \quad (198)$$

$$r^{-3z} r^{-6} r^{-3} r^{4\Delta_{\phi}} r^{\Delta_u} = r^0 \longrightarrow \Delta_u = 3z + 9 - 2(z+5) = z - 1 = 2 \quad (199)$$

$$m_{\phi}^2 = r^{\Delta_{m^2}} m_{\phi}^2 \quad (200)$$

$$r^{-z} r^{-2} r^{-1} r^{2\Delta_{\phi}} r^{\Delta_{m^2}} = r^0 \longrightarrow \Delta_{m^2} = z + 3 - z - 5 = -2 \quad (201)$$

5. Critical field theory

$$Z_{eff} \approx \int Dc_{\boldsymbol{p}s\sigma} D\phi_{\boldsymbol{q}} \exp\Big[- \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y c_{s\sigma}^{\dagger}(p_x, p_y, i\omega) \Big(- i c_f g^2 \mathrm{sgn}(\omega) |\omega|^{2/z} + s v_F p_x + \frac{p_y^2}{2m} \Big) c_{s\sigma}(p_x, p_y, i\omega)$$
$$- \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int_{0}^{\infty} dq_x \int_{-\infty}^{\infty} dq_y \phi(q_y, i\Omega) \Big(\gamma \frac{|\Omega|}{|q_y|^{z-2}} + v_\phi^2 q_y^2 \Big) \phi(-q_y, -i\Omega)$$
$$- \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int_{0}^{\infty} dq_x \int_{-\infty}^{\infty} dq_y g \phi(q_y, i\Omega) \sigma c_{s\sigma}^{\dagger}(p_x, p_y + q_y, i\omega + i\Omega) c_{s\sigma}(p_x, p_y, i\omega)$$
$$- \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \int_{0}^{\infty} dq_x \int_{-\infty}^{\infty} dq_y \int_{-\infty}^{\infty} \frac{d\Omega'}{2\pi} \int_{0}^{\infty} dq_x' \int_{-\infty}^{\infty} dq_y' \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \int_{0}^{\infty} dl_x \int_{-\infty}^{\infty} dl_y$$
$$\frac{u_\phi}{2} \phi(q_y + l_y, i\Omega + i\nu) \phi(q_y' - l_y, i\Omega' - i\nu) \phi(-q_y', -i\Omega') \phi(-q_y, -i\Omega) \Big] \tag{202}$$

a. Boson cutoff necessary.
b. u_ϕ is more irrelevant than the case of the Hertz-Moriya-Millis theory. Is it still dangerously irrelevant at this fixed point? ω/T scaling in the regime of intermediate temperatures?
c. Justification of the patch construction?

6. Scaling theory

$$\frac{dF(\delta, T, u)}{d \ln r} = \Big(\frac{\partial}{\partial \ln r} + \beta_\delta \frac{\partial}{\partial \ln \delta} + \beta_T \frac{\partial}{\partial \ln T} + \beta_u \frac{\partial}{\partial \ln u} \Big) F(\delta, T, u) = 0 \tag{203}$$

$$\frac{\partial F(\delta, T, u)}{\partial \ln r} = (d+z) F(\delta, T, u), \qquad \beta_\delta = \frac{\partial \ln \delta}{\partial \ln r} = \frac{1}{\nu}, \qquad \beta_T = \frac{\partial \ln T}{\partial \ln r} = z, \qquad \beta_u = \frac{\partial \ln u}{\partial \ln r} = -2(z+1) \tag{204}$$

$$F(\delta, T, u) = r^{-(d+z)} F(\delta r^{-\frac{1}{\nu}}, T r^{-z}, u r^{2(z+1)}) \tag{205}$$

$$r = T^{1/z} \longrightarrow F(\delta, T, u) = T^{-\frac{d+z}{z}} F(\delta T^{-\frac{1}{\nu z}}, 1, u T^{2\frac{z+1}{z}}) \tag{206}$$

The existence of a dangerously irrelevant operator gives rise to breakdown of ω/T scaling in dynamical susceptibility near quantum criticality. However, it vanishes faster than the case of the Hertz-Moriya-Millis theory. In this respect, the renormalization group flow of the dangerously irrelevant operator may not affect quantum critical physics relatively in the intermediate energy scale. We speculate the presence of a crossover line (temperature) from this novel fixed point to a Hertz-Moriya-Millis fixed point, where the dangerously irrelevant operator may start to work.

$$\chi(\omega, \delta, T, u) = T^{-\frac{\Delta_\chi}{z}} \chi(\omega/T, \delta T^{-\frac{1}{\nu z}}, 1, u T^{2\frac{z+1}{z}}) \tag{207}$$

$$\delta = 0 \longrightarrow \chi(\omega, 0, T, u) = T^{-\frac{\Delta_\chi}{z}} \chi(\omega/T, 0, 1, u T^{2\frac{z+1}{z}}) \tag{208}$$

VI. BEYOND HERTZ-MORIYA-MILLIS QUANTUM CRITICALITY

A. BKV (Belitz-Kirkpatrick-Vojta) singularity

B. Justification of the patch construction at quantum criticality

Ref. S.-S. Lee, Phys. Rev. B **78**, 085129 (2008)

a. Effective action

$$S_{eff} = S_0 + S_1 \tag{209}$$

$$S_0 = \sum_{\sigma=\pm} \int d\omega dk d\theta \psi_\sigma^\dagger(\omega, k, \theta)(i\omega + k)\psi_\sigma(\omega, k, \theta) + \int d\nu dq_l dq_t \frac{1}{2g^2}\phi(\nu, q_l, q_t; \theta)(\nu^2 + q^2)\phi(-\nu, -q_l, -q_t; \theta) \tag{210}$$

$$\begin{aligned}
S_1 &= -\frac{1}{(2\pi)^{3/2}} \sum_{\sigma=\pm} \int d\omega dk d\theta d\nu dq_l dq_t \phi(\nu, q_l, q_t; \theta) \\
&\sigma\psi_\sigma^\dagger(\omega + \nu/2, k + q_l/2, \theta + q_t/(2k_F))\psi_\sigma(\omega - \nu/2, k - q_l/2, \theta - q_t/(2k_F)) \\
&= -\frac{1}{(2\pi)^{3/2}} \sum_{\sigma=\pm} \int_{-\infty}^{\infty} d\omega \int_{-\Lambda_f/2}^{\Lambda_f/2} dk \int_{-\pi}^{\pi} d\theta \int_{-\infty}^{\infty} d\nu \int_0^{\Lambda_b} dqq \int_{-\pi}^{\pi} d\phi \\
&\Theta(\Lambda_f/2 - |k + q(\sin\phi)/2|)\Theta(\Lambda_f/2 - |k - q(\sin\phi)/2|) \\
&\phi(\nu, q, \theta - \phi)\sigma\psi_\sigma^\dagger(\omega + \nu/2, k + (\sin\phi)/2, \theta + q/(2k_F))\psi_\sigma(\omega - \nu/2, k - (\sin\phi)/2, \theta - q\cos\phi/(2k_F))
\end{aligned} \tag{211}$$

b. Double patch construction

$$\begin{aligned}
S_0 &= \sum_{s=\pm}\sum_{\sigma=\pm} \int_{-\infty}^{\infty} d\omega \int_{-\Lambda_f/2}^{\Lambda_f/2} dk \int_{-\pi/2}^{\pi/2} d\theta \psi_{\sigma s}^\dagger(\omega, k, \theta)(i\omega + sk)\psi_{\sigma s}(\omega, k, \theta) \\
&+ \int_{-\infty}^{\infty} d\nu \int_{-\Lambda_b}^{\Lambda_b} dq|q| \int_{-\pi/2}^{\pi/2} d\alpha \frac{1}{2g^2}\phi(\nu, q, \alpha)(\nu^2 + q^2)\phi(-\nu, -q, \alpha)
\end{aligned} \tag{212}$$

$$\begin{aligned}
S_1 &= -\frac{1}{(2\pi)^{3/2}} \sum_{s=\pm}\sum_{\sigma=\pm} \int_{-\infty}^{\infty} d\omega \int_{-\Lambda_f/2}^{\Lambda_f/2} dk \int_{-\pi/2}^{\pi/2} d\theta \int_{-\infty}^{\infty} d\nu \int_{-\Lambda_b}^{\Lambda_b} dq|q| \int_{-\Lambda_b/|q|}^{\Lambda_b/|q|} d\phi \\
&\phi(\nu, q, \theta - \phi)\sigma\psi_{\sigma s}^\dagger(\omega + \nu/2, k + s\phi/2, \theta + sq\phi/(2k_F))\psi_\sigma(\omega - \nu/2, k - s\phi/2, \theta - sq/(2k_F))
\end{aligned} \tag{213}$$

c. Quantum effective action in the one-loop level

$$\Gamma_0 = \sum_{s=\pm}\sum_{\sigma=\pm} \int d\omega dk d\theta \psi_{\sigma s}^\dagger(\omega, k, \theta)(ic\,\mathrm{sgn}(\omega)|\omega|^{2/3} + sk)\psi_{\sigma s}(\omega, k, \theta) + \int d\nu dq|q| d\alpha \phi^\dagger(\nu, q, \alpha)\Big(\gamma\frac{|\nu|}{|q|} + \chi q^2\Big)\phi(\nu, q, \alpha) \tag{214}$$

d. Scaling analysis

$$\omega = \omega'/b, \quad \nu = \nu'/b, \quad k = k'/b^{2/3}, \quad q = q'/b^{1/3}, \quad \Lambda_f = \Lambda_f'/b^{2/3}, \quad \Lambda_b = \Lambda_b'/b^{1/3} \tag{215}$$

$$\theta = \theta'/b^{1/3}, \qquad \phi = \phi'/b^{1/3} \tag{216}$$

$$\psi_{\sigma s}(\omega'/b, k'/b^{2/3}, \theta'/b^{1/3}) = b^{4/3}\psi_{\sigma s}'(\omega', k', \theta'), \qquad \phi(\nu'/b, q'/b^{2/3}, \phi'/b^{1/3}) = b^{4/3}\phi(\nu', q', \phi') \tag{217}$$

e. Effective field theory

$$S=\sum_{s=\pm}\sum_{\sigma=\pm}\int d\tau dxd\theta\psi^\dagger_{\sigma s}(\tau,x,\theta)(\partial_\tau-is\partial_x)\psi_{\sigma s}(\tau,x,\theta)+\frac{1}{2\Lambda_f}\int d\tau dxd\theta dq|q|\frac{1}{2g^2}\phi^\dagger_q(\tau,x,\theta)(-\partial_\tau^2+q^2)\phi_q(\tau,x,\theta)$$
$$-\frac{1}{(2\pi)^{1/2}}\sum_{s=\pm}\sum_{\sigma=\pm}\int d\tau dxd\theta dq\phi_q(\tau,x,\theta)\psi^\dagger_{\sigma s}(\tau,x,\theta+sq/(2k_F))\psi_{\sigma s}(\tau,x,\theta-sq/(2k_F)) \quad (218)$$

$$\text{Overlap area}\equiv\mathcal{S}_{\theta_1\theta_2}=\frac{\Lambda_f^2}{\Delta\theta},\qquad \Delta\theta=\theta_1-\theta_2,$$
$$\text{The ratio of this area to the area included in }\phi_q(\tau,x,\theta)\equiv\frac{\mathcal{S}_{\theta_1\theta_2}}{\mathcal{S}_G}=(\Lambda_f^2/\Delta\theta)/(\Lambda_f\Lambda_b),$$
$$\frac{\mathcal{S}_{\theta_1\theta_2}}{\mathcal{S}_G}\longrightarrow b^{-1/3}\frac{\mathcal{S}_{\theta_1\theta_2}}{\mathcal{S}_G}\quad\text{for a given }\Delta\theta \quad (219)$$

C. Failure of the self-consistent RPA theory in the large$-N$ limit

Ref. S.-S. Lee, Phys. Rev. B **80**, 165102 (2009)

1. Effective field theory

$$Z=\int D\psi_{\sigma s}D\phi\exp\Big[-\int_0^\beta d\tau\int_{-\infty}^\infty dx\int_{-\infty}^\infty dy\Big\{\psi^\dagger_{\sigma s}\Big(\partial_\tau-isv_F\partial_x-\frac{\partial_y^2}{2m}\Big)\psi_{\sigma s}-\frac{g}{\sqrt{N}}\phi\sigma\psi^\dagger_{\sigma s}\psi_{\sigma s}$$
$$+\phi(-\partial_\tau^2-v_\phi^2\partial_x^2-v_\phi^2\partial_y^2)\phi+m_\phi^2\phi^2+\frac{u_\phi}{2}\phi^4\Big\}\Big] \quad (220)$$

$$Z_{eff}\approx\int D\psi_{\sigma s}D\phi\exp\Big[-\int_{-\infty}^\infty\frac{d\omega}{2\pi}\int_{-\infty}^\infty dp_x\int_{-\infty}^\infty dp_y\psi^\dagger_{\sigma s}(p_x,p_y,i\omega)\Big(-ic_f\frac{g^2}{N}\text{sgn}(\omega)|\omega|^{2/z}+sv_Fp_x+\frac{p_y^2}{2m}\Big)\psi_{\sigma s}(p_x,p_y,i\omega)$$
$$-\int_{-\infty}^\infty\frac{d\Omega}{2\pi}\int_0^\infty dq_x\int_{-\infty}^\infty dq_y\phi(q_y,i\Omega)\Big(\gamma\frac{|\Omega|}{|q_y|^{z-2}}+v_\phi^2q_y^2+m_\phi^2\Big)\phi(-q_y,-i\Omega)$$
$$-\int_{-\infty}^\infty\frac{d\omega}{2\pi}\int_{-\infty}^\infty dp_x\int_{-\infty}^\infty dp_y\int_{-\infty}^\infty\frac{d\Omega}{2\pi}\int_0^\infty dq_x\int_{-\infty}^\infty dq_y\frac{g}{\sqrt{N}}\phi(q_y,i\Omega)\sigma\psi^\dagger_{\sigma s}(p_x,p_y+q_y,i\omega+i\Omega)\psi_{\sigma s}(p_x,p_y,i\omega)$$
$$-\int_{-\infty}^\infty\frac{d\Omega}{2\pi}\int_0^\infty dq_x\int_{-\infty}^\infty dq_y\int_{-\infty}^\infty\frac{d\Omega'}{2\pi}\int_0^\infty dq_x'\int_{-\infty}^\infty dq_y'\int_{-\infty}^\infty\frac{d\nu}{2\pi}\int_0^\infty dl_x\int_{-\infty}^\infty dl_y$$
$$\frac{u_\phi}{2}\phi(q_y+l_y,i\Omega+i\nu)\phi(q_y'-l_y,i\Omega'-i\nu)\phi(-q_y',-i\Omega')\phi(-q_y,-i\Omega)\Big] \quad (221)$$

$$z=3 \quad (222)$$

$$\omega=\frac{\omega'}{b},\qquad p_x=\frac{p_x'}{b^{2/3}},\qquad p_y=\frac{p_y'}{b^{1/3}},$$
$$\psi_{\sigma s}(i\omega'/b,p_x'/b^{3/2},p_y'/b^{1/3})=b^{4/3}\psi'_{\sigma s}(i\omega',p_x',p_y'),$$
$$\phi(i\omega'/b,p_x'/b^{3/2},p_y'/b^{1/3})=b^{4/3}\phi'(i\omega',p_x',p_y'),$$
$$g=g',\qquad u_\phi=b^{2/3}u_\phi' \quad (223)$$

2. 1/N expansion

a. The origin of the $1/N$ enhancement factor

$$\mathcal{L} = \psi^\dagger_{\sigma s}(\eta\partial_\tau - iv_x\partial_x - v_y\partial_y^2)\psi_{\sigma s} + \frac{g}{\sqrt{N}}\phi\psi^\dagger_{\sigma s}\psi_{\sigma s} + \phi(-\partial_y^2)\phi \tag{224}$$

$$g_0(k) = \frac{1}{i\eta k_0 + k_x + k_y^2}, \qquad D(k) = \frac{1}{\gamma\frac{|k_0|}{|k_y|} + k_y^2} \tag{225}$$

$$\Gamma(p, p+q) = -N^{-3/2}\int dk\int dl g_0(k)g_0(k+q)g_0(k+l)g_0(p+l)D(l)D(l-q) \tag{226}$$

$$\begin{aligned}
&\Gamma(0,q) = -N^{-3/2}\int dl_0 dl_y dk_0 \frac{F(l_0,l_y,k_0,q_0,q_y)}{l_y\delta_q + i\eta l_y q_0}, \qquad \delta_q = q_x + q_y^2,\\
&F(l_0,l_y,k_0,q_0,q_y) = 4\pi^3 i[\Theta(l_0+k_0-\Theta(l_0))][\Theta(k_0+q_0)-\Theta(k_0)][\Theta(q_y-l_y)-\Theta(q_y)]D(l)D(l-q),\\
&\Gamma(0,q) = -\frac{N^{-3/2}}{\eta q_0^{1/3}} f_1(q_y/q_0^{1/3}),\\
&f_1(t) = 4\pi^3\int_{-1}^{0}dx\int_0^{|x|}dy\int_1^\infty dz \frac{t^2(z-1)}{[\gamma y + (tz)^3][\gamma(1-y) + t^3(z-1)^3]}
\end{aligned} \tag{227}$$

$$g(k) = \frac{1}{i\eta k_0 + i\frac{c}{N}\mathrm{sgn}(k_0)|k_0|^{2/3} + k_x + k_y^2} \tag{228}$$

$$\begin{aligned}
&\Gamma(0,q) = -N^{-1/2} f_2(q_y/q_0^{1/3}),\\
&f_2(t) = \frac{4\pi^3}{c}\int_{-1}^{0}dx\int_0^{|x|}dy\int_1^\infty dz\\
&\frac{1}{|x+1|^{2/3} + y^{2/3} + |x+y|^{2/3} + (z-1)(|x+1|^{2/3} + |x|^{2/3})}\frac{1}{[\gamma y + (tz)^3][\gamma(1-y) + t^3(z-1)^3]}
\end{aligned} \tag{229}$$

$$1/\eta^n \text{singularity} = N^n \text{enhancement factor} \tag{230}$$

b. The origin of the $1/\eta$ singularity
(1) p and $p+q$ on the Fermi surface.

$$p_x + p_y^2 = 0, \qquad p_x + q_x + (p_y + q_y)^2 = 0 \tag{231}$$

(2) For a given q, k and $k+q$ on the Fermi surface.

$$k_x + k_y^2 = 0, \qquad k_x + q_x + (k_y + q_y)^2 = 0 \tag{232}$$

(3) As a result, $k = p$. (4) $k+l$ on the Fermi surface, tuned by l_x, and l_y free: one dimensional Fermi surface, described by l_y

$$k_x + l_x + (k_y + l_y)^2 = 0 \tag{233}$$

(5) Singular all four fermion propagators in an one dimensional manifold, embedded in the four dimensional space k and l, and referred to as a singular manifold. (6) The codimension of the singular manifold = 3: three dimensional integrals to cancel the IR divergence. (7) The strength of the singularity in the product of the four fermion propagators = 4. (8) The remaining singularity: $4 - 3 = 1 \longrightarrow 1/\eta$ singularity, replaced with N.

$$g_0(k) = \frac{1}{i\eta k_0 + k_x + k_y^2}, \qquad g_0(k+q) = \frac{1}{i\eta(k_0+q_0) + (k_x+q_x) + (k_y+q_y)^2},$$
$$g_0(k+l) = \frac{1}{i\eta(k_0+l_0) + (k_x+l_x) + (k_y+l_y)^2}, \qquad g_0(p+l) = \frac{1}{i\eta(p_0+l_0) + (p_x+l_x) + (p_y+l_y)^2} \tag{234}$$

c. A general rule for the Feynman diagram in $1/N$

(1) Single lines for fermion propagators and double lines for boson propagators (2) Vertex with the $q/\sqrt{N}$ factor (3) Factor N enhancement due to fermion loops (4) The enhancement factor $= N^{[I_f - 2L + n]}$ I_f: The number of internal fermion propagators = the strength of the $1/\eta$ singularity L: The number of loops (internal momenta) (2 = two dimensions) n: The number of closed single-line loops in the double line representation $2L - n$: The codimension of the singular manifold = the cancelation factor of the $1/\eta$ singularity $[x] = x$ if $x \geq 0$ and 0 if $x < 0$ (5) Each single line represents a momentum on the Fermi surface. (6) Momenta in the single lines connected to the external lines are uniquely fixed to make all fermions stay on the Fermi surface. (7) Momenta in the single lines forming closed loops are unfixed. (8) First diagram $N^{[4-(2\times 2-1)]} = N^1$ (four fermion propagators, two internal loops, and one dimensional singular manifold = one closed loop) (9) Second diagram $N^{[5-(2\times 3-2)]} = N^1$ (five fermion propagators, three internal loops, and two closed loops = two dimensional singular manifold)

D. Dimensional regularization for the Fermi-surface problem

Ref. S.-S. Lee, Phys. Rev. B **88**, 245106 (2013)

1. *Setup*

$$S = \sum_{s=\pm}\sum_{j=1}^{N}\int \frac{d^3k}{(2\pi)^3}\psi_{sj}^\dagger(k)(ik_0 + sk_1 + k_2^2)\psi_{sj}(k) + \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}(k_0^2 + k_1^2 + k_2^2)\phi(-k)\phi(k)$$
$$+\frac{g}{\sqrt{N}}\sum_{s=\pm}\sum_{j=1}^{N}\int \frac{d^3k d^3q}{(2\pi)^6}\phi(q)\psi_{sj}^\dagger(k+q)\psi_{sj}(k) \tag{235}$$

$$\Psi_j(k) = \begin{pmatrix} \psi_{+j}(k) \\ \psi_{-j}^\dagger(-k) \end{pmatrix} \tag{236}$$

$$S = \sum_{j=1}^{N}\int \frac{d^3k}{(2\pi)^3}\bar{\Psi}_j(k)[ik_0\gamma_0 + i(k_1 + k_2^2)\gamma_1]\Psi_j(k) + \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}(k_0^2 + k_1^2 + k_2^2)\phi(-k)\phi(k)$$
$$+\frac{g}{\sqrt{N}}\sum_{j=1}^{N}\int \frac{d^3k d^3q}{(2\pi)^6}\phi(q)\bar{\Psi}_j(k+q)i\gamma_1\Psi_j(k) \tag{237}$$

$$\gamma_0 = \sigma_y, \qquad \gamma_1 = \sigma_x, \qquad \bar{\Psi}_j(k) = \Psi_j^\dagger(k)\gamma_0 \tag{238}$$

$$S = \sum_{j=1}^{N}\int \frac{d^{d+1}k}{(2\pi)^3}\bar{\Psi}_j(k)(i\boldsymbol{\Gamma}\cdot\boldsymbol{K} + i\gamma_{d-1}\delta_k)\Psi_j(k) + \frac{1}{2}\int \frac{d^{d+1}k}{(2\pi)^3}(|\boldsymbol{K}|^2 + k_{d-1}^2 + k_d^2)\phi(-k)\phi(k)$$
$$+i\frac{g}{\sqrt{N}}\sqrt{d-1}\sum_{j=1}^{N}\int \frac{d^{d+1}k d^{d+1}q}{(2\pi)^{2d+2}}\phi(q)\bar{\Psi}_j(k+q)\gamma_{d-1}\Psi_j(k) \tag{239}$$

$$\boldsymbol{K} = (k_0, k_1, ..., k_{d-2}), \qquad \delta_k = k_{d-1} + k_d^2, \qquad \boldsymbol{\Gamma} = (\gamma_0, \gamma_1, ..., \gamma_{d-2}), \qquad \gamma_0 = \sigma_y, \qquad \gamma_{d-1} = \sigma_x \tag{240}$$

$$E_k = \pm\sqrt{\sum_{i=1}^{d-2} k_i^2 + \delta_k^2} \tag{241}$$

One dimensional Fermi surface embedded in the $d-$dimensional momentum space

$$k_i = 0, \quad \text{for } i = 1, ..., d-2, \qquad k_{d-1} = -\sqrt{d-1}k_d^2 \tag{242}$$

2. $d = 3$ and $N = 2$

$$S_\psi = \int \frac{d^4k}{(2\pi)^4} \sum_{s=\pm} \sum_{j=\uparrow,\downarrow}^{2} \psi_{sj}^\dagger(k)(ik_0 + sk_2 + \sqrt{2}k_3^2)\psi_{sj}(k) - k_1[\psi_{+\uparrow}^\dagger(k)\psi_{-\uparrow}^\dagger(-k) + \psi_{+\downarrow}^\dagger(k)\psi_{-\downarrow}^\dagger(-k) + H.c.] \tag{243}$$

$$i \begin{pmatrix} \psi_{+\uparrow}^\dagger(k) & \psi_{+\downarrow}^\dagger(k) \end{pmatrix} [\boldsymbol{d}(k) \cdot \boldsymbol{\sigma}]\sigma_y \begin{pmatrix} \psi_{-\uparrow}^\dagger(-k) \\ \psi_{-\downarrow}^\dagger(-k) \end{pmatrix} \tag{244}$$

$$\boldsymbol{d}(k) = ik_1\hat{y} \tag{245}$$

$p-$wave spin triplet superconducting state: $U(1) \times O(3) \times SU(2) \times Z_2^T \longrightarrow Z_2 \times O(3) \times U(1) \times Z_2^T$

3. Scaling analysis in $d > 5/2$

$$\begin{aligned} &\boldsymbol{K} = \boldsymbol{K}'/b, \qquad k_{d-1} = k_{d-1}'/b, \qquad k_d = k_d'/\sqrt{b}, \\ &\Psi_j(k) = b^{d/2+3/4}\Psi_j'(k'), \qquad \phi(k) = b^{d/2+3/4}\phi'(k') \end{aligned} \tag{246}$$

$$e' = b^{(1/2)(5/2-d)}e \tag{247}$$

$$\begin{aligned} S = &\sum_{j=1}^{N} \int \frac{d^{d+1}k}{(2\pi)^3} \bar{\Psi}_j(k)(i\boldsymbol{\Gamma} \cdot \boldsymbol{K} + i\gamma_{d-1}\delta_k)\Psi_j(k) + \frac{1}{2}\int \frac{d^{d+1}k}{(2\pi)^3} k_d^2 \phi(-k)\phi(k) \\ &+ i\frac{g}{\sqrt{N}}\mu^{\epsilon/2}\sqrt{d-1}\sum_{j=1}^{N} \int \frac{d^{d+1}kd^{d+1}q}{(2\pi)^{2d+2}} \phi(q)\bar{\Psi}_j(k+q)\gamma_{d-1}\Psi_j(k) \end{aligned} \tag{248}$$

4. Scaling analysis in $d < 5/2$

$$\begin{aligned} &\boldsymbol{K} = \boldsymbol{K}'/b^z, \qquad k_{d-1} = k_{d-1}'/b, \qquad k_d = k_d'/\sqrt{b}, \\ &\Psi_j(k) = b^{(d-1)z/2+5/4}\Psi_j'(k'), \qquad \phi(k) = b^{(d-1)z/2+5/4}\phi'(k') \end{aligned} \tag{249}$$

$$z = \frac{3}{3-2\epsilon}, \qquad \epsilon = \frac{5}{2} - d \tag{250}$$

$$S = \sum_{j=1}^{N} \int \frac{d^{d+1}k_b}{(2\pi)^3} \bar{\Psi}_{bj}(k_b)(i\mathbf{\Gamma} \cdot \boldsymbol{K}_b + i\gamma_{d-1}\delta_{k_b})\Psi_{bj}(k_b) + \frac{1}{2} \int \frac{d^{d+1}k_b}{(2\pi)^3} k_{bd}^2 \phi_b(-k_b)\phi_b(k_b)$$
$$+ i\frac{g_b}{\sqrt{N}}\sqrt{d-1}\sum_{j=1}^{N} \int \frac{d^{d+1}k_b d^{d+1}q_b}{(2\pi)^{2d+2}} \phi_b(q_b)\bar{\Psi}_{bj}(k_b+q_b)\gamma_{d-1}\Psi_{bj}(k_b) \tag{251}$$

$$S = S_r + S_{ct} \tag{252}$$

$$S_r = \sum_{j=1}^{N} \int \frac{d^{d+1}k}{(2\pi)^3} \bar{\Psi}_j(k)(i\mathbf{\Gamma} \cdot \boldsymbol{K} + i\gamma_{d-1}\delta_k)\Psi_j(k) + \frac{1}{2} \int \frac{d^{d+1}k}{(2\pi)^3} k_d^2 \phi(-k)\phi(k)$$
$$+ i\frac{g}{\sqrt{N}}\mu^{\epsilon/2}\sqrt{d-1}\sum_{j=1}^{N} \int \frac{d^{d+1}k d^{d+1}q}{(2\pi)^{2d+2}} \phi(q)\bar{\Psi}_j(k+q)\gamma_{d-1}\Psi_j(k) \tag{253}$$

$$S_{ct} = \sum_{j=1}^{N} \int \frac{d^{d+1}k}{(2\pi)^3} \bar{\Psi}_j(k)(iA_1\mathbf{\Gamma} \cdot \boldsymbol{K} + iA_2\gamma_{d-1}\delta_k)\Psi_j(k) + \frac{A_3}{2} \int \frac{d^{d+1}k}{(2\pi)^3} k_d^2 \phi(-k)\phi(k)$$
$$+ iA_4\frac{g}{\sqrt{N}}\mu^{\epsilon/2}\sqrt{d-1}\sum_{j=1}^{N} \int \frac{d^{d+1}k d^{d+1}q}{(2\pi)^{2d+2}} \phi(q)\bar{\Psi}_j(k+q)\gamma_{d-1}\Psi_j(k) \tag{254}$$

$$\text{Ward identity } \longrightarrow A_2 = A_4 \tag{255}$$

$$\begin{aligned} &\boldsymbol{K} = \frac{Z_2}{Z_1}\boldsymbol{K}_b, \qquad k_{d-1} = k_{bd-1}, \qquad k_d = k_{bd}, \\ &\Psi_j(k) = Z_\Psi^{-1/2}\Psi_b(k_b), \qquad \phi(k) = Z_\phi^{-1/2}\phi_b(k_b), \\ &Z_n = 1 + A_n, \qquad Z_\Psi = Z_2\Big(\frac{Z_2}{Z_1}\Big)^{d-1}, \qquad Z_\phi = Z_3\Big(\frac{Z_2}{Z_1}\Big)^{d-1} \end{aligned} \tag{256}$$

$$e_b = \mu^{\epsilon/2} Z_3^{-1/2}\Big(\frac{Z_2}{Z_1}\Big)^{(d-1)/2} e \tag{257}$$

$$z(e) = 1 - \frac{\partial \ln(Z_2/Z_1)}{\partial \ln \mu}, \qquad \eta_\Psi = \frac{1}{2}\frac{\partial \ln Z_\Psi}{\partial \ln \mu}, \qquad \eta_\phi(e) = \frac{1}{2}\frac{\partial \ln Z_\phi}{\ln \mu} \tag{258}$$

$$\beta(e) = \frac{\partial e}{\partial \ln \mu} = -\frac{\epsilon}{2}e + 0.02920\Big(\frac{3}{2} - \epsilon\Big)\frac{e^{7/3}}{N} - 0.01073\Big(\frac{3}{2} - \epsilon\Big)\frac{e^{11/3}}{N^2} \tag{259}$$

$$z = \frac{3}{3 - 2\epsilon}, \qquad \eta_\Psi = -\frac{\epsilon}{2} + 0.07541\epsilon^2, \qquad \eta_\phi = -\frac{\epsilon}{2} \tag{260}$$

E. Role of disorder in metallic quantum criticality

VII. MOTT QUANTUM CRITICALITY

A. Problem: Experiments

a. Phase diagram of $\kappa-$organic salts
b. Scaling phenomena (electrical transport) in the vicinity of metal-insulator transitions
c. Bad metal

B. DMFT (Dynamical mean-field theory) Mott quantum criticality

1. *Effective field theory and Mott transition as breakdown of the self-consistent pseudogap Kondo effect*

$$Z = \int Dc_{i\sigma} \exp\Big[- \int_0^\beta d\tau \Big\{ \sum_i c_{i\sigma}^\dagger (\partial_\tau - \mu) c_{i\sigma} - \frac{t}{\sqrt{z}} \sum_{ij} c_{i\sigma}^\dagger c_{j\sigma} - H.c. + u \sum_i n_{i\uparrow} n_{i\downarrow} \Big\} \Big] \tag{261}$$

$$Z = \int D\psi_\sigma Dc_{i\neq 0\sigma} \exp\Big[- \int_0^\beta d\tau \Big\{ \psi_\sigma^\dagger (\partial_\tau - \mu) \psi_\sigma + u n_\uparrow^\psi n_\downarrow^\psi - \frac{t}{\sqrt{z}} \sum_{i\in 0} \psi_\sigma^\dagger c_{i\sigma} - H.c.$$
$$+ \sum_{i\neq 0} c_{i\sigma}^\dagger (\partial_\tau - \mu) c_{i\sigma} - \frac{t}{\sqrt{z}} \sum_{ij\neq 0} c_{i\sigma}^\dagger c_{j\sigma} - H.c. + u \sum_{i\neq 0} n_{i\uparrow} n_{i\downarrow} \Big\} \Big] \tag{262}$$

$$S_{int} = - \int_0^\beta d\tau \Big(\frac{t}{\sqrt{z}} \sum_{i\in 0} \psi_\sigma^\dagger c_{i\sigma} + H.c. \Big) \tag{263}$$

$$\langle e^{-S_{int}} \rangle = \exp\Big\{ - \langle S_{int} \rangle + \frac{1}{2} \Big(\langle S_{int}^2 \rangle - \langle S_{int} \rangle^2 \Big) \Big\} \tag{264}$$

$$S_{eff}^{(2)} = -\frac{1}{2} \int_0^\beta d\tau \int_0^\beta d\tau' \frac{t^2}{z} \sum_{i\in 0} \sum_{j\in 0} \Big(\psi_\sigma^\dagger(\tau) \Big\langle c_{i\sigma}(\tau) c_{j\sigma'}^\dagger(\tau') \Big\rangle \psi_{\sigma'}(\tau') + c_{i\sigma}(\tau) \Big\langle \psi_\sigma^\dagger(\tau) \psi_{\sigma'}(\tau') \Big\rangle c_{j\sigma'}^\dagger(\tau')$$
$$- \Big\langle \psi_\sigma^\dagger(\tau) \psi_{\sigma'}(\tau') \Big\rangle \Big\langle c_{i\sigma}(\tau) c_{j\sigma'}^\dagger(\tau') \Big\rangle \Big) \tag{265}$$

$$S_{eff}^{(2)} = -\frac{1}{2} \int_0^\beta d\tau \int_0^\beta d\tau' \frac{t^2}{z} \Big(\psi_\sigma^\dagger(\tau) \Big\langle \sum_{i\in 0} c_{i\sigma}(\tau) c_{i\sigma}^\dagger(\tau') \Big\rangle \psi_\sigma(\tau') - \sum_{i\in 0} c_{i\sigma}^\dagger(\tau') \Big\langle \psi_\sigma^\dagger(\tau) \psi_\sigma(\tau') \Big\rangle c_{i\sigma}(\tau)$$
$$+ \Big\langle \psi_\sigma(\tau') \psi_\sigma^\dagger(\tau) \Big\rangle \Big\langle \sum_{i\in 0} c_{i\sigma}(\tau) c_{i\sigma}^\dagger(\tau') \Big\rangle \Big)$$
$$= \frac{1}{2} \int_0^\beta d\tau \int_0^\beta d\tau' \Big(\psi_\sigma^\dagger(\tau) G_L(\tau - \tau') \psi_\sigma(\tau') + \frac{t^2}{z} \sum_{i\in 0} c_{i\sigma}^\dagger(\tau') G_\psi(\tau' - \tau) c_{i\sigma}(\tau) - G_L(\tau - \tau') G_\psi(\tau' - \tau) \Big) \tag{266}$$

$$Z = \int D\psi_\sigma Dc_{i\neq 0\sigma} \exp\Big[- \int_0^\beta d\tau \Big\{ \psi_\sigma^\dagger (\partial_\tau - \mu) \psi_\sigma + u n_\uparrow^\psi n_\downarrow^\psi$$
$$+ \frac{1}{2} \int_0^\beta d\tau' \Big(\psi_\sigma^\dagger(\tau) G_L(\tau - \tau') \psi_\sigma(\tau') + \frac{t^2}{z} \sum_{i\in 0} c_{i\sigma}^\dagger(\tau') G_\psi(\tau' - \tau) c_{i\sigma}(\tau) - G_L(\tau - \tau') G_\psi(\tau' - \tau) \Big)$$
$$+ \sum_{i\neq 0} c_{i\sigma}^\dagger (\partial_\tau - \mu) c_{i\sigma} - \frac{t}{\sqrt{z}} \sum_{ij\neq 0} c_{i\sigma}^\dagger c_{j\sigma} - H.c. + u \sum_{i\neq 0} n_{i\uparrow} n_{i\downarrow} \Big\} \Big] \tag{267}$$

$$Z \approx \int D\psi_\sigma Dc_{\boldsymbol{k}\sigma} \exp\Big[-\int_0^\beta d\tau \Big\{ \psi_\sigma^\dagger (\partial_\tau - \mu)\psi_\sigma + u n_\uparrow^\psi n_\downarrow^\psi + \frac{1}{2}\int_0^\beta d\tau' \psi_\sigma^\dagger(\tau) G_L(\tau-\tau')\psi_\sigma(\tau') \\ -\frac{1}{2}\int_0^\beta d\tau' G_L(\tau-\tau') G_\psi(\tau'-\tau) \Big\} - \Big\{ \sum_{i\omega}\sum_{\boldsymbol{k}} c_{\boldsymbol{k}\sigma}^\dagger(i\omega)\Big(-i\omega - \mu + \epsilon_{\boldsymbol{k}} + \frac{t^2}{2} G_\psi(i\omega)\Big) c_{\boldsymbol{k}\sigma}(i\omega) + u\int_0^\beta d\tau \sum_{i\neq 0} n_{i\uparrow} n_{i\downarrow} \Big\} \Big] \tag{268}$$

$$\epsilon_{\boldsymbol{k}} = -2t \sum_{l=1}^{d} \cos k_l \tag{269}$$

$$Z \approx \int D\psi_\sigma(i\omega) Dc_{\boldsymbol{k}\sigma}(i\omega) \exp\Big[- \Big\{ \sum_{i\omega} \psi_\sigma^\dagger(i\omega) \Big(-i\omega - \mu + \frac{1}{2} G_L(i\omega) \Big) \psi_\sigma(i\omega) + u \int_0^\beta d\tau n_\uparrow^\psi n_\downarrow^\psi - \frac{1}{2}\sum_{i\omega} G_L(i\omega) G_\psi(i\omega) \Big\} \\ - \Big\{ \sum_{i\omega}\sum_{\boldsymbol{k}} c_{\boldsymbol{k}\sigma}^\dagger(i\omega)\Big(-i\omega - \mu + \epsilon_{\boldsymbol{k}} + \frac{t^2}{2} G_\psi(i\omega)\Big) c_{\boldsymbol{k}\sigma}(i\omega) + u\int_0^\beta d\tau \sum_{i\neq 0} n_{i\uparrow} n_{i\downarrow} \Big\} \Big] \tag{270}$$

$$Z \approx \int D\psi_\sigma(i\omega) Dc_{\boldsymbol{k}\sigma}(i\omega) \exp\Big[- \Big\{ \sum_{i\omega} \psi_\sigma^\dagger(i\omega) \Big(-i\omega - \mu + \frac{1}{2} G_L(i\omega) + \Sigma_L(i\omega) \Big) \psi_\sigma(i\omega) - \frac{1}{2}\sum_{i\omega} G_L(i\omega) G_\psi(i\omega) \Big\} \\ - \Big\{ \sum_{i\omega}\sum_{\boldsymbol{k}} c_{\boldsymbol{k}\sigma}^\dagger(i\omega)\Big(-i\omega - \mu + \epsilon_{\boldsymbol{k}} + \frac{t^2}{2} G_\psi(i\omega) + \Sigma(\boldsymbol{k}, i\omega)\Big) c_{\boldsymbol{k}\sigma}(i\omega) \Big\} \Big] \tag{271}$$

$$F_{eff}[G_L(i\omega), G_\psi(i\omega)] = -\frac{N_\sigma}{\beta} \sum_{i\omega} \Big\{ \ln \Big(-i\omega - \mu + \frac{1}{2} G_L(i\omega) + \Sigma_L(i\omega) \Big) + \frac{1}{2} G_L(i\omega) G_\psi(i\omega) \Big\} \\ -\frac{N_\sigma}{\beta} \sum_{i\omega}\sum_{\boldsymbol{k}} \ln \Big(-i\omega - \mu + \epsilon_{\boldsymbol{k}} + \frac{t^2}{2} G_\psi(i\omega) + \Sigma(\boldsymbol{k}, i\omega) \Big) \tag{272}$$

$$\frac{\partial F_{eff}[G_L(i\omega), G_\psi(i\omega)]}{\partial G_L(i\omega)} = 0 \longrightarrow G_\psi(i\omega) = \frac{1}{i\omega + \mu - \frac{1}{2} G_L(i\omega) - \Sigma_L(i\omega)}, \\ \frac{\partial F_{eff}[G_L(i\omega), G_\psi(i\omega)]}{\partial G_\psi(i\omega)} = 0 \longrightarrow G_L(i\omega) = \sum_{\boldsymbol{k}} \frac{1}{i\omega + \mu - \epsilon_{\boldsymbol{k}} - \frac{t^2}{2} G_\psi(i\omega) - \Sigma(\boldsymbol{k}, i\omega)} \tag{273}$$

$$\Sigma(\boldsymbol{k}, i\omega) \longrightarrow \Sigma_L(i\omega) \tag{274}$$

2. *Phase diagram and essential physics*

a. First order phase transition at low temperatures and the critical end point
b. Emergent local moments and their entropy contributions
c. Cluster extension: The role of RKKY correlations $\longrightarrow$ Phase diagram based on the entropy
d. Scaling phenomena (electrical transport) in the vicinity of metal-insulator transitions
e. Bad metal behaviors: Fate of electron quasiparticle excitations and their transport phenomena (Half filling vs. Away from half filling)

C. Spin-liquid Mott quantum criticality

1. Effective field theory and Mott transition as a Higgs transition

a. One-band Hubbard model

$$H = -t \sum_{ij} c_{i\sigma}^{\dagger} c_{j\sigma} - H.c. + u \sum_{i} n_{i\uparrow} n_{i\downarrow} \tag{275}$$

b. To take into account charge fluctuations only

$$\begin{aligned} Z &= \int Dc_{i\sigma} \exp\Big[- \int_{0}^{\beta} d\tau \Big\{ \sum_{i} c_{i\sigma}^{\dagger} (\partial_{\tau} - \mu) c_{i\sigma} - t \sum_{ij} c_{i\sigma}^{\dagger} c_{j\sigma} - H.c. + u \sum_{i} n_{i\uparrow} n_{i\downarrow} \Big\} \Big] \\ &= \int Dc_{i\sigma} D\varphi_{i} \exp\Big[- \int_{0}^{\beta} d\tau \Big\{ \sum_{i} c_{i\sigma}^{\dagger} (\partial_{\tau} - \mu - i\varphi_{i}) c_{i\sigma} - t \sum_{ij} c_{i\sigma}^{\dagger} c_{j\sigma} - H.c. + \frac{1}{2u} \sum_{i} \varphi_{i}^{2} \Big\} \Big] \end{aligned} \tag{276}$$

c. U(1) slave-rotor representation

$$c_{i\sigma} = e^{-i\theta_{i}} f_{i\sigma} \tag{277}$$

$$Z = \int Df_{i\sigma} D\theta_{i} D\varphi_{i} \exp\Big[- \int_{0}^{\beta} d\tau \Big\{ \sum_{i} f_{i\sigma}^{\dagger} (\partial_{\tau} - \mu - i\varphi_{i} - i\partial_{\tau}\theta_{i}) f_{i\sigma} - t \sum_{ij} f_{i\sigma}^{\dagger} e^{i\theta_{i}} e^{-i\theta_{j}} f_{j\sigma} - H.c. + \frac{1}{2u} \sum_{i} \varphi_{i}^{2} \Big\} \Big] \tag{278}$$

$$\begin{aligned} Z = \int Df_{i\sigma} D\theta_{i} D\varphi_{i} \exp\Big[- \int_{0}^{\beta} d\tau \Big\{ \sum_{i} f_{i\sigma}^{\dagger} (\partial_{\tau} - \mu - i\varphi_{i}) f_{i\sigma} - t \sum_{ij} f_{i\sigma}^{\dagger} e^{i\theta_{i}} e^{-i\theta_{j}} f_{j\sigma} - H.c. \\ + \frac{1}{2u} \sum_{i} (\partial_{\tau}\theta_{i} - \varphi_{i})^{2} \Big\} \Big] \end{aligned} \tag{279}$$

d. Hubbard-Stratonovich transformation for the hopping channel

$$\begin{aligned} Z = \int Df_{i\sigma} D\theta_{i} D\chi_{ij}^{f} D\chi_{ij}^{\theta} D\varphi_{i} \exp\Big[- \int_{0}^{\beta} d\tau \Big\{ \sum_{i} f_{i\sigma}^{\dagger} (\partial_{\tau} - \mu - i\varphi_{i}) f_{i\sigma} - t \sum_{ij} f_{i\sigma}^{\dagger} \chi_{ij}^{f} f_{j\sigma} - H.c. \\ + \frac{1}{2u} \sum_{i} (\partial_{\tau}\theta_{i} - \varphi_{i})^{2} - t \sum_{ij} e^{-i\theta_{j}} \chi_{ji}^{\theta} e^{i\theta_{i}} - H.c. + t \sum_{ij} \chi_{ij}^{f} \chi_{ji}^{\theta} + H.c. \Big\} \Big] \end{aligned} \tag{280}$$

e. Effective Lagrangian

$$\begin{aligned} L &= L_{f} + L_{\theta} + L_{\chi}, \\ L_{f} &= \sum_{i} f_{i\sigma}^{\dagger} (\partial_{\tau} - \mu - i\varphi_{i}) f_{i\sigma} - t \sum_{ij} f_{i\sigma}^{\dagger} \chi_{ij}^{f} f_{j\sigma} - H.c., \\ L_{\theta} &= \frac{1}{2u} \sum_{i} (\partial_{\tau}\theta_{i} - \varphi_{i})^{2} - t \sum_{ij} e^{-i\theta_{j}} \chi_{ji}^{\theta} e^{i\theta_{i}} - H.c., \\ L_{\chi} &= t \sum_{ij} \chi_{ij}^{f} \chi_{ji}^{\theta} + H.c. \end{aligned} \tag{281}$$

f. Mean-field theory for a spin-liquid state

$$L_{MF} = \sum_{i} f_{i\sigma}^{\dagger}(\partial_{\tau} - \mu_{r})f_{i\sigma} - t\chi_{f}\sum_{ij} f_{i\sigma}^{\dagger}f_{j\sigma} - H.c.$$
$$+\frac{1}{2u}\sum_{i}(\partial_{\tau}\theta_{i})^{2} - t\chi_{\theta}\sum_{ij} e^{-i\theta_{j}}e^{i\theta_{i}} - H.c. + 2ztL^{d}\chi_{f}\chi_{\theta} \tag{282}$$

g. Nonlinear $\sigma-$model representation

$$e^{i\theta_{i}} \longrightarrow b_{i}, \qquad |b_{i}|^{2} = 1 \tag{283}$$

$$Z_{MF} = \int Df_{i\sigma}Db_{i}\exp\Big[-\int_{0}^{\beta}d\tau\Big\{\sum_{i} f_{i\sigma}^{\dagger}(\partial_{\tau} - \mu_{r})f_{i\sigma} - t\chi_{f}\sum_{ij} f_{i\sigma}^{\dagger}f_{j\sigma} - H.c.$$
$$+\frac{1}{2u}\sum_{i}(-ib_{i}^{\dagger}\partial_{\tau}b_{i})^{2} - t\chi_{\theta}\sum_{ij} b_{j}^{\dagger}b_{i} - H.c. + \lambda\sum_{i}|b_{i}|^{2} + L^{d}(2zt\chi_{f}\chi_{\theta} - \lambda)\Big\}\Big] \tag{284}$$

h. Mean-field free energy

$$F_{MF}(\chi_{f},\chi_{\theta},\lambda) = -\frac{N}{\beta}\sum_{i\omega}\sum_{\boldsymbol{k}}\ln\Big(-i\omega - \mu_{r} - zt\chi_{f}\gamma_{\boldsymbol{k}}\Big) + \frac{1}{\beta}\sum_{i\Omega}\sum_{\boldsymbol{q}}\ln\Big\{\frac{\Omega^{2}}{2u} + \lambda - zt\chi_{\theta}\gamma_{\boldsymbol{q}}\Big\} + L^{d}(2zt\chi_{f}\chi_{\theta} - \lambda)$$
$$= -\frac{N}{\beta}\sum_{\boldsymbol{k}}\ln\Big\{1 + \exp\Big(-\beta[-zt\chi_{f}\gamma_{\boldsymbol{k}} - \mu_{r}]\Big)\Big\} + \frac{1}{\beta}\sum_{\boldsymbol{q}}\ln\Big\{1 - \exp\Big(-\beta\sqrt{2u(\lambda - zt\chi_{\theta}\gamma_{\boldsymbol{q}})}\Big)\Big\}$$
$$+\frac{1}{\beta}\sum_{\boldsymbol{q}}\ln\Big\{1 - \exp\Big(\beta\sqrt{2u(\lambda - zt\chi_{\theta}\gamma_{\boldsymbol{q}})}\Big)\Big\} + L^{d}(2zt\chi_{f}\chi_{\theta} - \lambda) \tag{285}$$

i. Self-consistent equations for order parameters

$$\frac{\partial F_{MF}(\chi_{f},\chi_{\theta},\lambda)}{\partial\chi_{f}} = 0 \longrightarrow \frac{N}{L^{d}}\sum_{\boldsymbol{k}} zt\gamma_{\boldsymbol{k}}\frac{1}{\exp\Big(\beta[-zt\chi_{f}\gamma_{\boldsymbol{k}} - \mu_{r}]\Big) + 1} = 2zt\chi_{\theta},$$
$$\frac{\partial F_{MF}(\chi_{f},\chi_{\theta},\lambda)}{\partial\chi_{\theta}} = 0 \longrightarrow \frac{1}{L^{d}}\sum_{\boldsymbol{q}}\Big\{\frac{uzt\gamma_{\boldsymbol{q}}}{\sqrt{2u(\lambda - zt\chi_{\theta}\gamma_{\boldsymbol{q}})}}\frac{1}{\exp\Big(\beta\sqrt{2u(\lambda - zt\chi_{\theta}\gamma_{\boldsymbol{q}})}\Big) - 1}$$
$$-\frac{uzt\gamma_{\boldsymbol{q}}}{\sqrt{2u(\lambda - zt\chi_{\theta}\gamma_{\boldsymbol{q}})}}\frac{1}{\exp\Big(-\beta\sqrt{2u(\lambda - zt\chi_{\theta}\gamma_{\boldsymbol{q}})}\Big) - 1}\Big\} = 2zt\chi_{f},$$
$$\frac{\partial F_{MF}(\chi_{f},\chi_{\theta},\lambda)}{\partial\lambda} = 0 \longrightarrow \frac{1}{L^{d}}\sum_{\boldsymbol{q}}\Big\{\frac{u}{\sqrt{2u(\lambda - zt\chi_{\theta}\gamma_{\boldsymbol{q}})}}\frac{1}{\exp\Big(\beta\sqrt{2u(\lambda - zt\chi_{\theta}\gamma_{\boldsymbol{q}})}\Big) - 1}$$
$$-\frac{u}{\sqrt{2u(\lambda - zt\chi_{\theta}\gamma_{\boldsymbol{q}})}}\frac{1}{\exp\Big(-\beta\sqrt{2u(\lambda - zt\chi_{\theta}\gamma_{\boldsymbol{q}})}\Big) - 1}\Big\} = 1 \tag{286}$$

j. Condensation of bosons

$$Z_{MF} = \int Df_{\boldsymbol{k}\sigma}Db_{\boldsymbol{q}}\exp\Big[-\Big\{\sum_{i\omega}\sum_{\boldsymbol{k}} f_{\boldsymbol{k}\sigma}^{\dagger}(-i\omega - \mu_{r} - zt\chi_{f}\gamma_{\boldsymbol{k}})f_{\boldsymbol{k}\sigma}$$
$$+\sum_{i\Omega}\sum_{\boldsymbol{q}} b_{\boldsymbol{q}}^{\dagger}\Big(\frac{\Omega^{2}}{2u} + \lambda - zt\chi_{\theta}\gamma_{\boldsymbol{q}}\Big)b_{\boldsymbol{q}} + \beta(\lambda - zt\chi_{\theta}\gamma_{\boldsymbol{q}=0})|b|^{2} + \beta L^{d}(2zt\chi_{f}\chi_{\theta} - \lambda)\Big\}\Big] \tag{287}$$

$$F_{MF}(\chi_f,\chi_\theta,\lambda) = -\frac{N}{\beta}\sum_{\boldsymbol{k}}\ln\Big\{1+\exp\Big(-\beta[-zt\chi_f\gamma_{\boldsymbol{k}}-\mu_r]\Big)\Big\} + \frac{1}{\beta}\sum_{\boldsymbol{q}}\ln\Big\{1-\exp\Big(-\beta\sqrt{2u(\lambda-zt\chi_\theta\gamma_{\boldsymbol{q}})}\Big)\Big\}$$
$$+\frac{1}{\beta}\sum_{\boldsymbol{q}}\ln\Big\{1-\exp\Big(\beta\sqrt{2u(\lambda-zt\chi_\theta\gamma_{\boldsymbol{q}})}\Big)\Big\} + (\lambda - zt\chi_\theta\gamma_{\boldsymbol{q}=0})|b|^2 + L^d(2zt\chi_f\chi_\theta-\lambda) \quad (288)$$

$$\frac{\partial F_{MF}(\chi_f,\chi_\theta,\lambda)}{\partial\chi_f} = 0 \longrightarrow \frac{N}{L^d}\sum_{\boldsymbol{k}} zt\gamma_{\boldsymbol{k}}\frac{1}{\exp\Big(\beta[-zt\chi_f\gamma_{\boldsymbol{k}}-\mu_r]\Big)+1} = 2zt\chi_\theta,$$
$$\frac{\partial F_{MF}(\chi_f,\chi_\theta,\lambda)}{\partial\chi_\theta} = 0 \longrightarrow \frac{1}{L^d}\sum_{\boldsymbol{q}}\Big\{\frac{uzt\gamma_{\boldsymbol{q}}}{\sqrt{2u(\lambda-zt\chi_\theta\gamma_{\boldsymbol{q}})}}\frac{1}{\exp\Big(\beta\sqrt{2u(\lambda-zt\chi_\theta\gamma_{\boldsymbol{q}})}\Big)-1}$$
$$-\frac{uzt\gamma_{\boldsymbol{q}}}{\sqrt{2u(\lambda-zt\chi_\theta\gamma_{\boldsymbol{q}})}}\frac{1}{\exp\Big(-\beta\sqrt{2u(\lambda-zt\chi_\theta\gamma_{\boldsymbol{q}})}\Big)-1}\Big\} = 2zt\chi_f - zt\gamma_{\boldsymbol{q}=0}|b|^2,$$
$$\frac{\partial F_{MF}(\chi_f,\chi_\theta,\lambda)}{\partial\lambda} = 0 \longrightarrow \frac{1}{L^d}\sum_{\boldsymbol{q}}\Big\{\frac{u}{\sqrt{2u(\lambda-zt\chi_\theta\gamma_{\boldsymbol{q}})}}\frac{1}{\exp\Big(\beta\sqrt{2u(\lambda-zt\chi_\theta\gamma_{\boldsymbol{q}})}\Big)-1}$$
$$-\frac{u}{\sqrt{2u(\lambda-zt\chi_\theta\gamma_{\boldsymbol{q}})}}\frac{1}{\exp\Big(-\beta\sqrt{2u(\lambda-zt\chi_\theta\gamma_{\boldsymbol{q}})}\Big)-1}\Big\} = 1-|b|^2,$$
$$\frac{\partial F_{MF}(\chi_f,\chi_\theta,\lambda)}{\partial|b|^2} = 0 \longrightarrow \lambda - zt\chi_\theta\gamma_{\boldsymbol{q}=0} = 0 \quad (289)$$

k. Electron Green's function

$$G_{el}(\boldsymbol{r}-\boldsymbol{r}',\tau-\tau') = -\langle T_\tau[c_\sigma(\boldsymbol{r},\tau)c_\sigma^\dagger(\boldsymbol{r}',\tau')]\rangle$$
$$= -\langle T_\tau[b^\dagger(\boldsymbol{r},\tau)f_\sigma(\boldsymbol{r},\tau)f_\sigma^\dagger(\boldsymbol{r}',\tau')b(\boldsymbol{r}',\tau')]\rangle \approx G_f(\boldsymbol{r}-\boldsymbol{r}',\tau-\tau')[G_b(\boldsymbol{r}'-\boldsymbol{r},\tau'-\tau)+|b|^2] \quad (290)$$

$$G_{el}(\boldsymbol{k},i\omega) = \frac{1}{\beta}\sum_{i\Omega}\sum_{\boldsymbol{q}} G_f(\boldsymbol{k}+\boldsymbol{q},i\omega+i\Omega)\Big(G_b(\boldsymbol{q},i\Omega)+\beta|b|^2\delta^{(d)}(\boldsymbol{q})\delta(\Omega)\Big)$$
$$= \frac{1}{\beta}\sum_{i\Omega}\sum_{\boldsymbol{q}} G_f(\boldsymbol{k}+\boldsymbol{q},i\omega+i\Omega)G_b(\boldsymbol{q},i\Omega) + |b|^2 G_f(\boldsymbol{k},i\omega) \quad (291)$$

$$G_f(\boldsymbol{k},i\omega) = \frac{1}{i\omega+\mu_r+zt\chi_f\gamma_{\boldsymbol{k}}}, \qquad G_b(\boldsymbol{q},i\Omega) = \frac{1}{\frac{\Omega^2}{2u}+\lambda-zt\chi_\theta\gamma_{\boldsymbol{q}}} \quad (292)$$

l. Transport near the spin-liquid Mott critical point

$$\sigma_{el} = \frac{\sigma_f\sigma_b}{\sigma_f+\sigma_b} \quad (293)$$

2. Renormalization group analysis and emergence of localized magnetic moments

a. Emergence of spin-liquid physics in $\kappa-$organic salts: NMR, specific heat, thermal conductivity, ...
b. Crossover from a spin-liquid state at low temperatures to a almost "decoupled" local moment state at high temperatures
c. Effective field theory for a U(1) spin-liquid state
d. Two-patch construction
e. Setup for the dimensional regularization
f. Renormalization group analysis I based on a stable spinon Fermi surface
g. Renormalization group analysis II based on a stable boson dynamics in the vicinity of the Higgs transition
h. Emergence of localized magnetic moments
i. Scaling theory for correlation functions

D. From a DMFT theory to a spin-liquid theory

VIII. DISCUSSION AND CONCLUSION

Hertz-Moriya-Millis theory is a standard theoretical framework for quantum criticality in metals. Within the self-consistent RPA (random phase approximation) analysis, critical order-parameter fluctuations become overdamped, described by Landau damping. As a result, the dynamical critical exponent is enhanced to result in the fact that critical dynamics of order parameter fluctuations is essentially mean-field-like since the critical field theory is above the upper critical dimension. Such mean-field-type critical dynamics does not respect the hyperscaling relation due to the presence of a dangerously irrelevant operator, responsible for the violation of the ω/T scaling behavior of the dynamical susceptibility for critical fluctuations, where ω is frequency and T is temperature. Even if low-energy critical electrons are taken into account fully self-consistently in the RPA level, the mean-field-type scaling theory with a dangerously irrelevant operator remains unchanged essentially except for the fact that the scaling dimension of the dangerously irrelevant operator becomes more positive and thus, more irrelevant.

Since the Yukawa coupling between low-energy electrons and critical order-parameter fluctuations is marginal at the critical point within the self-consistent RPA analysis, such an approximation scheme can be dangerous in the case when the fixed-point value does not reside within the convergence area for the self-consistent RPA analysis. In order to justify this approximation scheme, one may increase spin degeneracy from $\sigma =\uparrow, \downarrow$ to $\sigma = 1, ..., N$. Then, the interaction vertex is reduced from g to $g/\sqrt{N}$, and the self-consistent RPA analysis seems to be justified in the $N \to \infty$ limit, where any vertex corrections give rise to higher order contributions in $\mathcal{O}(1/N)$ and thus, self-energy corrections turn out to be in the leading order, referred to as the $1/N$ expansion.

Recently, S.-S. Lee has shown that the self-consistent RPA analysis cannot be justified even in the $N \to \infty$ limit. He starts from the Hertz-Moriya-Millis fixed point in the self-consistent RPA framework, where critical boson dynamics is described by Landau damping with the dynamical critical exponent $z = 3$ and the dynamics of low-energy critical electrons is given by the following non-Fermi liquid self-energy correction, $\Sigma(i\omega) \sim i(g^2/N)\text{sgn}(\omega)|\omega|^{2/3}$. Performing the scaling analysis which makes the self-consistent RPA critical field theory scale-invariant, he finds two essential aspects: First, the angular part of the momentum integral acquires an anomalous scaling dimension. Second, such an anomalous scaling exponent justifies the double-patch construction as a minimal effective field theory. In particular, the overlapping region between two different patches is shown to vanish in the IR (infrared) limit. Based on this effective critical field theory, S.-S. Lee investigated the stability of the self-consistent RPA fixed point. It turns out that the presence of the $1/N$ factor in the non-Fermi liquid self-energy correction spoils the structure of the $1/N$ expansion. In particular, he suggests the double-line representation for Feynman diagrams, where boson fluctuations are given by double lines and fermion excitations are described by single lines. As a result, he reveals that the number of decoupled fermion loops correspond to the enhancement factor of N, originating from the $1/N$ factor of the self-energy correction and identified with Fermi surface fluctuations. Although this counting rule turns out to break down beyond the single-patch approximation, this study proposes that vertex corrections should be taken into account properly in order to describe critical dynamics of low-energy order-parameter fluctuations and fermion excitations. We do not know how to incorporate such vertex corrections into the quantum criticality of the Hertz-Moriya-Millis fixed point systematically based on the field theoretical approach.

In order to make the physical description of metallic quantum criticality mathematically controllable, theoreticians have tried to find another expansion parameter beyond the $N \to \infty$ limit. Here, we focus on the dimensional regularization technique. The dimensional regularization technique is well known for bosonic quantum criticality. Although its conceptual aspect is completely clear, a concrete manipulation has not been performed for the Fermi-surface problem. S.-S. Lee proposed an interesting field-theoretical setup for the dimensional regularization technique to the Fermi-surface problem. We would like to call it "Graphenization" of the Fermi-surface problem. Maintaining the dimension of a Fermi surface with one dimension, he devises how to put the problem of spatially two-dimensional metallic quantum criticality into $d-$dimensions. Here, d counts only the spatial part. Suppose Ising nematic quantum criticality in two dimensions. In order to maintain the shape of the one-dimensional Fermi surface in three dimensions for example, one should gap out the band structure of electrons along the $z-$dimension. The resulting band structure turns out to describe p_z-wave superconductivity in thee dimensions. Within the dimensional regularization technique, the nematic quantum criticality in two dimensions can be achieved from the band structure of p_z-wave superconductivity in thee dimensions. In this situation the upper critical dimension of the Yukawa coupling between low-energy electrons and critical Ising nematic fluctuations is $d_c = 5/2$. Although it is questionable whether or not we are solving the same problem as the originally suggested one, the physical description is now completely justified at least mathematically, performing the renormalization group analysis in a slightly lower dimension than the upper critical dimension, i.e., $d = d_c - \varepsilon$ with $d_c = 5/2$ and $\varepsilon = 1/2$. The $\varepsilon-$expansion technique can be more justified than before.

The "graphenized" effective field theory of the double patch construction in $d-$dimensions allows an interacting fixed point for the Yukawa coupling constant. Critical dynamics of order-parameter fluctuations and non-Fermi liquid physics of low-energy electrons are described by the renormalization group analysis based on the dimensional regularization technique. Solving Callan-Symanzik equations gives scaling theories for correlation functions, identifying the nature of this novel interacting fixed point. The resulting interacting fixed point differs from the Hertz-Moriya-Millis critical point given by the self-consistent RPA analysis. First of all, the ω/T scaling behavior for the dynamic susceptibility of critical order-parameter fluctuations is expected to hold beyond the Hertz-Motiya-Millis framework. An essential point of the dimensional regularization technique is that the dynamical critical exponent is much less than the value of the self-consistent RPA theory. This is certainly expected due to the presence of pseudogap in the graphenization technique, responsible for the appearance of an interacting fixed point. Frankly speaking, it is not completely clear at all whether or not such an interacting fixed point reflects the nature of the originally proposed metallic quantum critical point. Suppose the Kondo problem. It is well understood that the nature of the quantum critical point between the local moment phase and the local Fermi-liquid state in the pseudogap Kondo model differs from that in a normal metallic host, where such a quantum critical point does not exist in the latter case. However, we reach the same renormalization group equation for the Kondo coupling constant if the pseudogap density-of-states parameter set to vanish and recover the finite density of states as in normal metals.

Here, we adopt the dimensional regularization technique for the renormalization group analysis. As discussed before, we consider an insulator-metal transition from a U(1) spin-liquid state with a spinon Fermi surface to a Fermi-liquid phase, given by the Higgs transition of bosonic charge degrees of freedom referred to as holons. The effective field theory for this spin-liquid Mott quantum criticality is as follows: First, critical spin dynamics is described by spin doublets interacting through low lying spin-singlet fluctuations, where spin doublets form a Fermi surface of spinons and low lying spin-singlets are expressed by U(1) gauge fluctuations. This Fermi-surface problem is exactly the same as that solved before by the dimensional regularization technique. Second, critical charge dynamics is described by sound modes interacting via low lying spin-singlet fluctuations, where sound modes are given by bosonic holons with the relativistic spectrum. This critical charge dynamics has never been taken into account on equal footing with the Fermi surface problem in a controllable way.

We start from the U(1) spin-liquid interacting fixed point as intensively discussed above, which occurs from the dimensional regularization technique for the sector of the spinon-gauge field problem, essentially the same as the Ising nematic quantum criticality problem. The appearance of such an interacting fixed point is based on the assumption for the stability of the spinon Fermi surface. It turns out that this spin-liquid fixed point is too stable to flow toward another fixed point. Since it is a fixed point for critical spinon dynamics, one may investigate the stability of such a fixed point, introducing the role of critical charge fluctuations into the spin-liquid fixed point. The self-interaction constant for the holon dynamics is relevant at the spin-liquid fixed point, where the upper critical dimension is $d_c = 3$ as expected. However, the upper critical dimension of the Yukawa coupling constant (the gauge charge) is 5/2, and thus, the expected screening effect for the holon self-interaction term does not arise at such a fractional dimension within the dimensional regularization scheme. As a result, the relevant four-boson interaction constant increases indefinitely, interpreted as gapping out from the spectrum. Critical spin dynamics with gapped charge fluctuations is identified with the U(1) spin-liquid interacting fixed point as the spinon Fermi-surface problem without introducing charge fluctuations.

Nonrenormalization for the charge dynamics in the dimensional regularization technique leads us to consider another fixed point as our starting point. Since the spin-liquid to Fermi-liquid insulator-metal quantum phase transition is described by the Higgs condensation transition in the holon dynamics, it may be natural to keep the boson dynamics at spin-liquid Mott quantum criticality. In other words, the relativistic holon spectrum in both $x-$ and $y-$ directions is assumed to be our starting fixed point of scale invariance. Then, we find that the spinon Fermi surface cannot be stabilized at this boson fixed point, where the curvature part of the spinon spectrum becomes irrelevant and the spinon dispersion shows the one-dimensional relativistic spectrum. The spinon dynamics remains itinerant along the direction of the Fermi velocity while spinons become localized along the direction of the Fermi surface. Critical spinon dynamics at UV (ultraviolet) is given by Luttinger liquid theory. The effective field theory is as follows: First, critical spinons are described by the one-dimensional Dirac spectrum, coupled with U(1) gauge fluctuations. Second, critical holons are described by the two-dimensional relativistic spectrum with their self interactions, coupled to U(1) gauge fluctuations. This critical field theory shows an emergent enhanced symmetry than that of the U(1) spin-liquid fixed point, that is, the conformal symmetry at UV beyond the U(1) spin-liquid fixed point.

We emphasize that the rotational symmetry does not break down although the spectrum is localized along one direction. We recall that the effective field theory is represented in the double-patch construction. The double-patched effective field theory should be taken into account for all angles of the Fermi surface, where other double-patched effective field theories do not communicate with each other as discussed before, thus regarded to be independent. As a result, the rotational symmetry is preserved.

Now, it is straightforward to apply the dimensional regularization technique for the renormalization group analysis to this conformal field theory. The one-dimensional spinon Fermi-surface with a flat band along the direction of the

Fermi surface remains unchanged in the dimensional regularization scheme. An essential point is that the upper critical dimension of the Yukawa coupling constant, i.e., the gauge charge is the same as the self-interaction coupling constant, given by $d_c = 3$. As a result, not only the gauge coupling constant but also the self-interaction coupling constant is screened to show an interacting fixed point at IR, certainly beyond the U(1) spin-liquid fixed point discussed before. More importantly, the conformal symmetry does not allow the emergence of the Landau damping term in the dynamics of U(1) gauge fluctuations. As a result, the critical spinon dynamics becomes completely localized even along the direction of the Fermi velocity. Localized magnetic moments emerge from itinerant fermions, completely nonperturbative and thus, not allowed usually in the perturbative or diagrammatic approach. Critical charge dynamics is also interesting and unexpected. Although the self-interaction constant of holons becomes reduced by screening, the relevant scaling in two dimensions at the tree level is expected to give a finite fixed point value for the self-interaction constant. However, it turns out that the self-interaction constant vanishes completely. It does not play any role in critical charge dynamics. On the other hand, the gauge coupling constant remains to be finite at IR, suggesting a novel IXY (inverted XY in the sense of duality) universality class, since the self-interaction constant vanishes.

The emergence of localized magnetic moments from itinerant fermions has been discussed recently in the critical field theory of fermions and order-parameter fluctuations with their Yukawa coupling interactions. An important assumption in this renormalization group analysis is that the Landau damping term does not arise due to a certain reason, not clarified in these previous studies. Although our renormalization group analysis gives essentially the same result of the emergence of localized magnetic moments, the present field theoretical construction serves more transparent physical mechanism for the absence of the Landau damping term and the emergent conformal symmetry plays an important role in the localization phenomenon.

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