Quantum Renormalization Group and Holography

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References :
arXiv : 1305.3908
arXiv : 1603.08509

AdS/CFT correspondence

[Maldacena]

- **Conjecture :**

  D-dim QFT = (D+1)-dim quantum gravity

  - The bulk space is emergent, and the geometry is dynamical
  - The bulk geometries encode states of the quantum field theory
  - Well tested for some supersymmetric field theories
    - N=4 Supersymmetric SU(N_c) gauge theory = IIB superstring theory in AdS^5xS^5

AdS/CFT Dictionary

[Gubser, Klebanov, Polyakov; Witten]

Low energy

(D+1)-dim space

$$O_n(x) \leftrightarrow j_n(x, z)$$

$$T_{\mu\nu}(x) \leftrightarrow g_{\mu\nu}(x, z)$$

z

High energy

x

D –dim flat space

$$\int D\phi(x) e^{iS_D[\phi(x)] + i\int J_n(x) O_n} = \int Dj(x,z) e^{iS_{D+1}[j(x,z)]}\Big|_{j_n(x,z=0)=J_n(x)}$$

$$\rightarrow e^{iS_{D+1}[\bar{j}(x,z)]}\Big|_{\bar{j}_n(x,z=0)=J_n(x)}$$

Goal

A first principle derivation of AdS/CFT correspondence, which allows one to find holographic duals for general QFTs*

*For general QFTs, holographic duals can be non-classical / non-local. However, we would like to find a general prescription to construct them.

What is behind the AdS/CFT correspondence? RG ≈ GR

- Radial direction in the bulk = length scale of QFT
- Bulk variables : scale dependent coupling functions
- Equations of motion in the bulk corresponds to the beta functions of QFT
- Radial evolution of the bulk fields correspond to the RG flow

The connection between RG and GR is incomplete

RG	GR
RG flow is classical : Given initial condition, coupling functions are deterministic without uncertainty	Bulk variables have quantum fluctuations

*In order to make the connection precise, RG should be promoted to quantum RG

Other related approaches

General Connection between holography and RG

- E. T. Akhmedov, Phys. Lett. B 442 (1998) 152
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- E. Mintun and J. Polchinski, arXiv:1411.3151
- …

Plan

- An introduction to quantum RG
  - RG flow as a wavefunction collapse
- An application of quantum RG
  - Vector model
  - Matrix model

From action to state

$$|S\rangle = \int D\phi \; e^{-S[\phi]} |\phi\rangle,$$

$$\langle \phi^{'} | \phi \rangle = \prod_i \delta(\phi_i^{'} - \phi_i)$$

- An action of QFT in D-dimensional space defines a D-dimensional quantum state
- The Boltzmann weight becomes wavefunction

Sources as variational parameters

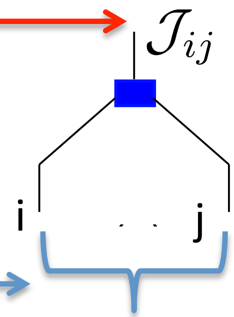
$$S = -\mathcal{J}^M \mathcal{O}_M$$

$$\big|\{\mathcal{J}\}\big\rangle = \int D\phi \; e^{\mathcal{J}^M \mathcal{O}_M} \big|\phi\big\rangle$$

- State can be labeled by the sources of operators

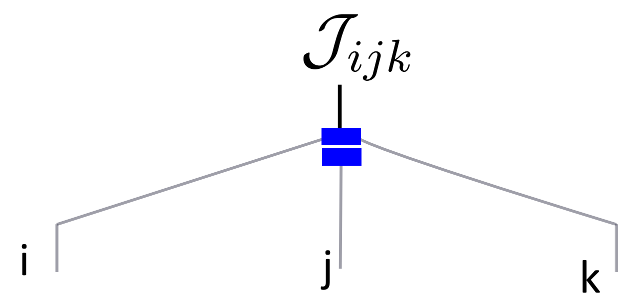
Tensor representation

$$e^{\mathcal{J}_{ij}\phi_i\phi_j}$$



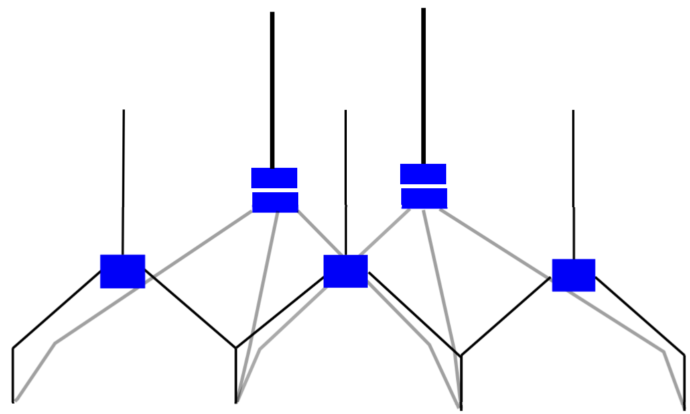
In general, O_M depends on multiple points in spacetime (e.g. bi-local operator in vector model, Wilson loop in gauge theory)

$$e^{\mathcal{J}_{ijk}(\phi_i\phi_j)(\phi_j\phi_k)}$$



O_M can be composite of multiple operators

Tensor representation



$$\left|\{\mathcal{J}\}\right\rangle = \int D\phi \; e^{\mathcal{J}^M \mathcal{O}_M} \left|\phi\right\rangle$$

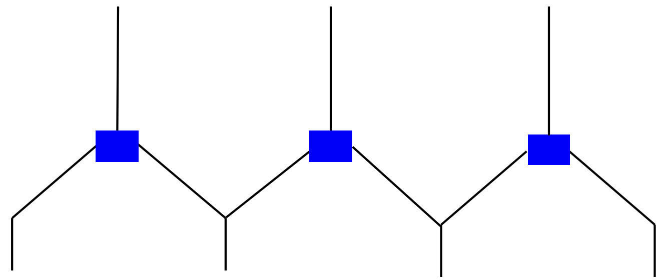
- Local action generates states given by a product of local tensors
- They are over-complete

Single-trace operator

$$\mathcal{O}_M = \sum c_M^{n_1,n_2,\cdots} O_{n_1} O_{n_2} \cdots$$

- Minimal set of operators of which all singlet operators can be written as polynomial

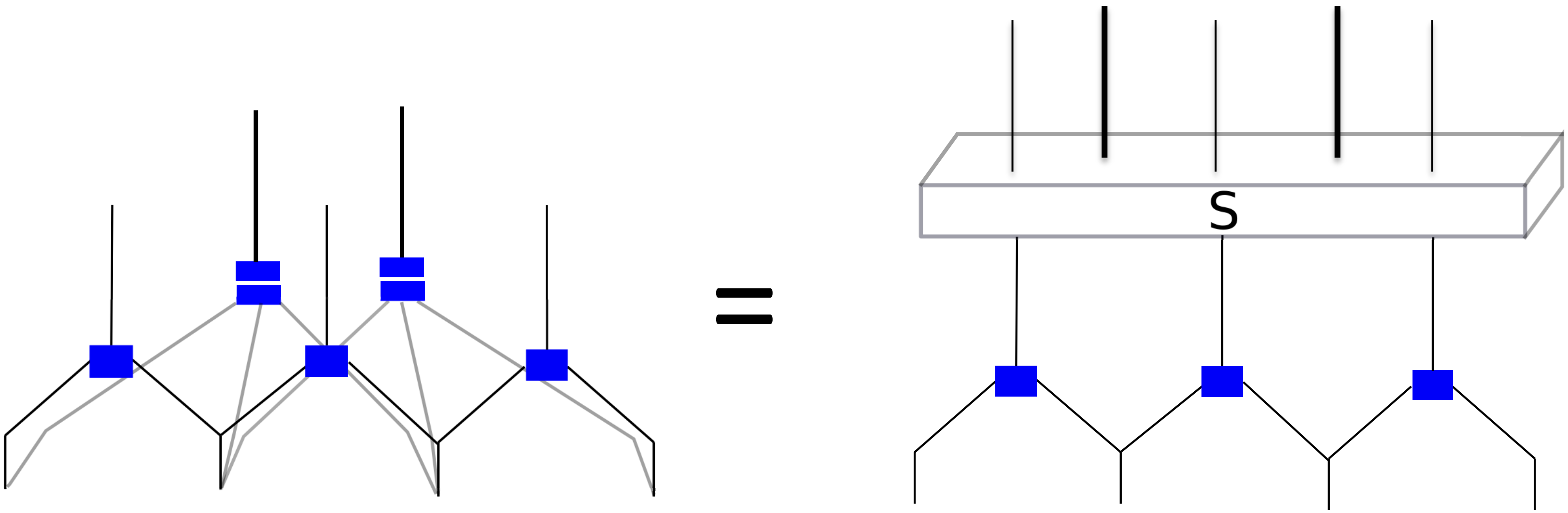
States generated from single-trace operators form a complete basis



$$|j\rangle = \int D\phi \; e^{j_n O_n} |\phi\rangle$$

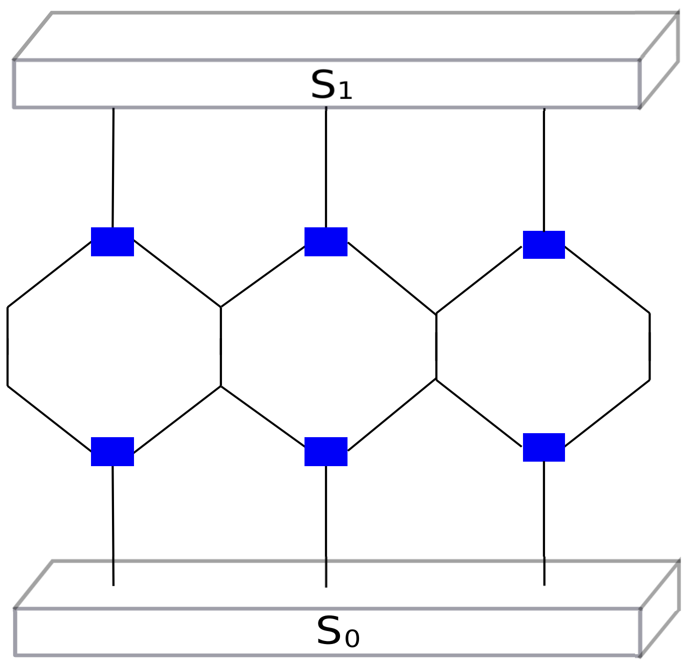
States generated from single-trace operators form a complete basis

$$\int D\phi \; e^{\sum_k \mathcal{J}^{n_1,n_2,..,n_k} O_{n_1} O_{n_2} .. O_{n_k}} \; |\phi\rangle = \int Dj \; \boxed{\Psi_S(\mathcal{J}, j)} \; |j\rangle$$



Partition function is an overlap between states

$$Z = \int D\phi \; e^{-(S_0+S_1)} = \langle S_0^* | S_1 \rangle$$



$$|S_0\rangle = \int D\phi \; e^{-S_0[\phi]} |\phi\rangle$$

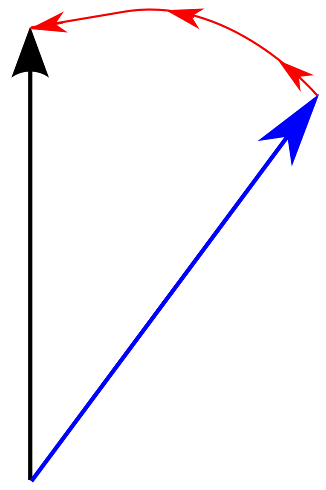
$$|S_1\rangle = \int D\phi \; e^{-S_1[\phi]} |\phi\rangle$$

RG flow as wave-function collapse

$$Z = \langle S_0 | S_1 \rangle = \langle S_0 | e^{-dz\hat{H}} | S_1 \rangle = \langle S_0 | S_1 + \delta S_1 \rangle$$

- $|S_0>$ is the ground state of H^+ with zero energy
- H acting on $|S_1>$ generates RG flow

$$Z = \langle S_0 | e^{-z\hat{H}} | S_1 \rangle$$



Example : Wilson-Polchinski RG equation

$$S_0 = \frac{1}{2}\int d^D k \; G_\Lambda^{-1}(k)\phi_k\phi_{-k}$$

$$S_1 = \text{ interactions}$$

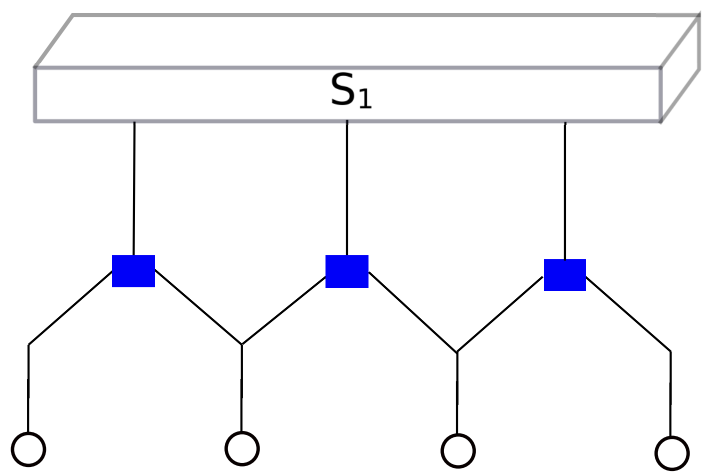
$$e^{-(S_1+\delta S_1)} = \left\langle \phi \middle| e^{-dz\hat{H}} \middle| S_1 \right\rangle$$

$$\hat{H} = \int dk \left[\frac{\tilde{G}(k)}{2}\hat{\pi}_k\hat{\pi}_{-k} - i\left(\frac{D+2}{2}\hat{\phi}_k + k\partial_k\hat{\phi}_k\right)\hat{\pi}_{-k}\right]$$

$$\tilde{G}(k) = \frac{\partial G_\Lambda(k)}{\partial \ln \Lambda}$$

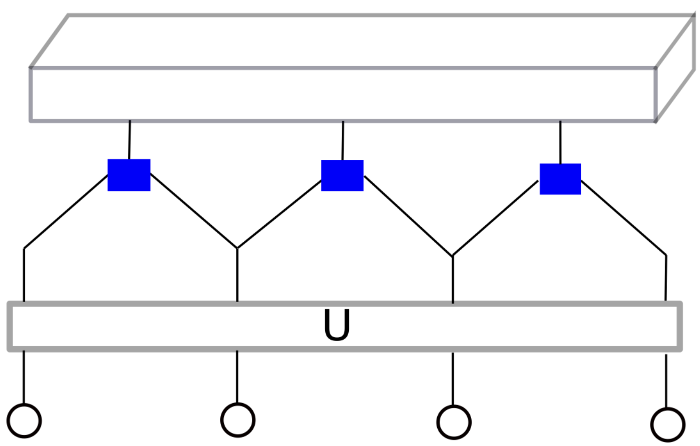
Direct product state for the reference state (tentative IR fixed point)

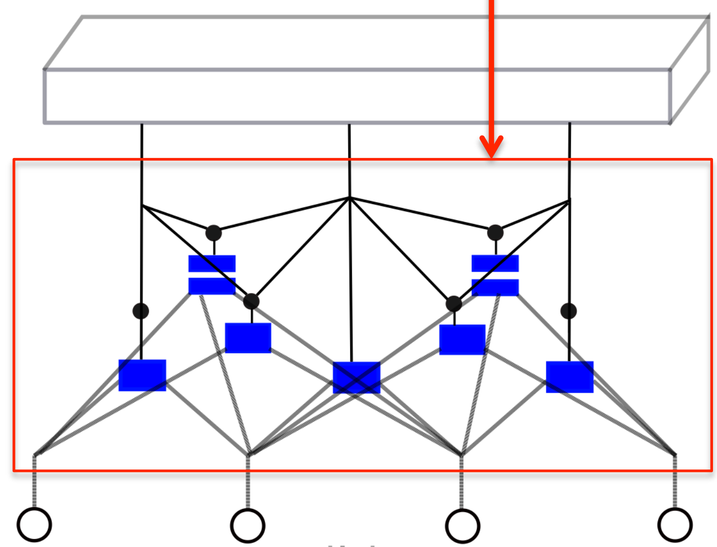
$$Z = \int Dj^{(0)} \left\langle S_0^* \middle| j^{(0)} \right\rangle \Psi(j^{(0)})$$



Coarse graining

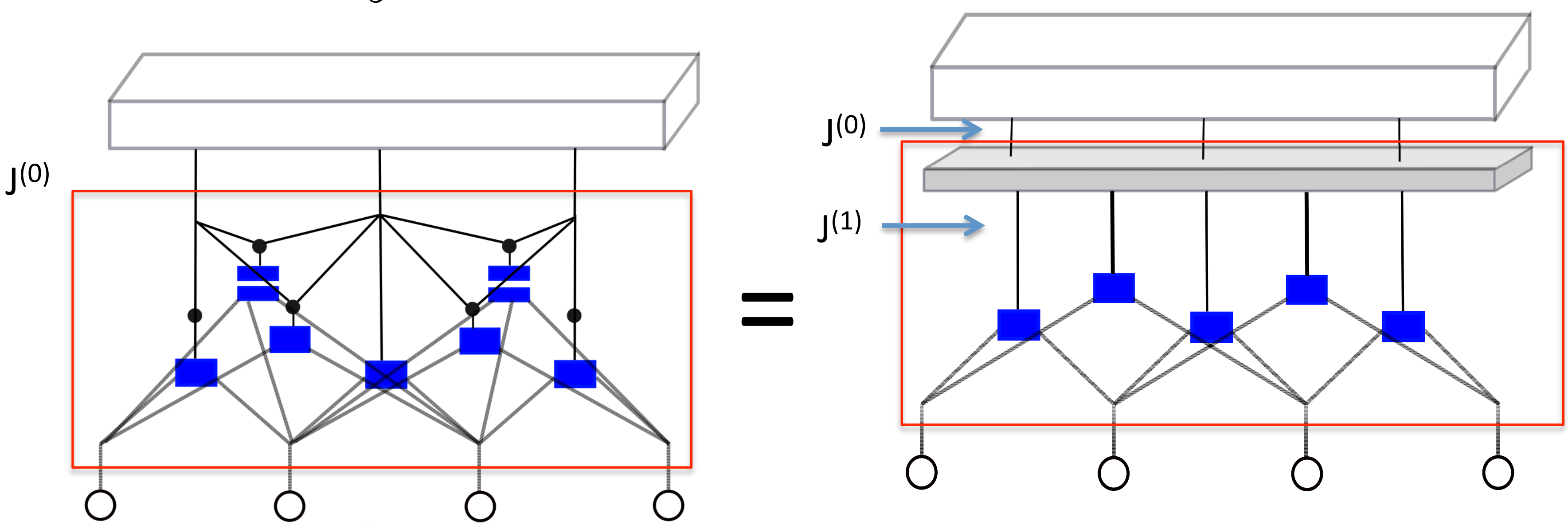
$$Z = \int Dj^{(0)} \; \langle S_0^* | e^{-dz\hat{H}} | j^{(0)} \rangle \Psi(j^{(0)})$$



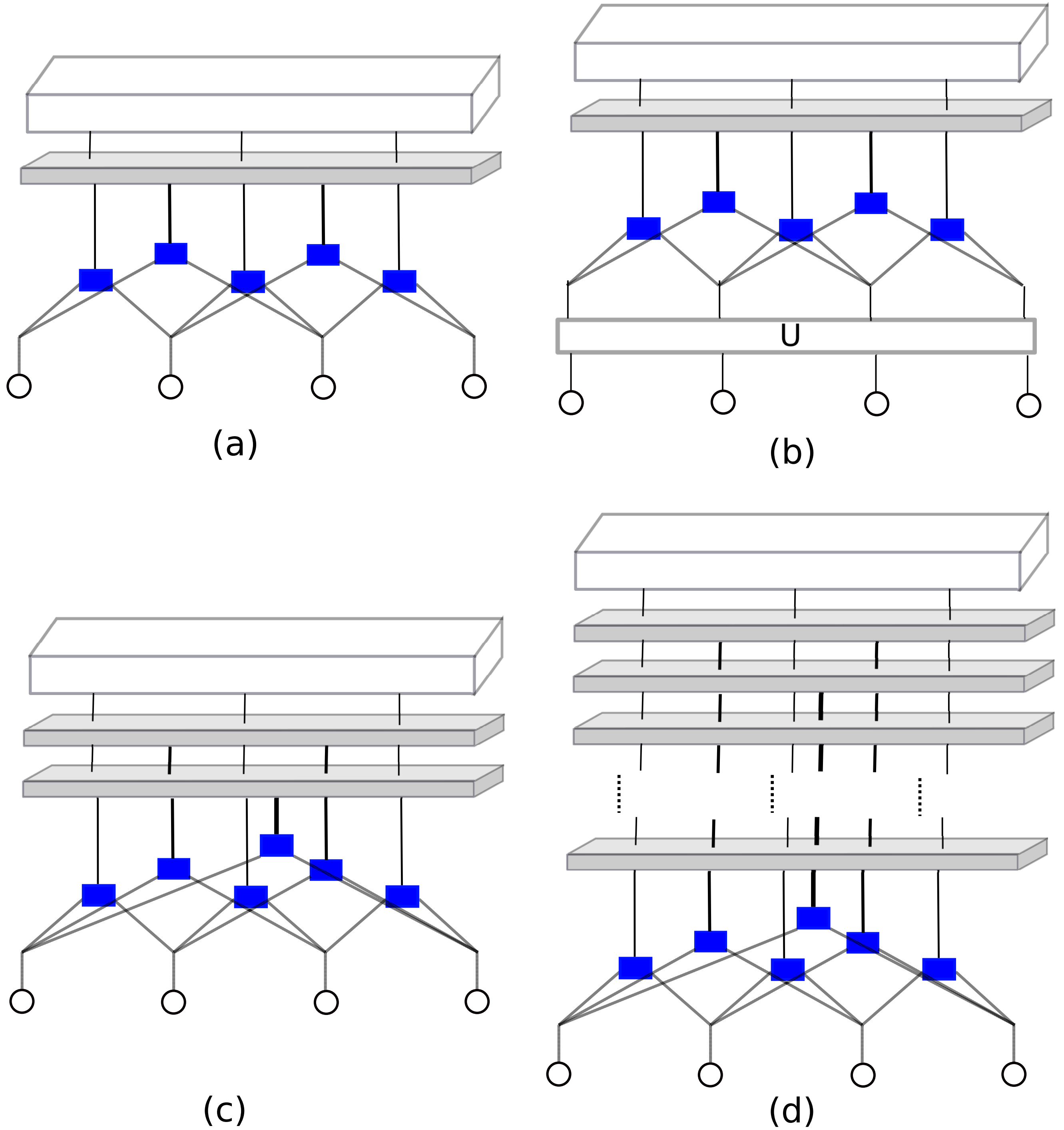


Quantum RG

$$e^{-dz\hat{H}}\left|j^{(0)}\right\rangle = \int Dj^{(1)}\; e^{-j_n^{*(1)}\left(j^{(1)n}-j^{(0)n}\right)-dz\mathcal{H}[j^{*(1)},j^{(0)}]}\left|j^{(1)}\right\rangle$$



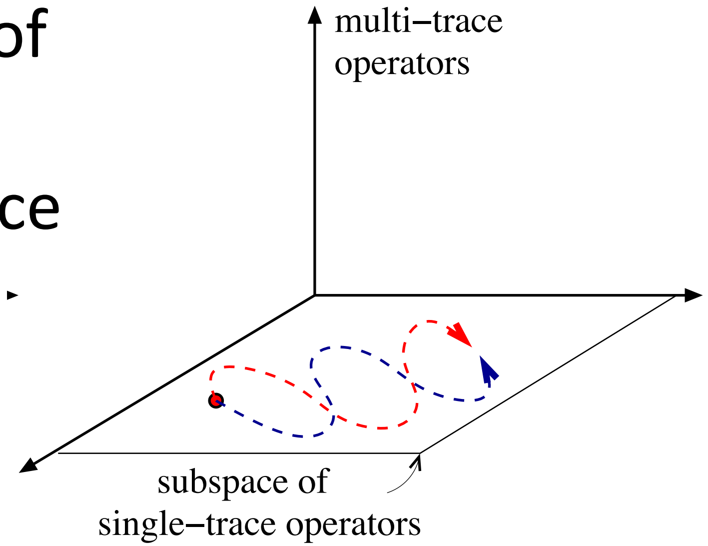
- State with multi-trace tensors can be written as a linear superposition of single-trace states
- Non-local single-trace tensors are generated


U
(a)
(b)
(c)
(d)

Quantum RG

$$Z = \int Dj^{'} Dj Dj(z) \ \ \Psi_0^*(j^{'}) \ e^{-\int dz (j^* \partial_z j + \mathcal{H}[j^*, j])} \Psi_1(j) \Big|_{j(0)=j, j(z)=j'}$$

- The RG flow is confined to the space of single-trace sources
- Sum over all RG path in the single-trace space
- Single-trace sources are promoted to quantum operators $[j^n, j_m^\dagger] = \delta_m^n$
- Quantum RG to Wilsonian RG is what quantum computer is to classical computer



Further comments

- The bulk tensor network involves single-trace tensors of all sizes (no pre-assigned local structure) : kinematic non-locality is a necessary condition for diffeomorphism invariance in the bulk
- The bulk theory include dynamical gravity : the source for single-trace energy momentum tensor (metric) gets promoted to dynamical variables
- Regularization of quantum gravity boils down to regularization of QFT

Example 1 : Lattice Vector Model

Example : Vector model

$$\mathcal{S} = \int d^D x \left[|\nabla \vec{\phi}|^2 + m^2 |\vec{\phi}|^2 + \frac{\lambda}{N} (|\vec{\phi}|^2)^2 \right]$$

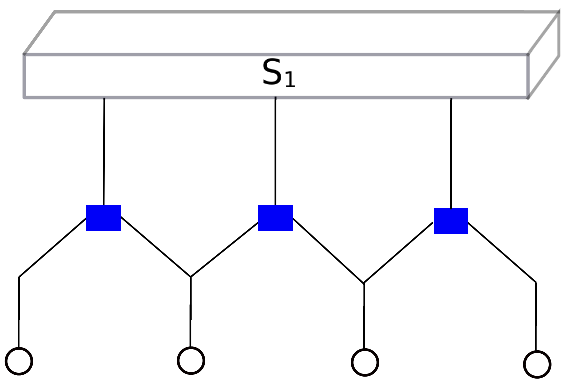
Lattice Regularization :

$$S_0 \; = \; m^2 \sum_i (\boldsymbol{\phi}_i^* \cdot \boldsymbol{\phi}_i)$$

$$S_1 \; = \; -\sum_{ij} t_{ij}^{(0)} \left(\boldsymbol{\phi}_i^* \cdot \boldsymbol{\phi}_j \right) + \frac{\lambda}{N} \sum_i (\boldsymbol{\phi}_i^* \cdot \boldsymbol{\phi}_i)^2$$

Example : Vector model

$$Z = \langle S_0 | t^{(0)} \rangle$$



Gapped phase (direct product state)

$$|S_0\rangle \; = \; \int D\boldsymbol{\phi}\; e^{-m^2 \sum_i \boldsymbol{\phi}_i^* \cdot \boldsymbol{\phi}_i} |\boldsymbol{\phi}\rangle,$$

Deformation to the gapped fixed point (entangled state)

$$|t^{(0)}\rangle \; = \; \int D\boldsymbol{\phi}\; e^{\; \sum_{ij} t_{ij}^{(0)} \boldsymbol{\phi}_i^* \cdot \boldsymbol{\phi}_j - \frac{\lambda}{N} \sum_i (\boldsymbol{\phi}_i^* \cdot \boldsymbol{\phi}_i)^2} |\boldsymbol{\phi}\rangle$$

Hamiltonian

$$\hat{H} = \sum_i \left[\frac{2}{m^2} \boldsymbol{\pi}_i \cdot \boldsymbol{\pi}_i^* + i(\boldsymbol{\phi}_i \cdot \boldsymbol{\pi}_i + \boldsymbol{\phi}_i^* \cdot \boldsymbol{\pi}_i^*) \right]$$

- H is not Hermitian, but has real eigenvalues (related to Hermitian through a similarity transformation)
- $|S_0\rangle$ is the ground state of H^+
- e^{-zH} gradually removes entanglement* in $|t^{(0)}\rangle$

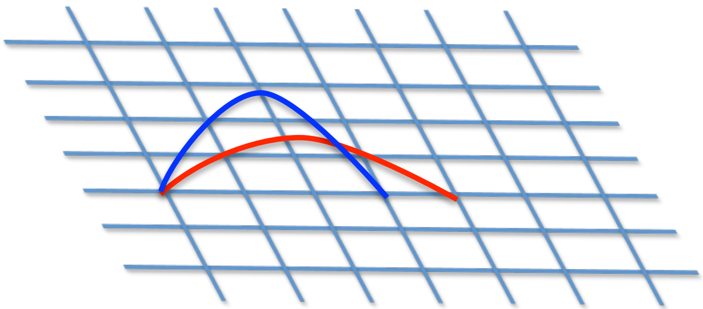
* Entanglement in spacetime

Bulk Hamiltonian (in a fixed gauge)

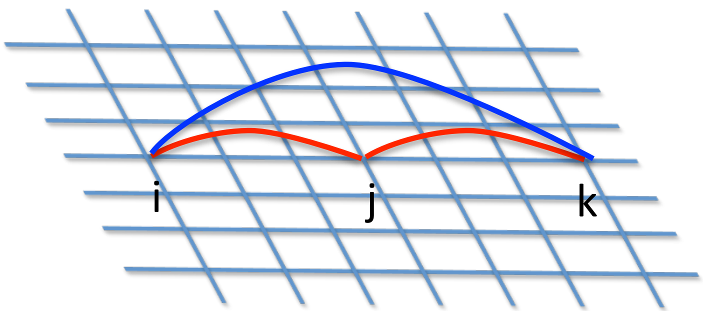
$$\hat{\mathcal{H}} = \sum_i \left[-\frac{2}{m^2} t_{ii} + \frac{4\lambda\left(1+\frac{1}{N}\right)}{m^2} t_{ii}^\dagger - 4\lambda \left(t_{ii}^\dagger\right)^2 - \frac{8\lambda^2}{m^2}\left(t_{ii}^\dagger\right)^3 \right] + \sum_{ij} \left[2 + \frac{4\lambda}{m^2}(t_{ii}^\dagger + t_{jj}^\dagger) \right] t_{ij}^\dagger \; t_{ij} - \frac{2}{m^2} \sum_{ijk} \left[t_{kj}^\dagger t_{ki} t_{ij} \right]$$

- t^{+}_{ij} (t_{ij}) creates (annihilates) a quantum of connectivity
- The Hamiltonian describes evolution of quantum geometry in the bulk

Background independence



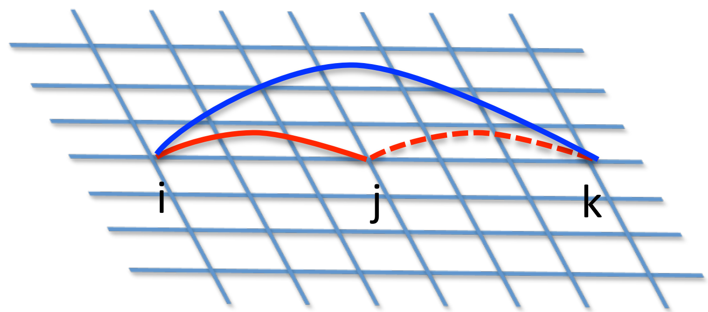
$$t^{\dagger}_{ik}t_{ij}$$



$$t^{\dagger}_{ik}t_{ij}t_{jk}$$

- There is no bare kinetic term for the bi-local object
- No pre-imposed background

Background independence



$$t^{\dagger}_{ik} t_{ij} t_{jk} \rightarrow t^{\dagger}_{ik} t_{ij} < t_{jk} >$$

- t_{ij} can move only in the presence of condensate
- The condensate, which is dynamical, determines the geometry on which t_{ij} propagates

Saddle point approximation

- In the large N limit, semi-classical RG path dominates the partition function
- At the saddle point, $t_{ij} \to \bar{t}_{ij}, \quad t^*_{ij} \to \bar{p}_{ij}$

$$\begin{aligned}
\partial_z \bar{t}_{ij} \quad &= -2\Big\{ \tfrac{2\lambda\,\delta_{ij}}{m^2} - \delta_{ij}\left[4\lambda + \tfrac{12\lambda^2}{m^2}\bar{p}_{ii}\right]\bar{p}_{ii} + \tfrac{2\lambda\,\delta_{ij}}{m^2}\textstyle\sum_k \left(\bar{t}_{ik}\bar{p}_{ik} + \bar{t}_{ki}\bar{p}_{ki}\right) \\
&\qquad + \left[1 + \tfrac{2\lambda}{m^2}\left(\bar{p}_{ii} + \bar{p}_{jj}\right)\right]\bar{t}_{ij} - \tfrac{1}{m^2}\textstyle\sum_k \bar{t}_{ik}\bar{t}_{kj} \Big\}, \\
\partial_z \bar{p}_{ij} \quad &= 2\left\{ -\tfrac{\delta_{ij}}{m^2} + \left[1 + \tfrac{2\lambda}{m^2}\left(\bar{p}_{ii} + \bar{p}_{jj}\right)\right]\bar{p}_{ij} - \tfrac{1}{m^2}\textstyle\sum_k \left(\bar{p}_{ik}\bar{t}_{jk} + \bar{t}_{ki}\bar{p}_{kj}\right) \right\}
\end{aligned}$$

Exact solution :

$$\begin{aligned}
\bar{T}_q(z) \;&=\; \frac{2\lambda}{m^2} + m^2 + \frac{2\lambda}{m^2}e^{-2z}(m^2\bar{p}_0(0) - 1) - m^2\frac{\delta^2 + q^2}{(1 - e^{-2z})(q^2 + \delta^2) + m^2 e^{-2z}}, \\
\bar{P}_q(z) \;&=\; \frac{e^{-2z}}{q^2 + \delta^2} + \frac{1 - e^{-2z}}{m^2}
\end{aligned}$$

Metric

- Fluctuations away from saddle point

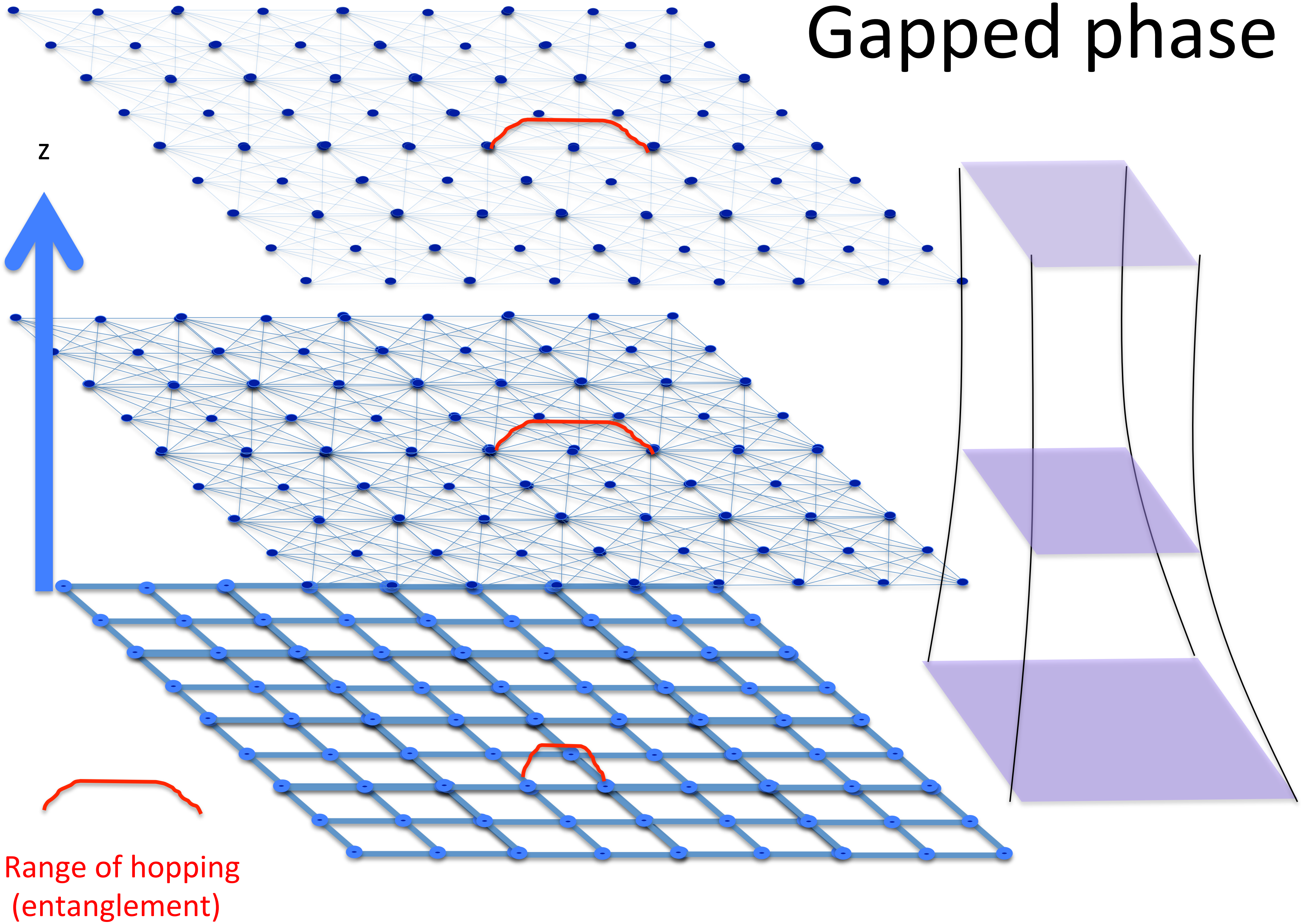
$$\tilde{t}_{ij} = t_{ij} - \bar{t}_{ij}$$

- Anti-symmetric component obeys a simple diffusive equation in the bulk

$$\tilde{t}^A_{ij} = \tilde{t}_{ij} - \tilde{t}_{ji}$$

$$\left(m\sqrt{g^{zz}}\partial_z - g^{\mu\nu}\partial_\mu\partial_\nu - g^{\mu\nu}\partial^{'}_\mu\partial^{'}_\nu + \ldots\right)\tilde{t}^A(x, x^{'}, z) = 0$$

Gapped phase



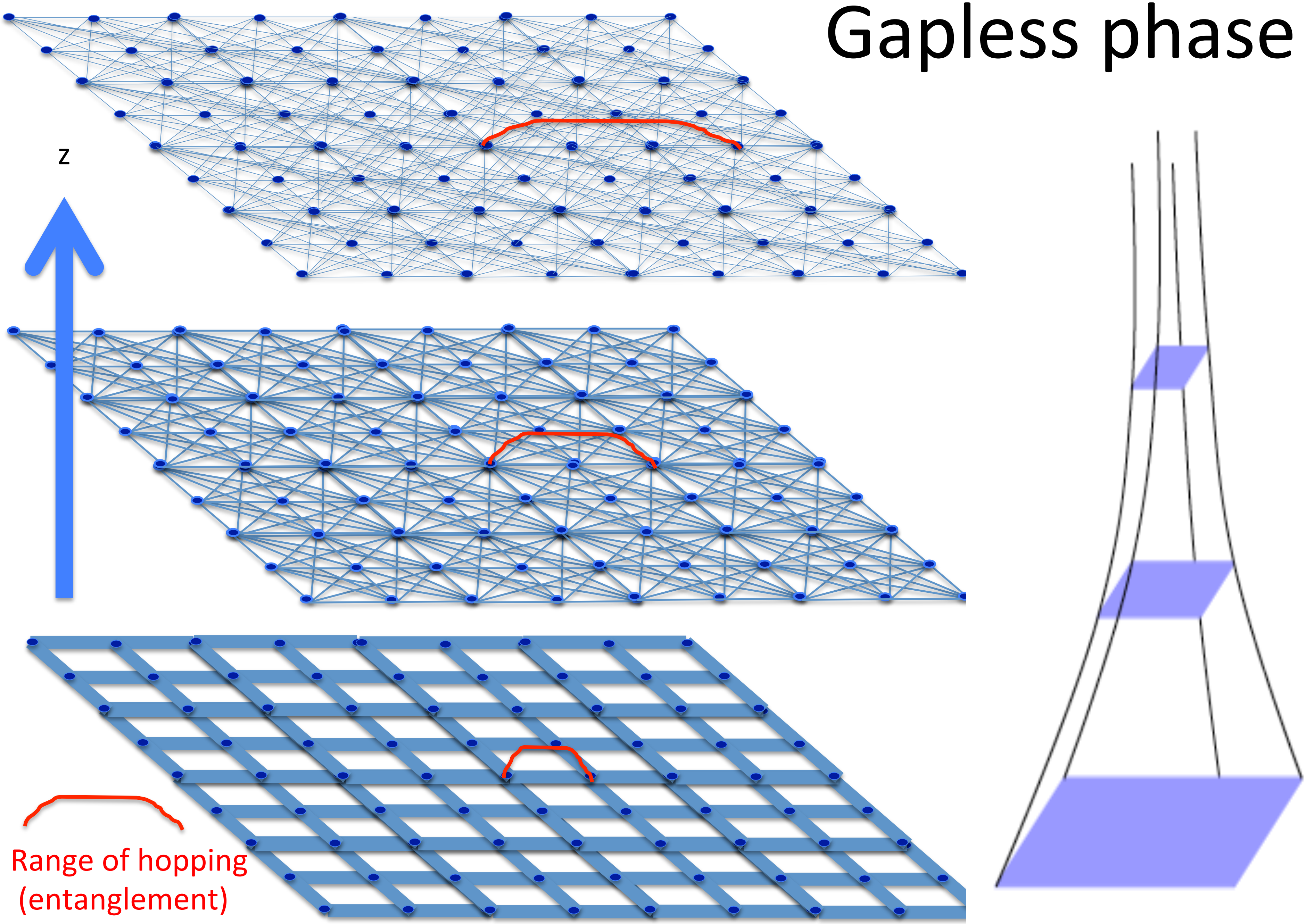
Gapped phase

- The range of entanglement (hopping) saturates in the large z limit
- The strength of hopping (entanglement) decays exponentially in z
- $e^{-z H} |S_1>$ is smoothly projected to the direct product state in the large z limit
- The bulk terminates at a finite proper distance
- The proper distance measures the complexity : # of RG steps needed to remove all entanglement

[Susskind]

$$ds^2 = \left(\frac{1}{1 + \left(\frac{\delta}{m} e^z\right)^2} \right)^2 \frac{dz^2}{m^2} + \left(\left(\frac{\delta}{m}\right)^2 + e^{-2z} \right) \sum_{\mu=0}^{D-1} dx^\mu dx^\mu .$$

Gapless phase



Gapless phase

- The range of entanglement (hopping) keep increasing with increasing z
- $e^{-z H} |S_1>$ can not be smoothly projected to the direct product state in the large z limit
- In the large z limit, the range of entanglement diverges : critical point -> Poincare horizon

$$ds^2 = \frac{dz^2}{m^2} + e^{-2z} \sum_{\mu=0}^{D-1} dx^\mu dx^\mu$$

- In metallic phase, horizon arises at finite z

[Q. Hu, SL, to appear]

Example 2 : A toy example

Matrix field theory which has no other operators with finite scaling dimension except for the energy-momentum tensor

D-dim matrix QFT on a curved background

$$Z[g^{(0)}] = \int D\Phi \quad e^{iS_1[\Phi; g^{(0)}(x)]}$$

- S_1 is an action which has only single-trace operators deformed by energy-momentum tensor
- This is equivalent to putting the theory on a curved background metric
- We assume that the theory is regularized respecting the D-dim. Diffeomorphism invariance

$$Z[g^{(0)}] = Z[g^{(0)\prime}]$$

Coarse graining

$$g^{(0)}_{\mu\nu}(x) \to g^{(0)}_{\mu\nu}(x) e^{-N^D(x)dz}$$

$$Z[g^{(0)}] = \int D\Phi \;\; e^{iS_1[\Phi; g^{(0)}(x)] + i\delta S'[T^{\mu\nu}; g^{(0)}]}$$

spacetime dependent speed of RG
[Osborn(94); SL(12)]

$$T^{\mu\nu} = \frac{1}{N^2}\frac{\delta S_1}{\delta g^{(0)}_{\mu\nu}}$$

$$\delta S'[T^{\mu\nu}; g^{(0)\mu\nu}] = dz N^2 \int d^D x \; N^D(x) \Big\{ \sqrt{|g^{(0)}|} \left(-C_0 + C_1^D \mathcal{R}(x; g^{(0)}]\right) - A_{\mu\nu} T^{\mu\nu} + \frac{B_{\mu\nu;\rho\sigma}}{2} T^{\mu\nu} T^{\rho\sigma} + .. \Big\}$$

Change of scale :
Warping factor

Double-trace operators

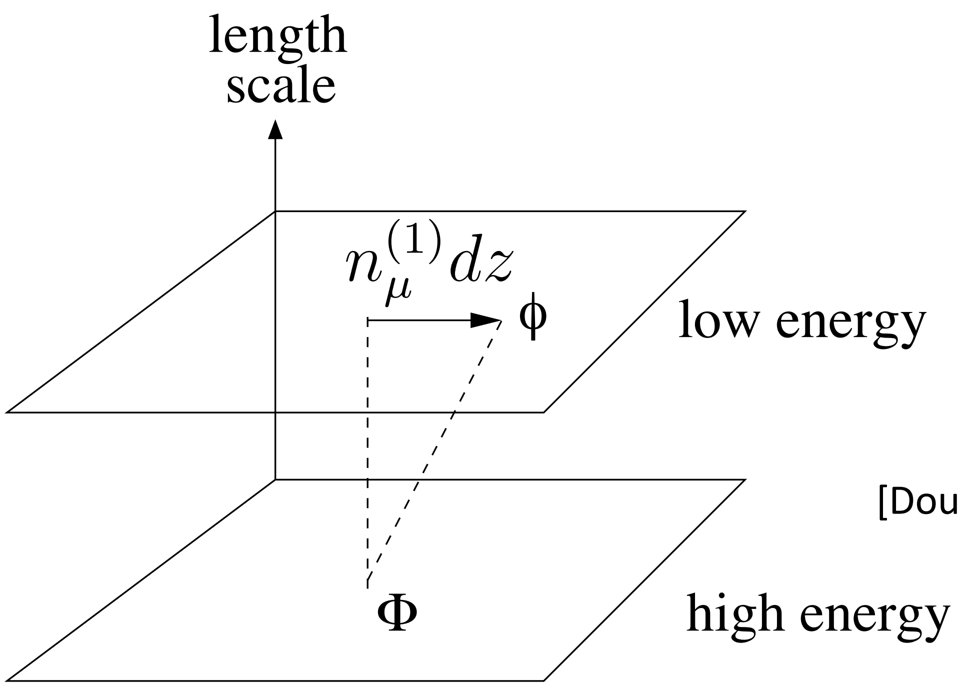
Higher derivative terms

Casimir energy
[Sakharov(67)]

Shift

$$Z[g^{(0)}] = \int D\Phi \;\; e^{iS_1[\Phi;g^{(0)}(x)]+i\delta S'[T^{\mu\nu};g^{(0)}]+i\delta S''[T^{\mu\nu};g^{(0)}]}$$

$$\delta S''[T^{\mu\nu}; g^{(0)\mu\nu}] \;=\; dzN^2 \int d^D x(\nabla^{(1)}_\mu n^{(1)}_\nu + \nabla^{(1)}_\nu n^{(1)}_\mu)T^{\mu\nu}$$



Shift of the coordinate of the low energy field relative to the coordinate of the high energy field

[Douglas, Mazzucato, and Razamat (11); SL (12)]

Auxiliary fields

$$Z[g^{(0)}] = \int Dg^{(1)}_{\mu\nu} D\pi^{(1)\mu\nu} D\Phi \quad e^{iN^2 \int d^D x \ \pi^{(1)\mu\nu}(g^{(1)}_{\mu\nu} - g^{(0)}_{\mu\nu})}$$

$$e^{i\delta S^{(1)'}[i/N^2 \delta/\delta g^{(1)}_{\mu\nu}; g^{(0)}]} e^{i\delta S^{(1)''}[i/N^2 \delta/\delta g^{(1)}_{\mu\nu}]} e^{iS_1[\Phi; g^{(1)}]}$$

- $T^{\mu\nu} = -i \dfrac{1}{N^2} \dfrac{\delta}{\delta g^{(1)}_{\mu\nu}}$
- $\pi^{(1)\mu\nu}$: Lagrangian multiplier
- Integration of $g^{(1)}_{\mu\nu}$ by parts : $\dfrac{\delta}{\delta g^{(1)}_{\mu\nu}} \rightarrow -i\pi^{(1)\mu\nu}$

Double trace operator : dynamical metric

$$Z[g^{(0)}] = \int Dg^{(1)}_{\mu\nu} D\pi^{(1)\mu\nu} D\Phi \;\; e^{iN^2 \int d^D x \;\; \pi^{(1)\mu\nu}(g^{(1)}_{\mu\nu} - g^{(0)}_{\mu\nu})}$$
$$\times e^{i\delta S'[\pi^{(1)\mu\nu}, g^{(0)}] + i\delta S''[\pi^{(1)\mu\nu}, g^{(0)}]} e^{iS_1[\Phi; g^{(1)}]}$$

$$\delta S' = dz N^2 \int d^D x \; N^D(x) \Big\{ \sqrt{|g^{(0)}|} \left(-C_0 + C^D_1 \mathcal{R}(x; g^{(0)}]\right)$$
$$+ A_{\mu\nu}\pi^{(1)\mu\nu} + \frac{B_{\mu\nu;\rho\sigma}}{2}\pi^{(1)\mu\nu}\pi^{(1)\rho\sigma} + ..\Big\}$$
$$\delta S'' = dz - N^2 \int d^D x (\nabla^{(1)}_\mu n^{(1)}_\nu + \nabla^{(1)}_\nu n^{(1)}_\mu)\pi^{(1)\mu\nu}$$

- Quadratic term in $\pi^{(1)\mu\nu}$ provides a Gaussian width for $g^{(1)}_{\mu\nu}$, which becomes a genuine fluctuating metric

Bulk action

$$S_{D+1} = \frac{N^2}{2\kappa^2} \int dz \int d^D x \left[\pi_{\mu\nu}\partial_z g^{\mu\nu} - N^D \mathcal{H} - N^\mu \mathcal{H}_\mu\right]$$

Casimir energy

Beta function of $T^{\mu\nu}T^{\rho\sigma}$

$$\mathcal{H} = -\sqrt{g}\left[C_0 + R^D + \frac{g^{-1}}{2}\left(\alpha\pi^2 - \pi^{\mu\nu}\pi_{\mu\nu}\right) + ..\right]$$

$$\mathcal{H}^\mu = -2\nabla_\nu \pi^{\mu\nu}$$

Not fixed by D-dimensional diff. inv.

- The linear term in $\pi^{\mu\nu}$ can be absorbed by a shift in $\pi^{\mu\nu}$ and a boundary term

First-class constraints

- Independence of partition function on RG schemes (speed of RG and shifts) ➔ (D+1)-constraints

$$< \mathcal{H}_M(x,z) >= \frac{1}{Z}\frac{\delta Z}{\delta N^M(x,z)} = 0 \qquad\qquad \mathcal{H} = 0, \quad \mathcal{H}_\mu = 0$$

M=0, 1, 2, …, (D-1), D $\quad N^D(x,z) \equiv \alpha(x,z)$ and $\mathcal{H}_D \equiv \mathcal{H}$

- The (D+1)-constraints are (classically) first-class

$$\frac{\partial}{\partial z} < \mathcal{H}_M(x,z) >= \int d^D y\, N^{M'}(y,z)\,\langle\{\mathcal{H}_M(x,z), \mathcal{H}_{M'}(y,z)\}\rangle = 0$$

$$\{\mathcal{H}_M(x,z), \mathcal{H}_{M'}(y,z)\} = 0$$

Einstein Gravity upto two derivatives

[SL, 1305.3908]

$$S_{D+1} = \frac{N^2}{2\kappa^2}\int dz \int d^D x \left[\pi_{\mu\nu}\partial_z g^{\mu\nu} - N^D \mathcal{H} - N^\mu \mathcal{H}_\mu\right]$$

$$= \frac{N^2}{2\kappa^2}\int d^{D+1}X \sqrt{|G|}\left(-\Lambda + {}^{(D+1)}\mathcal{R} + ..\right).$$

Casimir energy

Beta function of $T^{\mu\nu}T^{\rho\sigma}$

$$\mathcal{H} = -\sqrt{g}\left[C_0 + R^D + \frac{g^{-1}}{2}\left(\frac{\pi^2}{D-1} - \pi^{\mu\nu}\pi_{\mu\nu}\right) + ..\right]$$

$$\mathcal{H}^\mu = -2\nabla_\nu \pi^{\mu\nu}$$

Uniquely fixed by the first-class constraint condition

[Blas, Pujolas, Sibiryakov (09); Henneaux, Kleinschmidt and Gomez (10)]

Summary

- In quantum RG, only a subset of couplings are kept while those couplings are promotes to dynamical variables
- A bulk action determines the weight for each path in the path integral of RG paths
- The bulk theory describes dynamical geometry
- Quantum RG can be viewed as a smooth projection of wavefunction
- Obstruction to smooth projection of one phase to another phase manifests itself as a horizon in the bulk