AdS space and its dual theory

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@ Holography and Topology of Quantum Matter

List

- Brief review on the AdS/CFT correspondence
- holographic two-point correlation function at zero temperature
- holographic two-point correlation function at finite temperature (linear response theory)
- Conclusion

AdS geometry

- constantly curved space with a negative cosmological constant

Action for AdS_{d+1}

$$S = \frac{1}{16\pi G}\int d^{d+1}x\sqrt{-g}\ \left(\mathcal{R} - 2\Lambda\right)$$

with a negative cosmological constant

$$\Lambda = -\frac{d(d-1)}{2R^2}$$

Its equation of motion (Einstein equation), $0 = \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} + \Lambda g_{\mu\nu}$,

allows an AdS geometry

$$ds^2 = -\frac{r^2}{R^2}dt^2 + r^2\left(du^2 + u^2 d\Omega_{d-2}^2\right) + \frac{R^2}{r^2}dr^2$$

where $\mathcal{R} = -\frac{d(d+1)}{R^2}$

AdS(1,d) geometry

- can be is defined as a hypersurface in an one-dimensional higher flat space denoted by $\mathbf{R}^{2,d}$

Proper distance in the (d+2)-dimensional ambient flat space

$$ds^2 = -dy_{-1}^2 - dy_0^2 + dy_1^2 + \cdots + dy_d^2$$

which is invariant under the SO(2,d) Lorentz transformation

Then, the AdS_{d+1} geometry appears as the hypersurface satisfying

$$-y_{-1}^2 - y_0^2 + y_1^2 + \cdots + y_d^2 = -R^2$$

Since this constraint does not break the Lorentz symmetry of the ambient space, the resulting geometry also preserves the SO(2, d) symmetry which is nothing but the isometry of the AdS_{d+1} space.

Rewriting the metric of the ambient space in terms of coordinates satisfying the constraint

$$y_{-1} + y_d = R^2 r \quad \text{and} \qquad y^\mu = \frac{r}{R} x^\mu \ (\mu = 0, 1, \cdots, d-1)$$

one can reproduce the AdS_{d+1} metric

$$ds^2 = \frac{r^2}{R^2}\left(-dt^2 + d\vec{x}^2\right) + \frac{R^2}{r^2} dr^2$$

Using different coordinate, different AdS metrics can be obtained

$$ds^2 = -\frac{r^2 f_k(r)}{R^2} dt^2 + r^2 d\Sigma_k^2 + \frac{R^2}{r^2 f_k(r)} dr^2$$

where

$$f_k(r) = 1 + k\frac{R^2}{r^2}$$

- For $k = 0$, $d\Sigma_k^2$ implies the metric of $\mathbf{R}^{d-1}$
- For $k = 1$, $d\Sigma_k^2$ implies the metric of $\mathbf{S}^{d-1}$
- For $k = -1$, $d\Sigma_k^2$ implies the metric of $\mathbf{H}^{d-1}$

AdS/CFT correspondence (symmetry)

Classical SUGRA on AdS space-time	⟷ 1 to 1	Super-CFT at the AdS boundary (in a strong coupling regime)

AdS/CFT correspondence

Isometry of AdS_5	⟵	$SO(2,4)$	⟶	Conformal symmetry on $R^{1,3}$
Isometry of S^5	⟵	$SO(6)$	⟶	R-symmetry of N=4 SUSY

conformal symmetry in 1+3-dim. Space-time

Poincare group SO(1,3) + Scaling + Special conformal ➡ SO(2,4)

AdS/CFT correspondence (strong/weak duality)

AdS/CFT correspondence

$$Z_{gravity} \approx e^{-S_{on-shell}} \quad \overset{\text{one to one map}}{\longleftrightarrow} \quad Z_{gauge} = \langle e^{-S_{CFT}} \rangle$$

- this SUGRA solution is valid in the $gN \to \infty$ limit (classical limit, $\mathcal{R} \ll 1$)

- large t' Hooft coupling $\lambda = g_{YM}^2 N = gN$ → nonperturbatvie gauge theory

Applying the AdS/CFT correspondence to a deformed CFT

Deform the AdS geometry with a massive scalar field which breaks the conformal symmetry

$$S = \frac{1}{16\pi G}\int d^5x\sqrt{-g}\left(\mathcal{R} - 2\Lambda - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right)$$

If $-\frac{4}{R^2} < m^2 < 0$, ϕ corresponds to a relevant operator deforming the dual CFT.

Near the boundary

$$\phi = \frac{\phi_0}{r^{4-\Delta}}\,(1+\cdots) + \frac{\mathcal{O}}{r^\Delta}\,(1+\cdots) \quad \text{with} \quad 2<\Delta<4 \quad (m^2 = \Delta(\Delta-4))$$

when $r \to 0$, $\phi \to 0$ which does not affect the UV physics of the dual QFT (relevant).

In this case,

ϕ_0 : source of the dual operator

$\mathcal{O}$: dual operator

Δ : conformal dimension of the dual operator

The holographic renormalization

The boundary term of the scalar field reduces to

$$S_{bd} \sim \int d^4x \sqrt{-g}\, g^{rr} \phi\, \partial_r \phi$$

$$\approx \int d^4x \left(\frac{\phi_0^2}{r^{4-2\Delta}} + \phi_0 \mathcal{O} + \frac{\mathcal{O}^2}{r^{2\Delta-4}} + \cdots \right)$$

near the boundary ($r \to 0$) divergence finite term

In order to obtain a finite generating functional, we need to add an addition counter term.

After the appropriate holographic renormalization,

$$Z = e^{-S_{on-shell}}$$

with

$$S_{on-shell} = S_{CFT} - \int d^4x \phi_0 \mathcal{O}$$

which corresponds to the generating functional of the dual QFT.

One-point correlation function from the generating functional

$$\frac{\partial Z}{\partial \phi_0} = \langle \mathcal{O} \rangle|_{\phi_0 = 0}$$

n-point correlation function can be obtained by applying n-derivatives with respect to the source to the above generating functional

$$\frac{\partial^n Z}{\partial \phi_0^n} = \langle \mathcal{O}^n \rangle|_{\phi_0 = 0}$$

Two-point correlation function

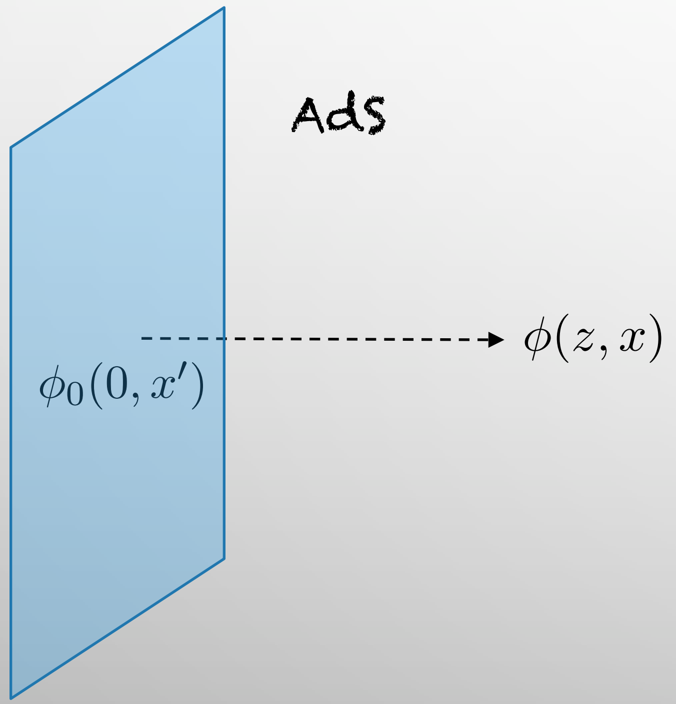
For convenience, introduce a new coordinate, $z = 1/r$

Then, the AdS metric becomes

$$ds^2 = \frac{dz^2 - dt^2 + d\vec{x}^2}{z^2}$$

and the massive scalar field is governed by

$$0 = \frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g}g^{\mu\nu}\partial_\nu \phi(z,t,\vec{x})\right) - m^2\phi(z,t,\vec{x})$$



Solution

$$\phi(z,x) = \int d^4x' \mathcal{D}_\phi\left(z,x;0,x'\right)\phi_0(0,x')$$

with Bulk-to-boundary propagator

$$\mathcal{D}_\phi\left(z,x;0,x'\right) \sim \left(\frac{z}{z^2+(x-x')^2}\right)^\Delta$$

Near the boundary ($z \to 0$),

the bulk-to-boundary propagator reduces to

$$\mathcal{D}_\phi\left(z, x; 0, x'\right) \approx \frac{z^\Delta}{(x-x')^{2\Delta}}$$

which leads to

$$\lim_{z\to 0} \phi(z,x) = z^\Delta \int d^4x' \frac{\phi_0(0,x')}{(x-x')^{2\Delta}}$$

Noting that the asymptotic behavior of $\phi(z,x)$ is given by

$$\phi(z,x) \approx \phi_0(x)\; z^{4-\Delta}\,(1+\cdots) + \mathcal{O}(x)\; z^\Delta\,(1+\cdots)$$

we can obtain

$$\mathcal{O}(x) \approx \int d^4x' \frac{\phi_0(0,x')}{(x-x')^{2\Delta}}$$

Then, the boundary (on-shell) action of matter becomes

$$S_{bd} \sim \int d^4x \int d^4x' \; \frac{\phi_0(0,x)\phi_0(0,x')}{(x-x')^{2\Delta}}$$

As a consequence, the two-point correlation function reads

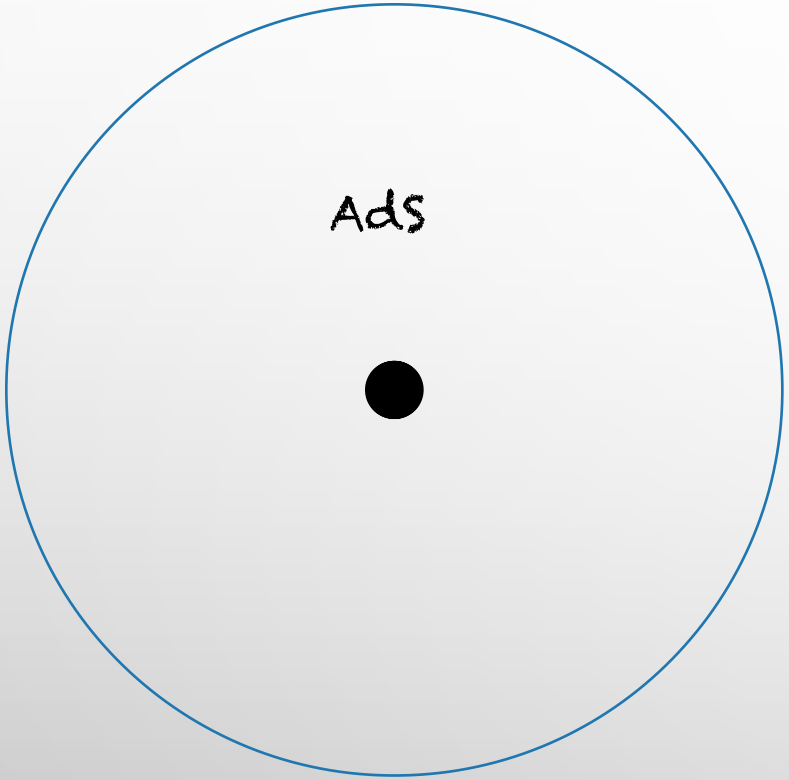
$$\langle \mathcal{O}(x)\mathcal{O}(x') \rangle = \frac{\partial}{\partial\phi_0(0,x')}\frac{\partial}{\partial\phi_0(0,x)} Z = \frac{1}{(x-x')^{2\Delta}}$$

which is exactly the two-point function expected in the CFT.

AdS black hole geometry

AdS space → ground state of dual QFT

AdS black hole geometry

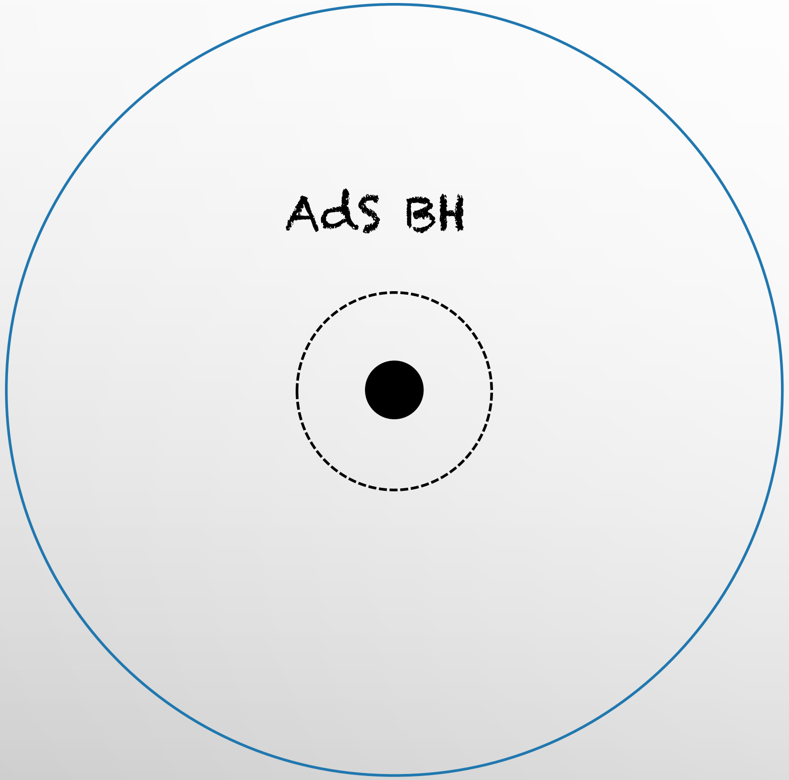


AdS space ➡ ground state of dual QFT

localized matter in the AdS space

➡ singularity at the center

AdS black hole geometry



AdS space ➡ ground state of dual QFT

localized matter in the AdS space

➡ singularity at the center

In order to avoid this singularity,

the black hole geometry is required

AdS black hole geometry

Equation of motion for a black hole

$$0 = \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} + \Lambda g_{\mu\nu}$$

Black hole geometry

$$ds^2 = -\frac{r^2 f_k(r)}{R^2} dt^2 + r^2 d\Sigma_k^2 + \frac{R^2}{r^2 f_k(r)} dr^2$$

with

$$f_k(r) = 1 + k\frac{R^2}{r^2} - \frac{m}{r^d}$$

where m is the black hole mass.

From now on, we focus on the case with $k = 0$ for simplicity.

Properties of a black hole

- Black hole horizon, r_h, where g_{tt} vanishes

- Hawking radiation (Hawking temperature)

$$T_H = \frac{1}{4\pi} \frac{\partial g_{tt}}{\partial r}\bigg|_{r=r_h}$$

- Bekenstein-Hawking entropy (area law)

$$S_{BH} = \frac{A}{4G}$$

- Black hole physics satisfies the thermodynamic law (macroscopic)

Mystery of black hole

1. Area law of the Bekenstein-Hawking entropy

2, What is the microscopic origin of the Bekenstein-Hawking entropy?

t Hooft proposed the holographic principle. The AdS/CFT correspondence is the realization of the holography

1. the area can be matched to the volume of the dual QFT (extensive).
2. the Bekenstein-Hawking entropy then counts the degrees of freedom of the dual QFT

AdS/CFT correspondence of a black hole geometry

Pure AdS space	⟺	groud state of dual QFT at zero temperature
AdS black hole	⟺	dual QFT at finite temperature

Black hole thermodynamics

For $k = 0$ and $d = 4$,

$$f_0(r) = 1 - \frac{m}{r^4}$$

- Black hole horizon : $r_h = m^{1/4}$
- Hawking temperature : $T_H = \frac{r_h}{\pi R^2}$
- Bekenstein-Hawking entropy : $S_{BH} = \frac{V}{4G} r_h^3$ with $V = \int d^3x$

- The first law of thermodynamics : $dE = T_H \, dS_{BH}$

  - internal energy : $E = \frac{3V}{16\pi G R^2} m = \frac{3V}{16\pi G R^2} T_H^4$
  - free energy : $F = E - T_H S_{BH} = -\frac{V}{16\pi G R^2} T_H^4$

Holographic renormalization

How can Black hole thermodynamics be identified with that of the dual QFT?

From the AdS/CFT correspondence $Z = e^{-S_{on-shell}} = \left\langle e^{-S_{CFT}} \right\rangle$
one can evaluate the nonperturbative partition function (or generating functional) of the dual QFT and then extract many physical information from it

Now, let us calculate the on-shell gravity action.

When evaluating the on-shell gravity action, we encounter several problems.

- not well-defined variation
- divergence

Gibbons-Hawking term

In the gravity action

$$S = \frac{1}{16\pi G}\int d^{d+1}x\sqrt{-g}\ (\mathcal{R} - 2\Lambda)$$

the curvature scalar involves

$$\mathcal{R} = (\partial^{\mu} g_{\mu\nu})^2 + \cdots + \partial^{\mu}\partial^{\nu} g_{\mu\nu}$$

Although this boundary term does not affect the bulk equation of motion, it causes problem for defining the variation of the boundary action.

$$\delta S = \delta S_{bulk} + \delta S_{bd}$$

equation of motion

To get rid of such a problematic term, one should add a Gibbons-Hawking term

$$S_{GB} = \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^4x\sqrt{\gamma}\ \mathcal{K}$$

Counter terms

Although the Gibbons-Hawking term leads to a well-defined action variation, the on-shell gravity action usually diverges at the boundary due to the integration over r .

$$S = \frac{1}{16\pi G}\int d^4x \int_0^\infty dr\sqrt{-g}\,(\mathcal{R} - 2\Lambda)$$

$$= \frac{1}{16\pi G}\int d^4x \int_0^\infty dr\ r^3\left(-\frac{8}{R^2}\right)$$

On the dual QFT, the previous divergence corresponds to the UV divergence which should be removed by adding appropriate counter terms similar to the renormalization procedure of an ordinary QFT. ➡ Holographic renormalization

Counter terms for the AdS geometry

$$S_{ct} = \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^4x\sqrt{\gamma}\left(\frac{3}{R}+\frac{R}{4}\mathcal{R}^{(4)}\right)$$

$\mathcal{R}^{(4)}$: intrinsic curvature of the boundary space

The renormalized on-shell gravity action

$$S_{re} \equiv \beta F = S_G + S_{GH} + S_{ct}$$

Energy-momentum tensor of the dual QFT

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{\gamma}}\frac{\delta F}{\delta\gamma^{\mu\nu}} = \frac{1}{8\pi G}\left[-\mathcal{K}_{\mu\nu}+\gamma_{\mu\nu}\mathcal{K}+\frac{3}{R}\gamma_{\mu\nu}-\frac{R}{2}\left(\mathcal{R}^{(4)}_{\mu\nu}-\frac{1}{2}\gamma_{\mu\nu}\mathcal{R}^{(4)}\right)\right]$$

Dual QFT of the AdS black hole for k=0

thermodynamic quantities derived from the holographic renormalization

$$
\begin{aligned}
F &= \frac{S_{re}}{\beta} = -\frac{V}{16\pi G R^2} m \\
E &= \int d^3x \sqrt{\gamma}\gamma^{00} T_{00} = \frac{3V}{16\pi G R^2} m \\
P^i &= -\int d^3x \sqrt{\gamma}\gamma^{ii} T_{ii} = \frac{V}{16\pi G R^2} m
\end{aligned}
$$

which perfectly matches with the black hole thermodynamics $(m = T_H^4)$.

- the above results consistent with the black hole thermodynamic law
- the thermal entropy exactly coincides with the Bekenstein-Hawking entropy

Linear response theory

In the low energy limit,

after applying a time varying source, we can investigate the linear response of the medium

$$\langle \mathcal{O} \rangle_{\mathrm{QFT}} = -\chi \partial_t \phi_0(t)$$

Transport coefficients

- typical parameters of an effective low energy description

- Once they are specified, they completely determine the macroscopic behavior of the medium

Ex) Transport coefficient of the electromagnetic theory

applying a time dependent vector potential in x-direction to conductor

$$J_x = \sigma \, \underline{\partial_t A_x(t)} \quad (= E_x)$$

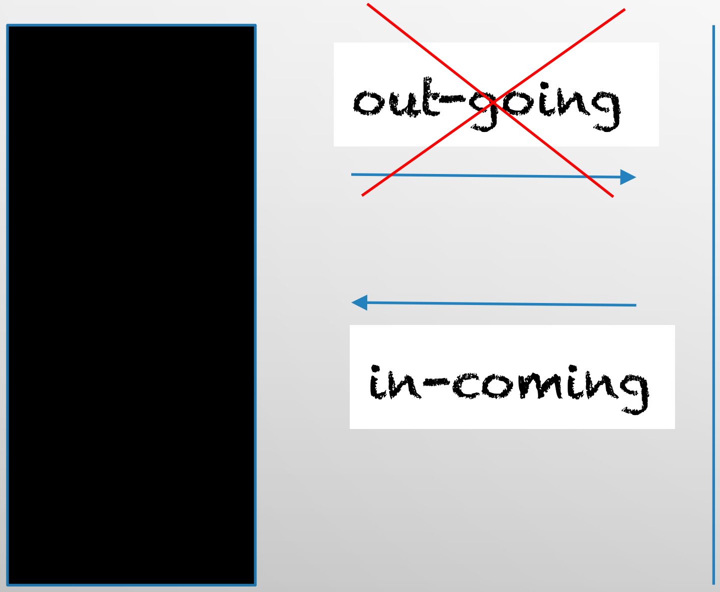
σ : conductivity

In general, the exact microscopic theory of the AdS/CFT correspondence is not known except several cases (N=4 SYM, ABJM, and their deformations).

In such a situation, macroscopic properties governed by thermodynamics and linear response theory do not rely on the microscopic details and become helpful to understand the dual QFT.

Two-point function at finite temperature

We can also apply the holographic technique to obtain a two-point function at zero temperature. In this case, it should be notee that the existence of a black hole requires a different boundary condition (incomng boundary condition) unlike the zero temperature case.



$$\Phi = \phi^{(in)} + \phi^{(out)}$$

with

$$\phi^{(out)} = \left(\phi^{(in)}\right)^{*}$$ (unitary)

In-coming boundary condition $\phi^{(out)} = 0$

breaks the unitarity (leads to dissipation)

Green function

$$G_{unitary} = G_{advanced} + G_{retarded}$$

$$G^R(\omega, \boldsymbol{q}) = -i \int d^4x \, e^{-iq\cdot x} \, \theta(t) \langle [\hat{\mathcal{O}}(x), \, \hat{\mathcal{O}}(0)] \rangle \, ,$$
$$G^A(\omega, \boldsymbol{q}) = i \int d^4x \, e^{-iq\cdot x} \, \theta(-t) \langle [\hat{\mathcal{O}}(x), \, \hat{\mathcal{O}}(0)] \rangle \, .$$

Transport coefficient

$$\chi = \frac{\mathrm{Im}\left[G_{retarded}\right]}{\omega} = \frac{1}{\omega} \langle \mathcal{O}\mathcal{O} \rangle$$

with

$$\langle \mathcal{O}\mathcal{O} \rangle = \frac{\partial^2 S_{on}}{\partial\phi_0 \partial\phi_0}$$

Transport coefficients of the dual QFT

Turn on the U(1) vector fluctuations on the AdS black hole background

$$S_M = -\frac{1}{4g_4^2}\int d^4x\sqrt{-g}F^{\mu\nu}F_{\mu\nu}$$

After taking the $A_r = 0$ gauge and the following Fourier mode expansions

$$A_i(t,\mathbf{x},r) = \int \frac{d\omega d^2q}{(2\pi)^3}e^{-i(\omega t-\mathbf{q}\cdot\mathbf{x})}A_i(\omega,\mathbf{q},r)$$

with the momentum only along y direction like $\mathbf{q} = (0,q)$

the equations of motion can be divided into two parts: the longitudinal and transverse one

- Longitudinal modes A_t and A_y

$$0 = b^2\omega A_t' + gqA_y'$$
$$0 = b^2 A_t'' + 2bb' A_t' - \frac{1}{g}(q\omega A_y + q^2 A_t)$$
$$0 = gA_y'' + g'A_y' + \frac{1}{g}(\omega q A_t + \omega^2 A_y),$$

$$\Rightarrow \quad 0 = A_t''' + \frac{F'(z)}{F(z)} A_t'' + \frac{\tilde{\Lambda}^2}{F^2(z)} \left[\tilde{\omega}^2 - F(z)\tilde{q}^2\right] A_t'$$

- Transverse mode A_x

$$0 = A_x'' + \frac{g'}{g} A_x' + \frac{1}{g^2}\left[\omega^2 - q^2 \frac{g}{b^2}\right] A_x.$$

In the hydrodynamic limit ($\tilde{\omega} << 1$ and $\tilde{q}^2 << 1$), we can solve these equations perturbatively

$$A_t'(z) = (1 - z^d)^\nu G(z),$$

$$G(z) = G_0(z) + \tilde{\omega} G_1(z) + \tilde{q}^2 G_2(z) + \tilde{\omega}\tilde{q}^2 G_3(z) + \cdots$$

Boundary conditions

- Incoming BC at the horizon, which breaks the unitarity of the dual QFT.
- Dirichlet BC at the asymptotic boundary, which fixes the source of dual QFT.

The retarded Green function of longitudinal modes

in the low frequency and low momentum limit (hydrodynamic limit),
the retarded Green functions become

$$\mathcal{G}_{tt} = -\frac{1}{g_4^2}\left[\frac{q^2}{i\omega - \left(\frac{\tilde{\Lambda}}{T_H}\right)q^2}\right],$$

$$\mathcal{G}_{yy} = -\frac{1}{g_4^2}\left[\frac{\omega^2}{i\omega - \left(\frac{\tilde{\Lambda}}{T_H}\right)q^2}\right],$$

$$\mathcal{G}_{ty} = \mathcal{G}_{yt} = -\frac{1}{g_4^2}\left[\frac{\omega q}{i\omega - \left(\frac{\tilde{\Lambda}}{T_H}\right)q^2}\right],$$

The longitudinal Green function has a charge diffusive pole governed by the following dispersion relation

$$\omega = -iDk^2, \qquad k = k_z$$

with

$$D = \frac{\tilde{\Lambda}}{T_H} = \frac{(-\Lambda)}{16\pi T_H}\frac{(4+\eta^2)^2}{4-\eta^2}.$$

the charge diffusion constant

- The charge diffusion constant implies that the quasi normal mode (charge current of the dual QFT) eventually diffuses away back into the thermal equilibrium with a half-life time

$$t_{1/2} = \frac{1}{D\, q^2}$$

The retarded Green function of transverse mode

$$\mathcal{G}_{xx} = \frac{1}{g_4^2}\left[i\omega - \left(\frac{\tilde{\Lambda}}{T_H}\right) q^2\right]$$

- There is no pole.
- The DC conductivity is given by

$$\sigma = \lim_{\omega \to 0} \mathrm{Re}\left(\frac{\mathcal{G}_{xx}}{i\omega}\right) = \frac{1}{g_4^2}$$

Important holographic results

In RHIC experiment, quark-gluon plasma showed a very small ratio between the shear viscosity and thermal entropy.

Intriguingly, the holographic calculation leads to such a small ratio

Ratio between Shear viscosity and entropby

$$\frac{\eta}{S} = \frac{1}{4\pi}$$

Conclusions

- The AdS/CFT correspondence provides a new way to understand a strongly interacting QFT.

- Although the microscopic details of the dual QFT are not known, many macroscopic properties can be understood by the holographic methods.

- It would be helpful to understand various macroscopic properties of the real physical systems in nuclear and condensed matter physics.

THANK YOU FOR YOUR ATTENTION!