HOLOGRAPHIC RENORMALIZATION AND ENTANGLEMENT ENTROPY

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@ **Lecture series on**

1. Beyond Landau Fermi liquid and BCS superconductivity near quantum criticality

2. Real-space renormalization group approach

List

Lecture 1. Holographic renormalization

- Brief review on the AdS/CFT correspondence

- Renormalization schemes of the dual gravity

Lecture 2. Holographic entanglement entropy

- Central charge of the dual CFT

- a- and F-theorem

Motivation

One of the most remarkable successes in the AdS/CFT correspondence is the microscopic derivation of the Bekenstein-Hawking entropy for a BPS black hole

$$S_{BH} = \frac{A}{4G}$$

This idea relates the gravitational entropy to the degeneracy of the dual quantum field theory with its microscopic description.

On the other hand, there exists a different kind of entropy called the entanglement entropy in quantum mechanical systems which measures the entanglement between quantum states.

Ryu and Takayanagi proposed the formula following the black hole entropy

$$S = \frac{\text{Area of } \gamma}{4G}$$

The goal of this work is to figure out the entanglement entropy in the strong coupling regime following the AdS/CFT correspondence.

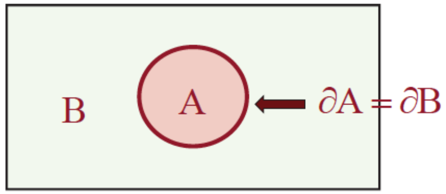
Review of the holographic entanglement entropy

The entanglement entropy measures

how closely and quantumly a given wave function is entangled.

Definition of EE (entanglement entropy)

- Divide a quantum system into two parts, A and B.



$$H_{tot} = H_A \otimes H_B \quad .$$

- Reduced density matrix of the subsystem A : $\rho_B = \mathrm{Tr}_A \rho_{tot}$

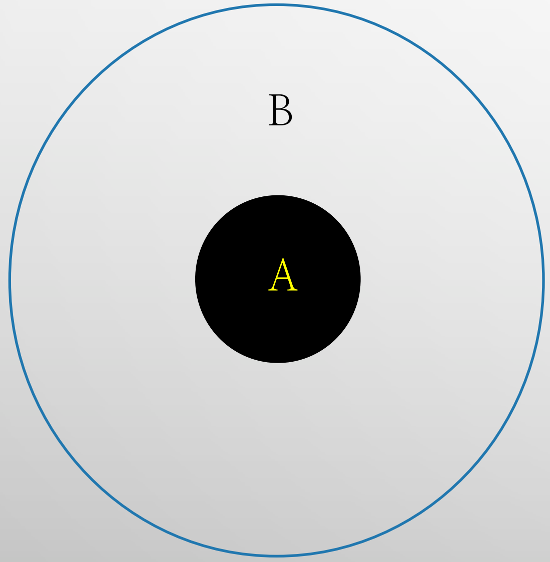
- The entanglement entropy (EE)

$$S_B = -\mathrm{Tr}_B \rho_B \log \rho_B$$

which is proportional to the area of the entangling surface (∂A)

S_B describes the quantum entanglement detected by an observer who is only accessible to the subsystem B and can not receive any signal from A.

This is similar to the Bekenstein-Hawking entropy of the black hole.
Since an observer sitting in the outside of the horizon, B, can not receive any information from A, we can regard A as a black hole and the boundary of A as the black hole horizon.



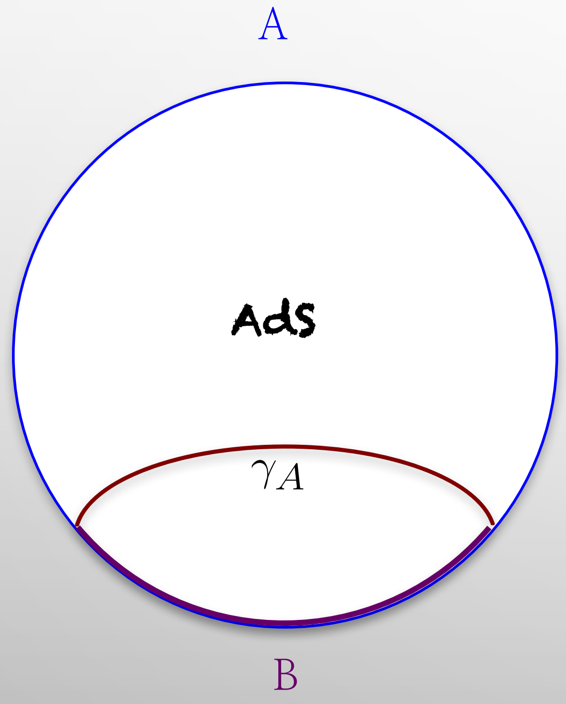
1. The area law of the entanglement entropy is also similar to that of the black hole entropy

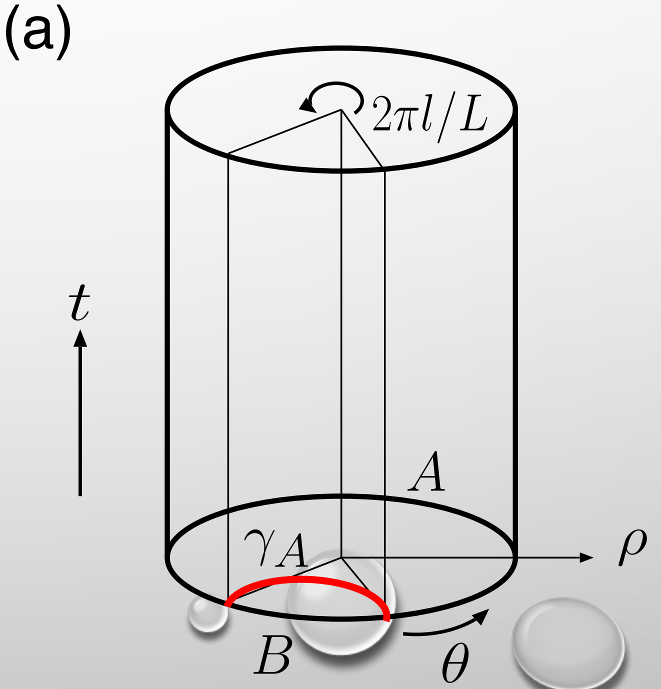
2. The entanglement entropy is utilized to figure out the black hole entropy

Due to the similarity to the black hole,
Ryu and Takayanagi [2006] proposed the holographic entanglement entropy (hEE)
following the AdS/CFT correspondence

the EE of a d-dimensional CFT can be evaluated by the area of the minimal surface in the d+1-dimensional dual AdS gravity

$$S_E = \frac{Area(\gamma_A)}{4G}$$





2-dim. CFT result [Calabrese-Cardy, 2004]

It is known that the entanglement entropy of the 2-dim CFT is given by

$$S_E = \frac{c}{3}\log\left(\frac{L}{\pi\epsilon}\sin\frac{\pi l}{L}\right) \approx \frac{c}{3}\log\frac{l}{\epsilon}$$

where l and L are the length of the subsystem A and the total system.
ϵ is a UV cutoff (lattice spacing) and c is the central charge of the CFT.

Away from criticality (fixed point), the entanglement entropy is replaced by

$$S_E = \frac{c}{6}\mathcal{A}\log\frac{\xi}{\epsilon}$$

where ξ is the correlation length and $\mathcal{A}$ is the number of the boundary points of A ($\mathcal{A} = 2$ in the setup).

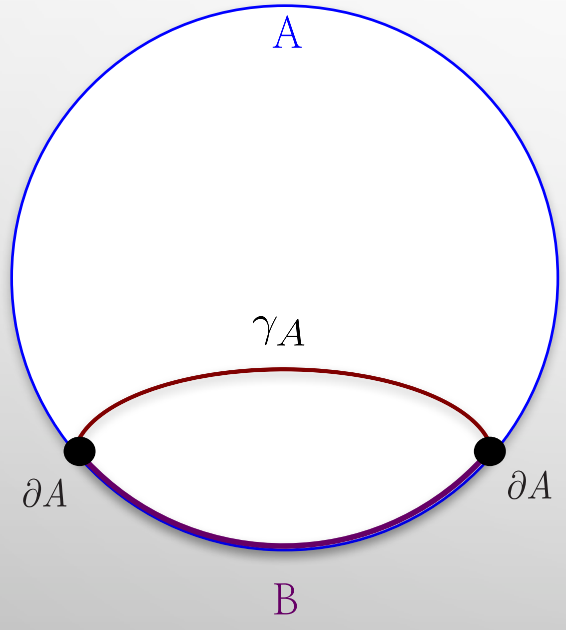
This is due to the infinite conformal symmetry and modular invariance of a 2-dim. CFT defined on the torus.

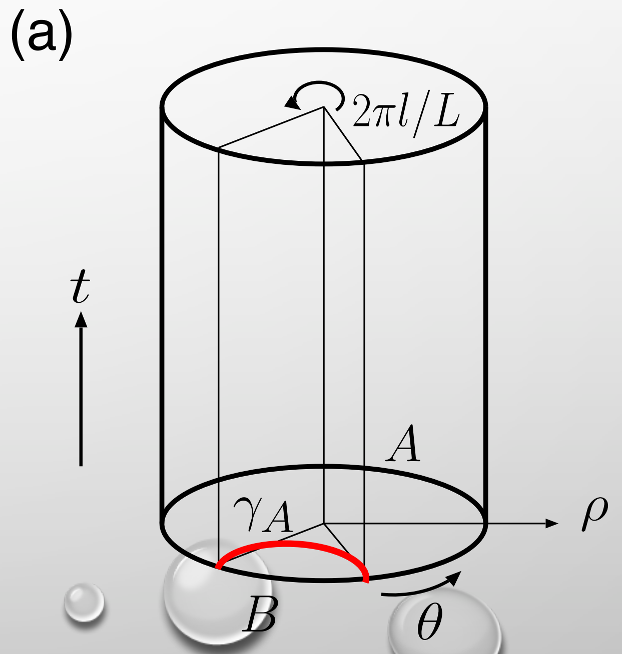
AdS(3)/CFT(2)

Assume that a 2-dim. dual CFT theory lives on R (time) X S (spatial) and S is divided into two subsystems, A and B (∂A is two points).

Then, on the gravity side, the static minimal surface γ_A with the same boundary ∂A is given by a geodesic curve in AdS(3).

The area of γ_A is proportional to the EE (HEE, holographic EE).





3-dim. AdS metric

$$ds^2 = \frac{1}{z^2}\left(-dt^2 + dx^2 + dz^2\right)$$

Induced metric on the minimal surface at a given time

$$ds^2 = \frac{z'^2 + 1}{z^2} dx^2$$

The entanglement entropy is given by

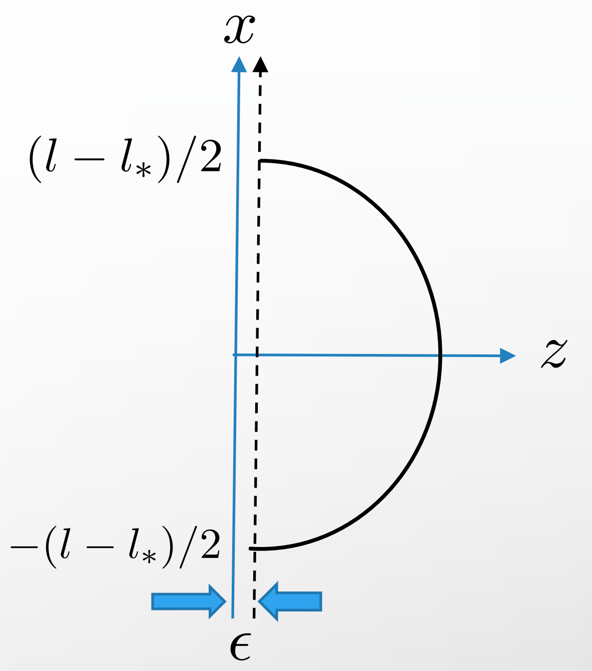
$$S_E = \frac{1}{4G}\int_{-l/2}^{l/2} dx \frac{\sqrt{z'^2+1}}{z}$$

The minimal surface satisfies the following equation

$$0 = zz'' + z'^2 + 1$$

Solution

$$z = \sqrt{\frac{l^2}{4} - x^2} \qquad \text{with} \qquad l_* = \frac{2\epsilon^2}{l}$$



The entanglement entropy for $\epsilon \to 0$

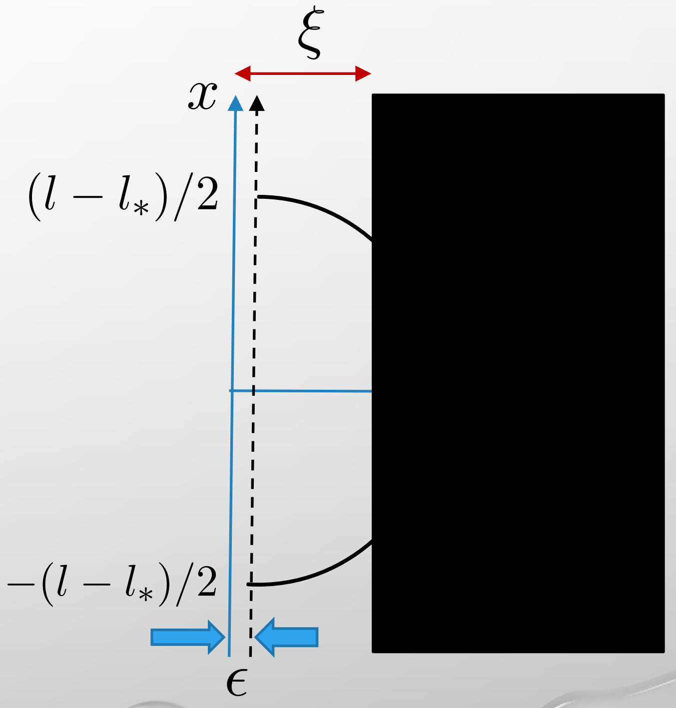
$$S_E = \frac{1}{2G} \log \frac{l}{\epsilon}$$

Using the fact

$$c = \frac{3R}{2G} \qquad \text{where we take } R = 1$$

we finally obtain

$$S_E = \frac{c}{3} \log \frac{l}{\epsilon}$$



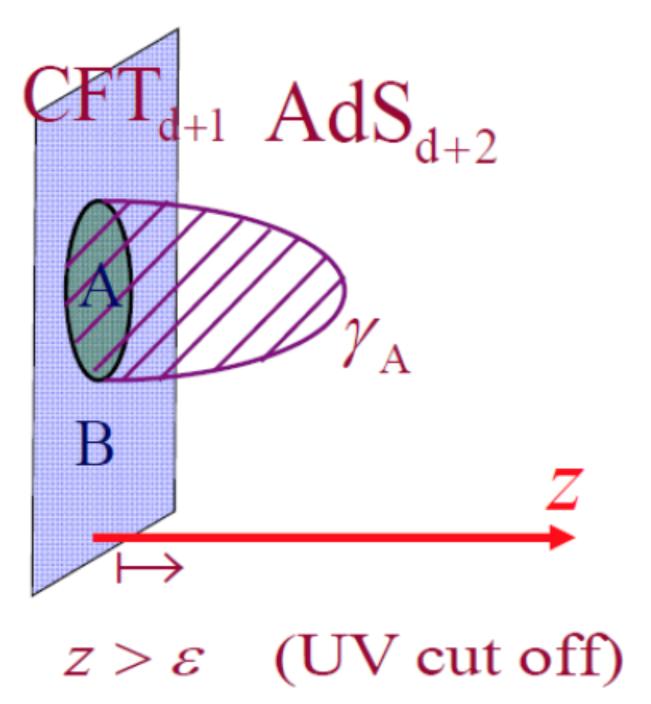
Introducing the IR cutoff, ξ

the similar calculation leads to

$$S_E \approx \frac{c}{3} \log \frac{\xi}{\epsilon}$$

Even in higher dimensions and in the strong coupling regime,

one can easily apply the Ryu-Takayanagi formula



γ_A is given by a co-dimension 2 surface

In the AdS/CFT context,

the entanglement entropy is geometrized as a minimal surface area.

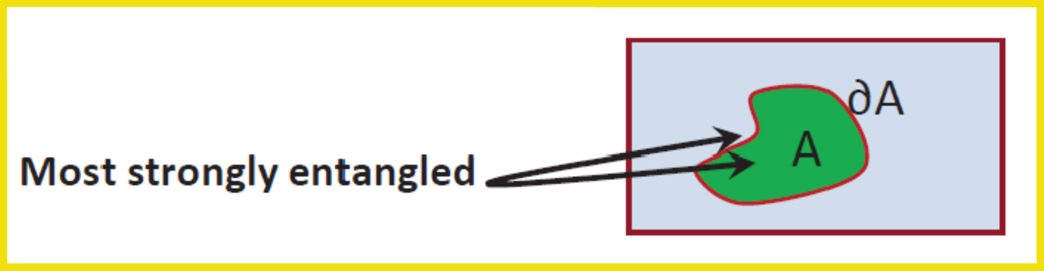
Aspects of the holographic entanglement entropy

GENERAL PROPERTIES OF THE ENTANGLEMENT ENTROPY

1) Area law of the entanglement entropy

The leading term of the entanglement entropy is provided by the short distance interaction between two subsystems near the boundary. In the continuum limit, this term causes a UV divergence and its coefficient is proportional to the area of the entangling surface ∂A (UV cutoff sensitive, regularization scheme dependent).

$$S_E \sim \frac{Area(\partial A)}{\epsilon^{d-1}} + \text{ subleading finite terms}$$

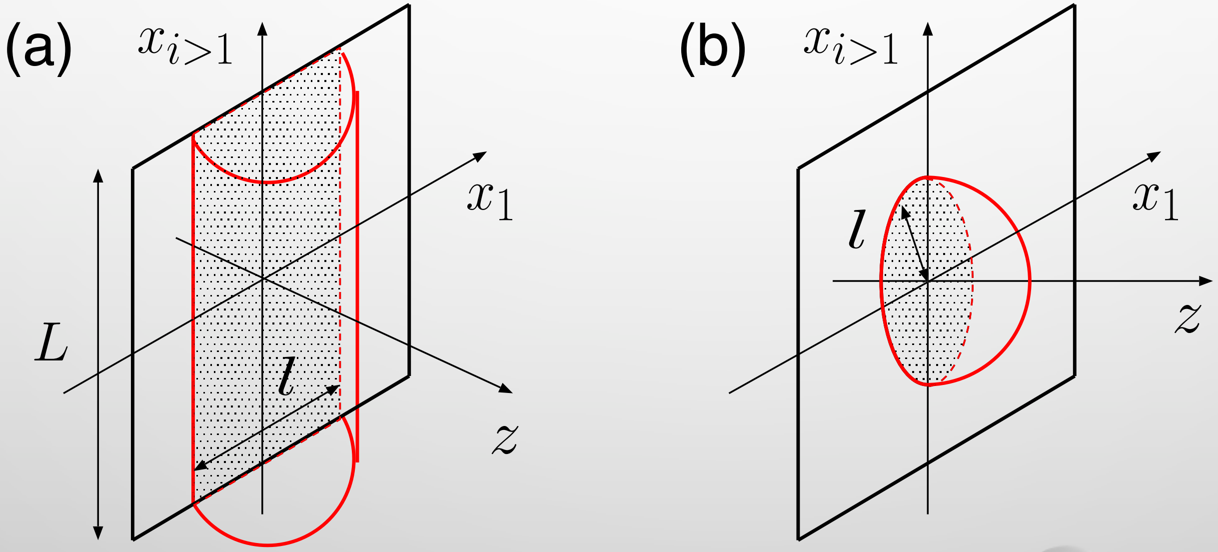


2) Subleading finite terms

There exists the terms not relying on a UV cutoff, which can provide an important physical information associated with the long range correlations.

In general, the entanglement entropy crucially depends on the shape and size of the entangling surface.

In AdS_{d+1} , let us take into account the HEE of a strip (a) and disk (b)



(a) Area of a strip

$$A = \frac{2R^{d-1}}{d-2}\left(\frac{L}{\epsilon}\right)^{d-2} - \frac{2^{d-1}\pi^{(d-1)/2}R^{d-1}}{d-2}\left[\frac{\Gamma\left(\frac{d}{2(d-1)}\right)}{\Gamma\left(\frac{1}{2(d-1)}\right)}\right]^{d-1}\left(\frac{L}{l}\right)^{d-2}$$

where L and ϵ indicate the size of the total system and a UV cutoff respectively.

- This result represents the entanglement entropy of vacuum states.
- There is no logarithmic term except for d=2.

(b) Area of a disk

which depends on the dimensionality

(i) for d=odd

$$A = \Omega_{d-2}\left[\frac{1}{d-2}\left(\frac{l}{\epsilon}\right)^{d-2} + F + \mathcal{O}\left(\frac{\epsilon}{l}\right)\right]$$

- No logarithmic term
- There exists a constant term, F, which is identified with a free energy of the 3-dimesional dual CFT for d=3.
- For d=3,

  F is the exact same as the free energy of 3-dim. CFT which has been checked by the comparison with the localization result.

(ii) for d=even

$$A = \Omega_{d-2}\left[\frac{1}{d-2}\left(\frac{l}{\epsilon}\right)^{d-2} + a'\ \log\left(\frac{l}{\epsilon}\right) + \mathcal{O}(1)\right]$$

with

$$a' = (-)^{d/2-1}\frac{(d-3)!!}{(d-2)!!}$$

- There exists **a universal logarithmic term**. Its coefficient is universal in that it is **independent of the regularization scheme**.

- The coefficient of the logarithmic term is independent of the entangling surface area, which is related to the **a-type anomaly**.

- Weyl anomaly of 4-dim. CFT,

$$\langle T^\alpha_\alpha \rangle = -\frac{c}{8\pi}W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} + \frac{a}{8\pi}\tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}$$

with

$$\begin{aligned} W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2, \\ \tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \end{aligned}$$

- As a consequence, *the logarithmic term is related to the anomaly and crucially depends on the dimension and shape of the entangling surface.*

c-theorem by Zamoldchikov

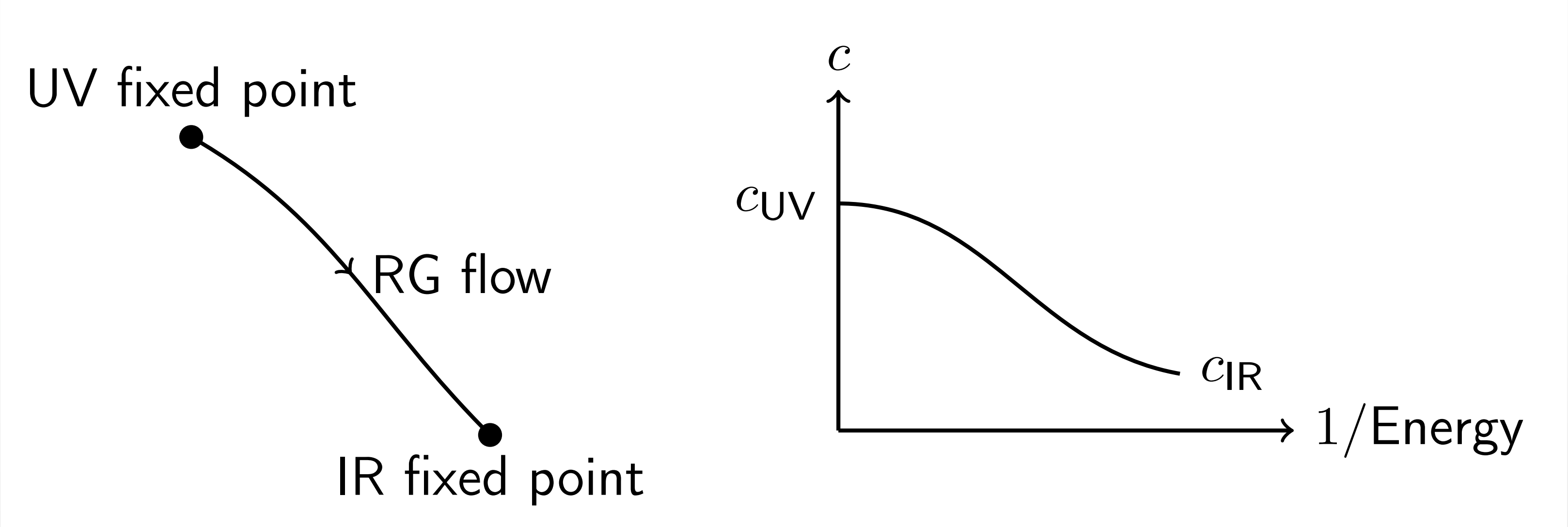
When a 2-dim. CFT is deformed by a relevant operator, it flows a new IR fixed point. In this case, the central charge, which describes degrees of freedom of a system, monotonically decreases along the RG flow.

In higher dimensional theory, is there a theorem similar to the C-theorem?

- For d=4, there exists two central charges, a and c. It has been believed that the a-type anomaly satisfies the c-theorem (*a-theorem*).
- For d=3, it has been conjectured that the free energy monotonically decreases along the RG flow (*F-theorem*).

F-theorem in 3-dim. CFT [Jafferis-Klebanov-Pufu-Safdi 2011, Myers-Sinha 2010]

RG flow under a relevant deformation



$$F_{\mathrm{UV}}(\mathbb{S}^3) \geq F_{\mathrm{IR}}(\mathbb{S}^3)\ , \quad F = -\log Z(\mathbb{S}^3)$$

HEE with a spherical entangling surface in a 3-dim. CFT

$$S_E = \alpha \frac{l}{\epsilon} - F(\mathbf{S}^3)$$

Renormalized entanglement entropy [Liu-Mezei 12]

$$\mathcal{F} \equiv \left(l \frac{\partial}{\partial_l} - 1 \right) S_E = -\log Z^{(ren)} \left(\mathbf{S^3} \right)$$

Using the strong sub-additivity and Lorentz invariance [Casini-Huerta 12]
it has been proved to be

$$\frac{\partial \mathcal{F}}{\partial l} = l \frac{\partial^2 S_E}{\partial l^2} \leq 0$$

When l increases (from UV to IR),

the renormalized entanglement entropy (renormalized free energy on $\mathbf{S}^3$)
monotonically decreases (F-theorem).

Additional logarithmic correction to the entanglement entropy

Recently, it has been argued that if a CFT is deformed by a relevant operator with the specific conformal dimension $\Delta = \frac{d+2}{2}$, an additional logarithmic term appears.

Unlike the previous logarithmic term associated with the central charge, it occurs regardless of the dimension and shape of the entangling surface.

Its coefficient is proportional to the entangling surface area

$$\delta S \sim \lambda^2 A_\Sigma \log\left(\lambda^{2/(d-2)}\epsilon\right)$$

where λ, A_Σ and ϵ imply the coupling constant, the entangling surface area and a UV cutoff respectively.

In a CFT on a Euclidean space,

- The action is given by a functional of the metric and fields.

- If the system resides in the vacuum state $|0\rangle$,
  the reduced density matrix in a subsystem A is represented as

$$\rho = \mathrm{Tr}_{\bar{A}} \, |0\rangle \langle 0| \equiv e^{-K},$$

  where K is a Hermitian operator called the modular Hamiltonian

- The modular Hamiltonian plays an important role in studying the entanglement entropy, becausea unitary operator $U(s) = \rho^{is} = \epsilon^{-iKs}$ generates a symmetry of the subsystem

$$\mathrm{Tr}\,(\rho\, \mathcal{O}) = \mathrm{Tr}\,(\rho\, U(s)\mathcal{O}U(-s))$$

- Usually, the modular Hamiltonian is non-local except several specific cases

$$\begin{aligned} K &= 2\pi\Omega_2 \int_{\rho\leq l} d\rho\, \rho^2\, \frac{l^2-\rho^2}{2l} T_{00} \quad \text{for a spherical entangling surface} \\ &= -2\pi \int_{\Sigma} d^{d-2}x \int_0^{\infty} dx_1\, x_1\, T_{00}, \quad \text{for a planar entangling surface} \end{aligned}$$

- The entanglement entropy in terms of the modular Hamiltonian

$$S = -\mathrm{Tr}\,(\rho \log \rho) = \langle 0\,|K|\,0\rangle$$

- When a CFT is deformed by a relevant operator $\mathcal{O}$,

$$K = K_0 + \lambda\mathcal{O} \equiv K_0 + \lambda \int d^4x\ \mathcal{O}(x),$$

- If λ is small, the entanglement entropy change is given by

$$\delta S = \frac{1}{2}\Big(\langle K_0\mathcal{O}\mathcal{O}\rangle - \langle \mathcal{O}\mathcal{O}\rangle\ \Big)\,\lambda^2 + \cdots .$$

In the weak coupling limit,

Now, let us consider a 4-dim. free massless fermion theory and deform it by a fermion mass term, $\lambda = m$ and $\mathcal{O} = \bar{\psi}\psi$(a relevant operator with the conformal dimension $\Delta = 3$)

Using the free fermion propagator

$$\langle \bar{\psi}(x)\psi(0) \rangle = \frac{1}{S_4}\frac{\gamma_\mu x^\mu}{x^4}$$

where γ_μ is a Euclidean gamma matrix and S_4 denotes a solid angle.

The small change of the entanglement entropy at second order leads to the expected logarithmic correction

$$\delta S = \frac{1}{12\pi}\, m^2 A_\Sigma \log{(m\epsilon)}$$

In the strong coupling limit,

Following the AdS/CFT correspondence,

a massive bulk scalar field corresponds to a scalar operator whose conformal dimensional is determined by

$$m_\phi^2 = \Delta\left(\Delta - d\right)$$

In the strong coupling limit, the dual gravity of the pervious fermion mass deformation is governed by

$$S = \int d^{d+1}x\sqrt{-G}\left[\frac{1}{2\kappa^2}\left(\mathcal{R} - 2\Lambda\right) - \frac{1}{2}\left(\partial\phi\right)^2 - \frac{1}{2}m_\phi^2\phi^2\right]$$

with

$$m_\phi^2 = -3.$$

Under the following metric ansatz

$$ds^2 = \frac{1}{z^2}\left[f(z)\left(-dt^2 + d\vec{x}^2\right) + dz^2\right]$$

Einstein equation and equation of motion of a massive scalar field become

$$\begin{aligned}
0 &= \kappa^2 f\left(3\phi^2 - z^2\phi'^2\right) - 6z\left(zf'' - 3f'\right), \\
0 &= \frac{6f'^2}{f^2} - \frac{3\kappa^2\phi^2}{z^2} - \frac{24f'}{zf} - \kappa^2\phi'^2, \\
0 &= 2z^2 f'\phi' + f\left(3\phi + z(z\phi'' - 3\phi')\right),
\end{aligned}$$

In the asymptotic region, the perturbative solution is given by

$$\begin{aligned}
f(z) &= 1 - \frac{\kappa^2}{6}m_q^2 z^2 + \frac{1}{36}\left(\kappa^4 m_q^4 - 9\kappa^2 m_q\sigma - 3\kappa^4 m_q^4 \log z\right) z^4 + \mathcal{O}(z^6), \\
\phi(z) &= m_q z + \left(\sigma + \frac{1}{3}\kappa^2 m_q^3 \log z\right) z^3 + \mathcal{O}(z^5).
\end{aligned}$$

where m_q and $\sigma = \langle\bar{\psi}\psi\rangle$ indicate the fermion mass and chiral condensate with $\Delta = 3$.

Parmetrizing a spherical entangling surface

$$0 \le \rho \le l \quad \text{and} \quad z = z(\rho),$$

the area of the minimal surface, whose boundary coincides with the entangling surface, is described by

$$A = \Omega_2 \int_0^l d\rho \, \frac{\rho^2 f(z)}{z^3} \sqrt{f(z) + z'^2}$$

Using dimensionless small parameters, $m_q l$ and σl^3, the deformed minimal surface near the known solution, $z_0(r)$, can be expanded into

$$z(\rho) = z_0(\rho) + m_q^2 l^2 \ z_2(\rho) + \cdots$$

where $z_0(\rho) = \sqrt{l^2 - \rho^2}$ represents the geodesic of a particle in the AdS space.

Near the asymptotic boundary,

the first correction of the minimal surface deformation reads

$$z_2 \approx \frac{\kappa^2(l-\rho)^{3/2}\left(-13\sqrt{2}+6\sqrt{2}\log 2-6\sqrt{2}\log(l-\rho)+6\sqrt{2}\log l\right)}{72\sqrt{l}}.$$

Substituting these solutions into the area formula, the entangling surface area reads

$$A = A_0 + A_{20} + A_{22}$$

with

$$\begin{aligned} A_0 &= \frac{l^2\Omega_2}{2\epsilon^2} - \frac{\Omega_2}{2}\log\left(\frac{l}{\epsilon}\right) + \mathcal{O}(1), \\ A_{20} &= \frac{1}{6}\kappa^2 m_q^2 l^2 \Omega_2 \log(m_q\epsilon) + \mathcal{O}(1), \\ A_{22} &= \frac{1}{12}\kappa^2 m_q^2 l^2 \Omega_2 \log(m_q\epsilon) + \mathcal{O}(1). \end{aligned}$$

Here, A_0 is associated with the entanglement entropy of the undeformed CFT.

$$S_0 \equiv \frac{2\pi A_0}{\kappa^2} = \frac{\pi l^2 \Omega_2}{\kappa^2 \epsilon^2} - \frac{\pi \Omega_2}{\kappa^2} \log\left(\frac{l}{\epsilon}\right) + \mathcal{O}(1),$$

while A_{20} and A_{22} imply the contributions from the metric and minimal surface deformation, respectively.

As a consequence,
the logarithmic correction caused by the relevant deformation becomes in the strong coupling limit

$$\delta S = \frac{2\pi}{\kappa^2}\left(A_{20} + A_{22}\right) = \frac{\pi}{2} m_q^2 A_\Sigma \log\left(m_q \epsilon\right),$$

where the area of the entangling surface is given by $A_\Sigma = l^2 \Omega_2$.

This additional logarithmic term is proportional to the entangling surface area unlike the one associated with the central charge.

Thermodynamics-like law of the entanglement entropy

It has been found that the entanglement entropy of an excited state in a strip region follows the thermodynamics-like law after defining an appropriate entanglement temperature [Bhattacharya-Nozaki-Takayanagi-Ugajin, 2013]

$$\Delta E = T_E \Delta S.$$

In this case, the entanglement temperature has the form

$$T_E \equiv \frac{\Delta E}{\Delta S} = \frac{1}{\lambda}\frac{1}{l},$$

Where λ is a non-universal constant relying on details of the microscopic theory.

Focusing on the size dependence of the entanglement temperature,
it is proportional to the inverse of the system size, $T_E \sim 1/l$, regardless of the shape of the entangling surface and microscopic details of the underlying theory.

The similar universal structure also occurs in black hole physics.

Bekenstein bound

Through a thought experiment for black hole thermodynamics, the Bekenstein bound has been proposed as a universal bound of the thermal entropy in flat space.

When an object is absorbed into a black hole, the entropy of an object increases the black hole area due to the generalized second law of thermodynamics. The increased entropy is bounded by the absorbed energy

$$\Delta S \leq \lambda l \Delta E,$$

where l and λ are a typical size of the system and a non-universal numerical factor of order one.

The Bekenstein bound is universal in that it is independent of the microscopic detail up to λ.

- The entanglement entropy has been proposed as the origin of black hole entropy.

- Let us try to understand the Bekenstein bound from the entanglement entropy bound which is applicable not only a thermal system but also to a quantum system.

Relative entropy

- When two states are in the same Hilbert space, the relative entropy gives rise to a fundamental statistical measure of their distance.

- If two reduced density matrices are denoted by ρ_1 and ρ_0, the relative entropy $S(\rho_1|\rho_0)$ is defined as

$$S(\rho_1|\rho_0) \equiv \mathrm{Tr}\left(\rho_1 \log \rho_1\right) - \mathrm{Tr}\left(\rho_1 \log \rho_0\right).$$

Here we can identify ρ_0 with the reduced density matrix of a ground or thermal state, while ρ_1 is the one for a quantumly or thermally excited state.

If there exists a parameter connecting two reduced density matrices such that $\rho_1 = \rho_1(\lambda)$ and $\rho_0 = \rho_1(0)$, the relative entropy usually has a non-negativity value

$$S(\rho_0|\rho_0) = 0 \quad \text{and} \quad S(\rho_1|\rho_0) > 0 \text{ for } \rho_0 \neq \rho_1 .$$

Thus, ρ_0 corresponds to a minimum point.

Using the definition of the entanglement entropy, the relative entropy can be reexpressed as

$$S(\rho_1|\rho_0) = \Delta K - \Delta S,$$

where variations of the modular Hamiltonian and entanglement entropy are given by

$$\Delta K = \mathrm{Tr}\,(\rho_1 K) - \mathrm{Tr}\,(\rho_0 K) \quad \text{and} \quad \Delta S = S(\rho_1) - S(\rho_0).$$

The non-negativity of the relative entropy leads to the following relation

$$\Delta K \geq \Delta S,$$

which is a generalized Bekenstein bound holding for any region in QFT.

Universality of the entanglement temperature

Following the simple dimension counting of a relativistic QFT, we can guess that the increased modular Hamiltonian is proportional to the increased energy

$$\Delta K = \lambda\, l \Delta E$$

with a non-universal numerical factor λ.

Substituting this relation into the entanglement entropy bound,
we finally arrive at the Bekenstein bound working in a QFT.

When the entanglement entropy bound is saturated, $\Delta K = \Delta S$, we can reinterpret it as the thermodynamics-like law. In this case, the entanglement temperature founded by Bhattacharya, Nozaki, Takayanagi and Ugajin naturally appears

$$T_E \equiv \frac{\Delta E}{\Delta S} = \frac{1}{\lambda l}.$$

Notice

A general modular Hamiltonian is not known except several simple cases.
One of them is the case with a spherical entangling surface.

Now, we check that the modular Hamiltonian is related to the energy, $\Delta K = \lambda\, l \Delta E$, in a nontrivial but physically interesting theory.

A charged dilatonic black brane geometry, whose dual QFT has a Fermi sea and massless fluctuations on the Fermi surface, is described by [Gubser-Rocha, 2009]

$$S = \frac{1}{2\kappa^2}\int d^5x\sqrt{-g}\left[\mathcal{R} - \frac{1}{4}e^{4\phi}F_{\mu\nu}F^{\mu\nu} - 12\partial_\mu\phi\partial^\mu\phi + \frac{1}{R^2}\left(8e^{2\phi}+4e^{-4\phi}\right)\right].$$

If there exists a nontrivial ϕ approaching to zero at the asymptotic boundary, there exists an asymptotic AdS solution.

The charged dilatonic black brane solution

$$ds^2 = r^2 e^{2A(r)} \left(-f(r)dt^2 + d\vec{x}^2\right) + \frac{e^{2B(r)}}{r^2 f(r)} dr^2,$$
$$A = A_t dt,$$

with

$$\phi(r) = \frac{1}{6}\log\left(1+\frac{Q^2}{8mr^2}\right),$$
$$A(r) = \frac{1}{3}\log\left(1+\frac{Q^2}{8mr^2}\right),$$
$$B(r) = -\frac{2}{3}\log\left(1+\frac{Q^2}{8mr^2}\right),$$
$$f(r) = 1 - \frac{m}{r^4\left(1+\frac{Q^2}{8mr^2}\right)^2},$$
$$A_t = 2\kappa^2\mu - \frac{Q}{2r^2\left(1+\frac{Q^2}{8mr^2}\right)},$$

where m, μ, and Q indicate the charged black brane's mass, chemical potential, and charge density, respectively.

Using the holographic renormalization,
the grand potential of the grand canonical ensemble becomes

$$\Omega = -\frac{\pi^4 V_3}{2\kappa^2} T_H^4 - 2\pi^2\kappa^2 V_3 T_H^2 \mu^2 - \frac{10}{3}\kappa^6 V_3 \mu^4,$$

other thermodynamic quantities satisfying the first law of thermodynamics are given by

$$\begin{aligned}
E &= \frac{3\pi^4 V_3}{2\kappa^2} T_H^4 + 6\pi^2\kappa^2 V_3 T_H^2 \mu^2 + \frac{14}{3}\kappa^6 V_3 \mu^4, \\
P &= \frac{\pi^4}{2\kappa^2} T_H^4 + 2\pi^2\kappa^2 T_H^2 \mu^2 + \frac{10}{3}\kappa^6 \mu^4, \\
S_{BH} &= \frac{2\pi^4 V_3}{\kappa^2} T_H^3 + 4\pi^2\kappa^2 V_3 T_H \mu^2, \\
\frac{N}{V_3} &= 4\pi^2\kappa^2 T_H^2 \mu + \frac{40}{3}\kappa^6 \mu^3,
\end{aligned}$$

where E, P, S_{BH} and N/V_3 indicate the energy, pressure, entropy and charge density respectively.

If the quantum state is excited without the change of the chemical potential, the modular Hamiltonian is associated with the previous stress tensor

$$K|_\mu = 2\pi\Omega_2 \int_{\rho\leq l} d\rho\ \rho^2\ \frac{l^2-\rho^2}{2l}\ T_{00}|_\mu$$

where $T_{00}|_\mu$ indicates the energy density at a given μ and Ω_2 is the solid angle of the spherical entangling surface.

Since T_{00} is uniform in this model, the modular Hamiltonian can be rewritten as

$$K|_\mu = \frac{2\pi}{5} l\ E|_\mu\,,$$

where the energy contained in the ball-shaped region is given by $E|_\mu = \Omega_2 \int_{\rho\leq l} d\rho\ \rho^2\ T_{00}|_\mu$.

- This relation shows how the modular Hamiltonian is related to the energy over the interior of the sphere.
- This is the expected form and shows that the entanglement entropy bound is equivalent to the Bekenstein bound except that the former is working even in a quantum system.

When μ is fixed, the increased modular Hamiltonian becomes

$$\Delta K|_{\mu} \equiv K(T_H, \mu) - K(0, \mu) = \frac{\pi^5 l^4 \Omega_2}{5\kappa^2} T_H^4 + \frac{4\pi^3 \kappa^4 l^4 \Omega_2}{5\kappa^2} \mu^2 T_H^2.$$

Using the holographic technique, the HEE becomes

$$\begin{aligned} S(T_H, \mu) \quad = \quad & \frac{\pi l^2 \Omega_2}{\kappa^2 \epsilon^2} + \frac{\pi \Omega_2}{\kappa^2} \log\left(\frac{\epsilon}{l}\right) - \frac{\pi \Omega_2}{2\kappa^2}(1 + 2\log 2) \\ & + \frac{4\pi l^2 \Omega_2}{3\kappa^2} \kappa^4 \mu^2 - \frac{8\pi l^4 \Omega_2}{45\kappa^2} \kappa^8 \mu^4 + \frac{\pi l^4 \Omega_2}{5\kappa^2} \left(\pi^2 T_H^2 + 2\kappa^4 \mu^2\right)^2, \end{aligned}$$

which is the entanglement entropy of the excited state with the chemical potential.

At a given chemical potential,
the increased entanglement entropy up to l^4 order is given by

$$\Delta S|_{\mu} \equiv S(T_H, \mu) - S(0, \mu) = \frac{\pi^5 l^4 \Omega_2}{5\kappa^2} T_H^4 + \frac{4\pi^3 \kappa^4 l^4 \Omega_2}{5\kappa^2} \mu^2 T_H^2,$$

which is the exact same as the previous increased modular Hamiltonian.

When higher order corrections are ignored,
the saturated entanglement entropy bound leads to the thermodynamics-like law

$$\Delta K|_{\mu} = \Delta S|_{\mu} = \frac{1}{T_E} \Delta E|_{\mu} ,$$

with

$$T_E = \frac{5}{2\pi l} .$$

Recently, it has been shown that <u>*the linearized Einstein equation of the AdS geometry can be reproduced from the entanglement entropy's thermodynamics-like law of the dual QFT*</u> [Raamsdonk and et.al, 2013].

$$K|_{\mu} \quad = \quad 2\pi\Omega_2 \int_{\rho \le l} d\rho \, \rho^2 \, \frac{l^2 - \rho^2}{2l} \, T_{00}|_{\mu}$$

$dK = dS_E$

$$\delta \langle T_{00} \rangle = \frac{d^2 - 1}{2\pi\Omega_{d-2}} \lim_{l \to 0} \frac{\delta S_E}{l^d}$$

u^{μ} boosting

$$u^{\mu} u^{\nu} \delta \langle T_{\mu\nu} \rangle = \frac{d^2 - 1}{2\pi\Omega_{d-2}} \lim_{l \to 0} \frac{\delta S_E^{(u)}}{l^d}$$

Holographic renomalization

$$\delta \langle T_{\mu\nu} \rangle = \frac{d l^{d-3}}{16\pi G} \gamma_{\mu\nu}^{(d)}$$

: linearized Einstein equation

Under the global quench

Suppose that K_0 is the modular Hamiltonian of the undeformed theory. Then, the reduced density matrix is given by

$$\rho_0 = \frac{e^{-K_0}}{\mathrm{Tr}\, e^{-K_0}}.$$

In a ball-shaped region, the modular Hamiltonian is related to the energy, $K_0 = \frac{E}{T_E}$.

Now, let us deform this theory by a relevant number operator, N,

$$K = K_0 - \frac{\mu_E}{T_E} N,$$

where the entanglement chemical potential, μ_E, accounts for how the global quench modifies the modular Hamiltonian and entanglement entropy.

Then, the reduced density matrix of the deformed theory becomes

$$\rho = \frac{e^{-K}}{\mathrm{Tr}\, e^{-K}}.$$

The non-negativity of the relative entropy gives rise to a generalized entanglement entropy bound

$$\Delta K = \frac{\Delta E}{T_E} - \frac{\mu_E}{T_E}\Delta N \geq \Delta S \quad \text{or} \quad \Delta E \geq T_E \Delta S + \mu_E \Delta N.$$

When the generalized entanglement entropy bound is saturated,
the entanglement entropy satisfies a generalized thermodynamics-like law

$$dE = T_E dS_E + \mu_E dN$$

THANK YOU FOR YOUR ATTENTION!