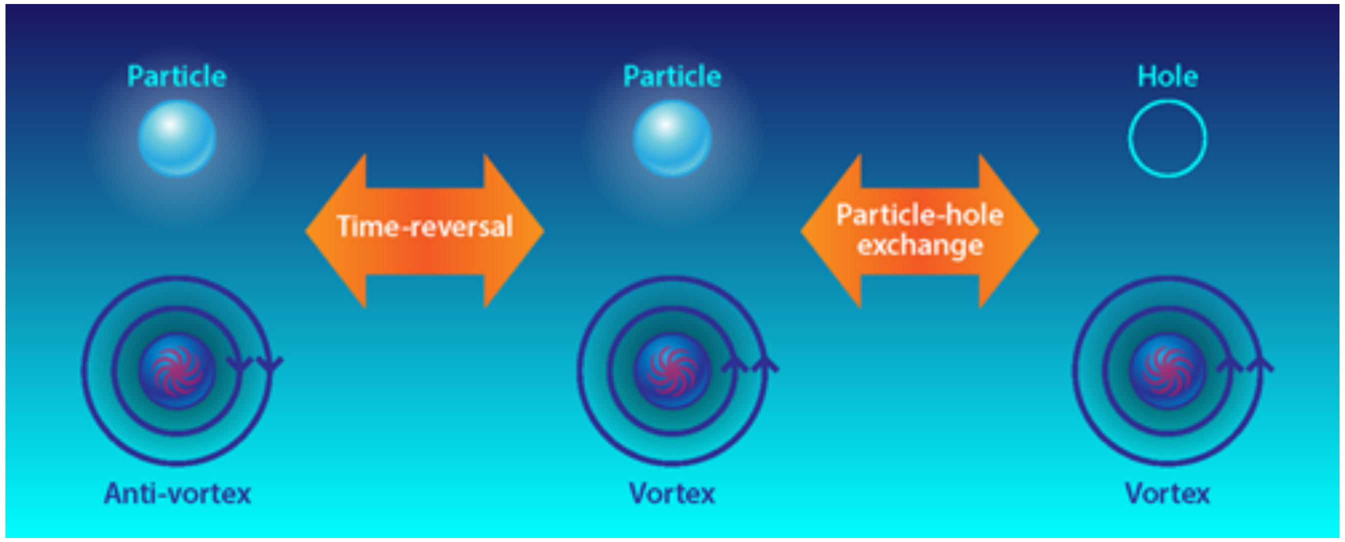
Bosonization and Mirror Symmetry



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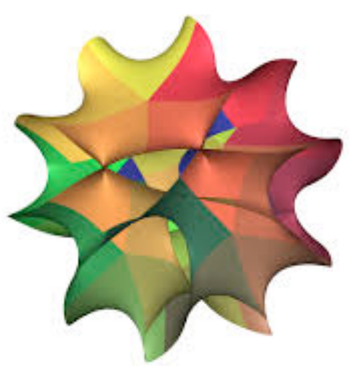
(based on arXiv:1608.05077 with Mulligan, Torroba, Wang; see also our earlier paper arXiv:1506.01376)

I. Introduction

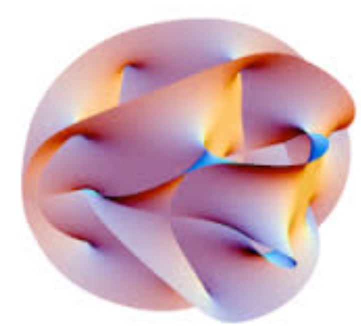
Duality in quantum field theory has been a powerful tool in many contexts.

Since the mid-1990s, studies of duality in particle theory focused on supersymmetric QFTs.

* mirror symmetry of 2d sigma models



A Calabi-Yau
and its mirror



* Seiberg duality of 4d N=1 gauge theories

$$SU(N_c),\ N_f \text{ flavors} \leftrightarrow\ SU(N_f - N_c)\ N_f \text{ flavors} + \text{meson}$$

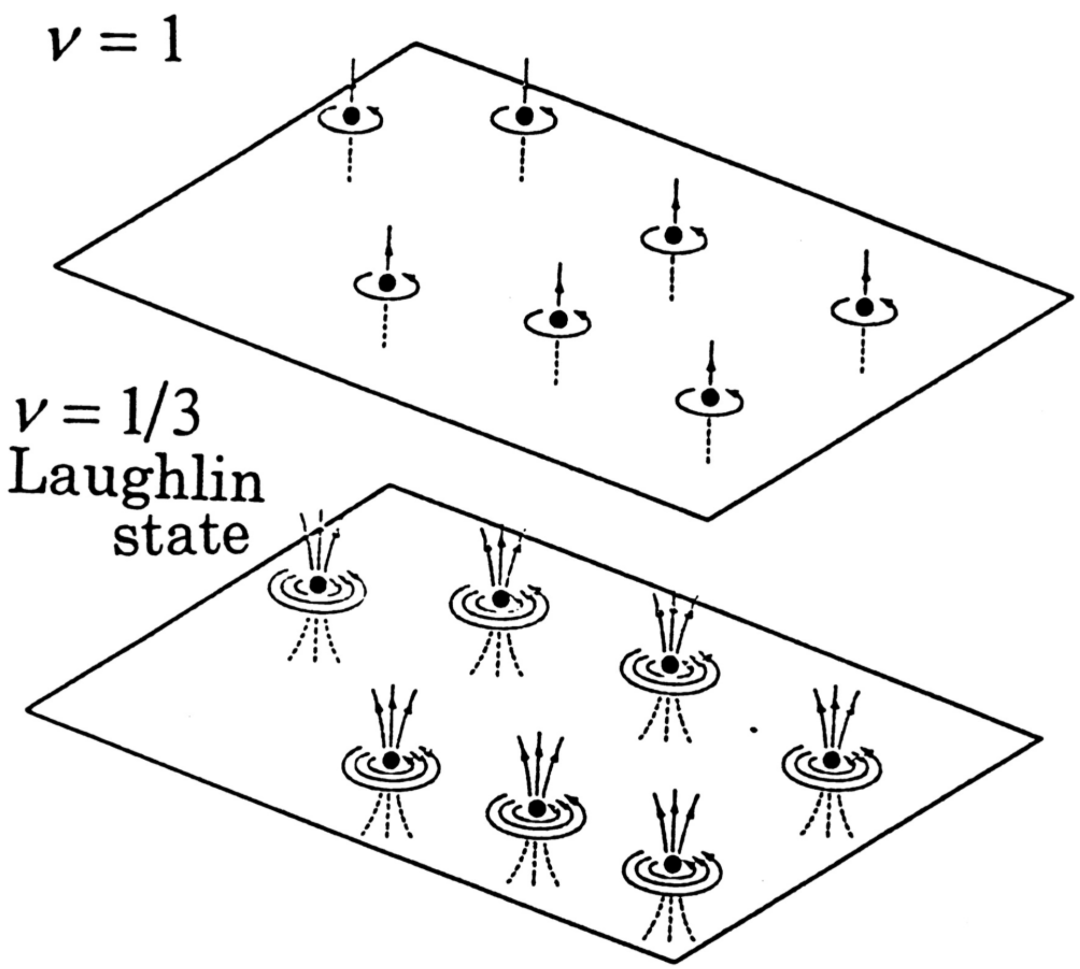
However, the subject predates fancy constructions in string theory and supersymmetric QFT.

Kramers-Wannier duality of the Ising model and many other examples arise in condensed matter physics. (More relevant to talk: bosonization/Luttinger liquid).

Today, I'll give a very brief description of a derivation of a duality of interest in condensed matter, starting from a classic example of mid 1990s supersymmetric dualities.

The duality I'll be focusing on relates a theory of free fermions to a scalar QED3 theory.

One place where such dualities may be relevant is in understanding the fractional quantum Hall effect.



Electrons moving in a magnetic field can be dressed by a certain number of flux quanta each. The resulting quasiparticle sees different flux density and in general has different statistics, which can be helpful.

* Can map FQHE to IQHE
* Can map 1/2 - filled LL to (non)-Fermi liquid (?)

In fact our work was motivated by duality conjectures made roughly in that context, by D.T. Son and by Metlitski, Senthil, Vishwanath, and Wang.

c.f. Senthil, Seiberg, Witten, Wang; Karch, Tong

II. Mirror symmetry of 3d N=4 gauge theories

We will start with a well studied and (fairly) rigorously understood duality from high-energy physics, the mirror symmetry of 3d N=4 gauge theories.

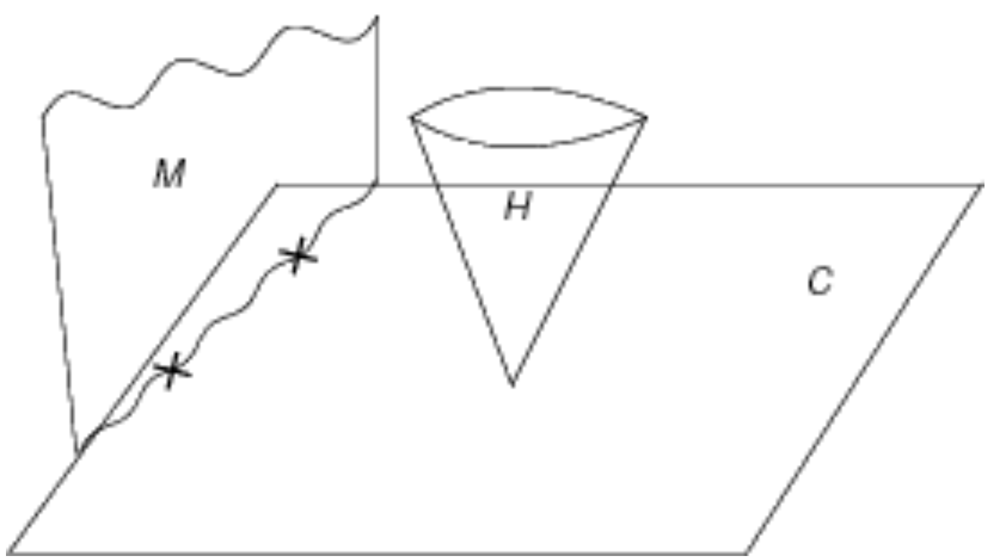
These can be thought of as the dimensional reductions of (perhaps more familiar) 4d N=2 theories.

The two types of supermultiplets that arise are:

$$\text{Vectormultiplet} : (A_\mu, \sigma, \phi), (\lambda, \psi_\phi)$$

$$\text{Hypermultiplet} : (q, \psi_q), (\tilde{q}, \psi_{\tilde{q}})$$

N=4 theories have moduli spaces of vacua whose basic structure is:



Higgs and Coulomb branches, which are hyperKahler manifolds. We will not have mixed branches today.

Mirror symmetry is a symmetry of pairs of 3d N=4 gauge theories A and B, where

$$\mathrm{Higgs}(A) = \mathrm{Coulomb}(B)$$

$$\mathrm{Higgs}(B) = \mathrm{Coulomb}(A)$$

Intriligator, Seiberg

Example:

We will be satisfied today with using just the simplest, prototypical example of 3d mirror symmetry.

Theory A: Free hypermultiplet
Theory B: QED with one charged hyper

The Lagrangian of theory A is

$$L^{(\mathrm{A})} = \sum_{\pm} \left(|\partial_\mu v_\pm|^2 + i\bar{\Psi}_\pm \, \partial\!\!\!/ \Psi_\pm \right)$$

The +/- labels the charge under an important symmetry. A table of the fields and their transformations under the various symmetries is:

	$U(1)_R$	$U(1)_A$	$U(1)_J$
v_+	1	-1	1
v_-	1	-1	-1
Ψ_+	0	-1	1
Ψ_-	0	-1	-1

(there are really two SU(2)s, but only the Cartan properties matter for us)

The Lagrangian and table of symmetries of theory B:

$$L^{(\mathrm{B})} = L_V(\mathcal{V}) + L_H(\mathcal{Q},\mathcal{V}),$$

$$L_V(\mathcal{V}) = \frac{1}{g^2}\left(-\frac{1}{4}f_{\mu\nu}^2 + \frac{1}{2}(\partial_\mu\phi_{ij})^2 + i\bar{\lambda}_{ia}\ \partial\!\!\!/\lambda_{ia} + \frac{1}{2}D_{(ab)}^2\right)$$

$$L_H(\mathcal{Q},\mathcal{V}) = |D_\mu u_a|^2 + i\bar{\psi}_i\ D\!\!\!\!/\psi_i - \phi_{ij}^2|u_a|^2 - \phi_{ij}\bar{\psi}_i\psi_j + \sqrt{2}(i\lambda_{ia}u_a^*\psi_i + \mathrm{h.c.}) + D_{(ab)}u_a^* u_b\,.$$

	$U(1)_R$	$U(1)_A$	$U(1)_J$	$U(1)_a$
u_+	0	1	0	1
u_-	0	1	0	-1
ψ_+	-1	1	0	1
ψ_-	-1	1	0	-1
$e^{2\pi i\gamma/g^2}$	0	0	1	0
σ	0	0	0	0
ϕ	2	-2	0	0
λ	1	0	0	0
ψ_ϕ	1	-2	0	0

$$f_{\mu\nu} \equiv \epsilon_{\mu\nu\rho}\partial^\rho\gamma$$

$$J_\mu^{U(1)_J} \equiv \frac{1}{2\pi}\partial_{\mu\nu\rho}\partial^\nu a^\rho$$

"topological current"
vortices carry charge

Slogan: following $U(1)_J$ charge, particles of theory A are vortices of theory B.

The moduli spaces are quite simple:

The tree level moduli space in theory A is $\mathbb{R}^4$.
As the theory is free, there are no corrections.

On the other hand, theory B has a Coulomb branch parametrized by σ, ϕ and the dual photon. Its geometry receives quantum corrections.

In the IR limit where $g \to \infty$, there is a symmetry exchanging these two moduli spaces of vacua.

More formal formulation:

We can promote the topological U(1) to a full background vector multiplet.

$$\mathcal{L}^{(A)}(\mathcal{Q},\hat{\mathcal{V}}_J)=\mathcal{L}^{\mathcal{H}}(\mathcal{Q},\hat{\mathcal{V}}_J)=\int d^4\theta\left(V_+^\dagger e^{2\hat{V}_J}V_++V_-^\dagger e^{-2\hat{V}_J}V_-\right)+\int d^2\theta\,\sqrt{2}i\hat{\Phi}_JV_+V_-+\text{h.c.}$$

In theory B, it enters subtly through a BF term:

$$\mathcal{L}^{(B)}(\mathcal{U},\mathcal{V},\hat{\mathcal{V}}_J)=\mathcal{L}^{\mathcal{V}}(\mathcal{V})+\mathcal{L}^{\mathcal{H}}(\mathcal{U},\mathcal{V})-\mathcal{L}_{BF}^{\mathcal{N}=4}(\mathcal{V},\hat{\mathcal{V}}_J)\,.$$

$$\mathcal{L}_{BF}^{\mathcal{N}=4}(\mathcal{V}^{(1)},\mathcal{V}^{(2)})=\frac{1}{2\pi}\int d^4\theta\,V^{(1)}\,\Sigma^{(2)}-\frac{1}{2\pi}\int d^2\theta\,\Phi^{(1)}\Phi^{(2)}+\text{h.c.}$$

$$\begin{aligned}\mathcal{L}_{BF}^{\mathcal{N}=2}(V^{(1)},V^{(2)})&=\frac{1}{2\pi}\int d^4\theta\,V^{(1)}\,\Sigma^{(2)}\\&=\frac{1}{2\pi}\left(\epsilon^{\mu\nu\rho}A_\mu^{(1)}\partial_\nu A_\rho^{(2)}+D^{(1)}\sigma^{(2)}+D^{(2)}\sigma^{(1)}+\frac{1}{2}(\bar{\lambda}^{(1)}\lambda^{(2)}+\bar{\lambda}^{(2)}\lambda^{(1)})\right)\end{aligned}$$

In this fancier formulation, the formal statement of mirror symmetry is that

$$Z^{(A)}[\hat{\mathcal{V}}_J] = Z^{(B)}[\hat{\mathcal{V}}_J]\,.$$

Our plan now is to consider what happens when we similarly promote the other U(1) symmetries.

We can then consider perturbations by background values of their $\hat{\sigma}, \hat{D}$ fields.

This will lead to supersymmetry breaking, and allow us to infer a non-supersymmetric duality.

III. Perturbations to the basic N=4 duality

A. Promoting other global symmetries

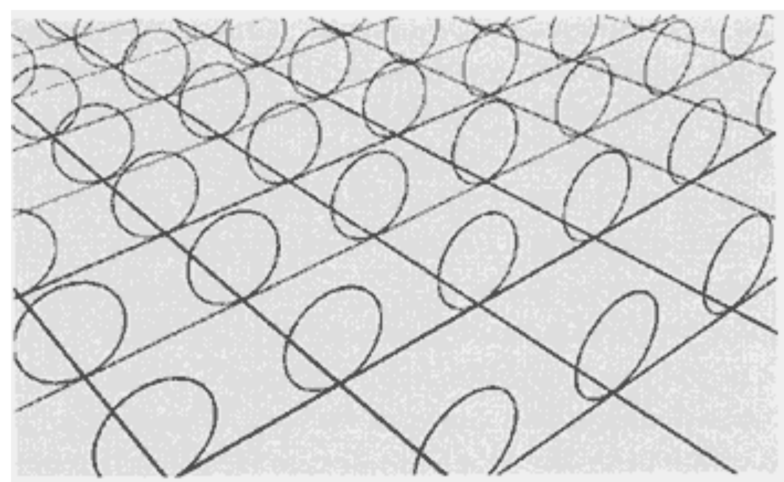
Using the tables of charges, it is straightforward to see what happens when we promote the $U(1)_A$ symmetry to a full background multiplet $\hat{V}_A$:

$$\mathcal{L}^{(A)}(\mathcal{Q}, \hat{V}_A) = \int d^4\theta \left(V_+^\dagger e^{2(\hat{V} - \hat{V}_A)} V_+ + V_-^\dagger e^{-2(\hat{V} + \hat{V}_A)} V_- \right)$$

$$\begin{aligned}\mathcal{L}^{(B)}(\mathcal{U}, \mathcal{V}, \hat{V}_A) = & \frac{1}{4g^2} \int d^2\theta\, W_\alpha^2 + \text{h.c.} + \frac{1}{g^2} \int d^4\theta \Phi^\dagger e^{-4\hat{V}_A} \Phi \\ & + \int d^4\theta \left(U_+^\dagger e^{2(V + \hat{V}_A)} U_+ + U_-^\dagger e^{-2(V - \hat{V}_A)} U_- \right) - \frac{1}{2\pi} \int d^4\theta\, V\, \hat{\Sigma}\,.\end{aligned}$$

Promoting the R-symmetry is a bit harder, because it does not commute with supersymmetry. However, the basic elements of the map are easy to infer:

* $\hat{A}_R$ couples to j_R on both sides of the duality



* The coupling to $\hat{\sigma}_R$ is more subtle. Think of the 3d theory as a dimensional reduction of a 4d theory. Then this field is the 4th component of the gauge field, and couples to the 4th component of the appropriate current.

With this understanding, the mirror duality is promoted to:

$$Z^{(A)}[\hat{V}_J, \hat{V}_A, \hat{V}_R] = Z^{(B)}[\hat{V}_J, \hat{V}_A, \hat{V}_R]\,.$$

B. First step: a simpler N=2 theory

As the first step on our road to a non-supersymmetric duality, lets break the N=4 to an N=2 theory with a single chiral multiplet. Looking at the charges:

	$U(1)_R$	$U(1)_A$	$U(1)_J$
v_+	1	-1	1
v_-	1	-1	-1
Ψ_+	0	-1	1
Ψ_-	0	-1	-1

we see it will be interesting to consider a perturbation of the form

$$|\hat{\sigma}_A - \hat{\sigma}_J| \ll \hat{\sigma}_A \sim \hat{\sigma}_J.$$

Since we know that σ fields coupled to charged scalars via:

$$\sigma^2 q^2 |\phi|^2$$

type couplings, this will give a large mass to v_- (and its superpartner). The remaining light fields & charges:

	$U(1)_R$	$U(1)_A$	$U(1)_J$
V_+	1	-1	1
v_+	1	-1	1
Ψ_+	0	-1	1

The resulting theory A Lagrangian for light fields is:

$$\mathcal{L}^{(A)}_{\text{chiral}} = |D_{\hat{A}_J - \hat{A}_A + \hat{A}_R} v_+|^2 - \left((\hat{\sigma}_J - \hat{\sigma}_A + \hat{\sigma}_R)^2 + \hat{D}_J - \hat{D}_A \right) |v_+|^2$$

$$+ i\bar{\Psi}_+ \not{D}_{\hat{A}_J - \hat{A}_A} \Psi_+ - (\hat{\sigma}_J - \hat{\sigma}_A)\bar{\Psi}_+\Psi_+ + \frac{1}{8\pi} k^{(A)}_{MN} \hat{A}_M d\hat{A}_N,$$

Here,

$$\hat{A}_M = (\hat{A}_J, \hat{A}_A, \hat{A}_R)$$

$$k^{(A)}_{MN} = \text{sgn}(\hat{\sigma}^0_A) \begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The k-matrix arises from integrating out the fermion Ψ_- ; in general integrating out a fermion induces a change in the k-matrix

$$k_{ij} = \frac{1}{2} \sum_f q_i^f q_j^f \,\mathrm{sgn}(m_f)$$

Under further perturbations, vacuum stability of this theory will require

$$m^2_{v_+} = (\hat{\sigma}_J - \hat{\sigma}_A + \hat{\sigma}_R)^2 + \hat{D}_J - \hat{D}_A \geq 0.$$

Otherwise, v_+ will condense and run away to the cutoff.

We can map over the perturbation to theory B as well. The analysis is a bit more involved, but in the end the light fields are the u_-, ψ_- multiplet along with a (shifted) sigma field. The effective Lagrangian is:

$$\begin{aligned}\mathcal{L}^{(B)}_{\text{chiral}} &= \frac{1}{2g^2_{\text{eff}}}\left((\partial\tilde{\sigma})^2 + D^2\right) + |D_{-a+\hat{A}_A}u_-|^2 + \bar{\psi}_- i\not{D}_{-a+\hat{A}_A-\hat{A}_R}\psi_- - ((\tilde{\sigma}-\hat{\sigma}_A)^2 - D + \hat{D}_A)|u_-|^2 \\ &- 8\pi\bar{\psi}_-\psi_-|u_-|^2 - (-\tilde{\sigma}+\hat{\sigma}_A-\hat{\sigma}_R)\bar{\psi}_-\psi_- + \frac{1}{8\pi}(a+\hat{A}_A-\hat{A}_R)d(a+\hat{A}_A-\hat{A}_R) \\ &+ \frac{1}{4\pi}(\tilde{\sigma}+\hat{\sigma}_A)(D+\hat{D}_A) - \frac{1}{8\pi}\hat{A}_R d\hat{A}_R - \frac{1}{8\pi}(2\hat{A}_A-\hat{A}_R)d(2\hat{A}_A-\hat{A}_R) \\ &- \frac{1}{2\pi}\left(\hat{A}_J da + \hat{D}_J\tilde{\sigma} + D\hat{\sigma}_J\right) . \qquad (3.16)\end{aligned}$$

The duality we've exhibited here is one between a single free chiral N=2 superfield, and N=2 QED with a charged chiral multiplet.

C. Breaking supersymmetry

The final step is to break supersymmetry. We do this by turning on a $\hat{D}_J$ term.

In theory A, as v_+ carries positive topological charge, this results in a massless Fermi theory:

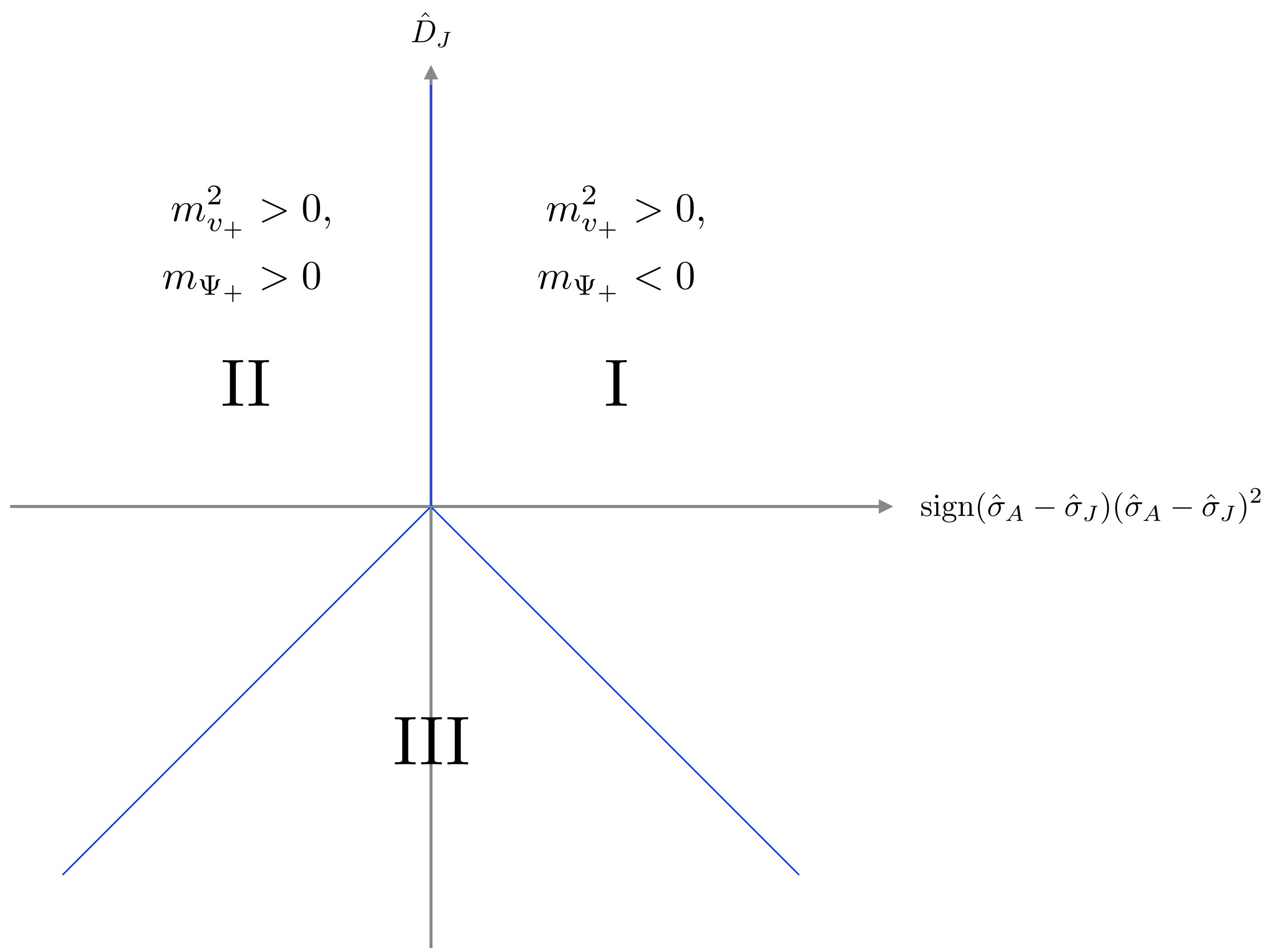
$$\hat{\mathcal{L}}^{(A)}_{\text{Dirac}} = \bar{\Psi}_+ i \not{D}_{\hat{A}_J - \hat{A}_A} \Psi_+ - m_{\Psi_+} \bar{\Psi}_+ \Psi_+ + \frac{k^{\text{crit}}_{MN}}{8\pi} \hat{A}_M d\hat{A}_N$$

$$k^{\text{crit}}_{MN} = \begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We can consider varying away from the critical point by slightly detuning $\hat{\sigma}_A - \hat{\sigma}_J$, giving a mass to the fermion. The resulting k-matrices are:

$$k^{(A)}_{MN} = \begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mathrm{sgn}(\hat{\sigma}_J - \hat{\sigma}_A) \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

Phase diagram:



Now, we can consider the same perturbation to theory B. This theory is strongly coupled. But we can use macroscopic considerations:

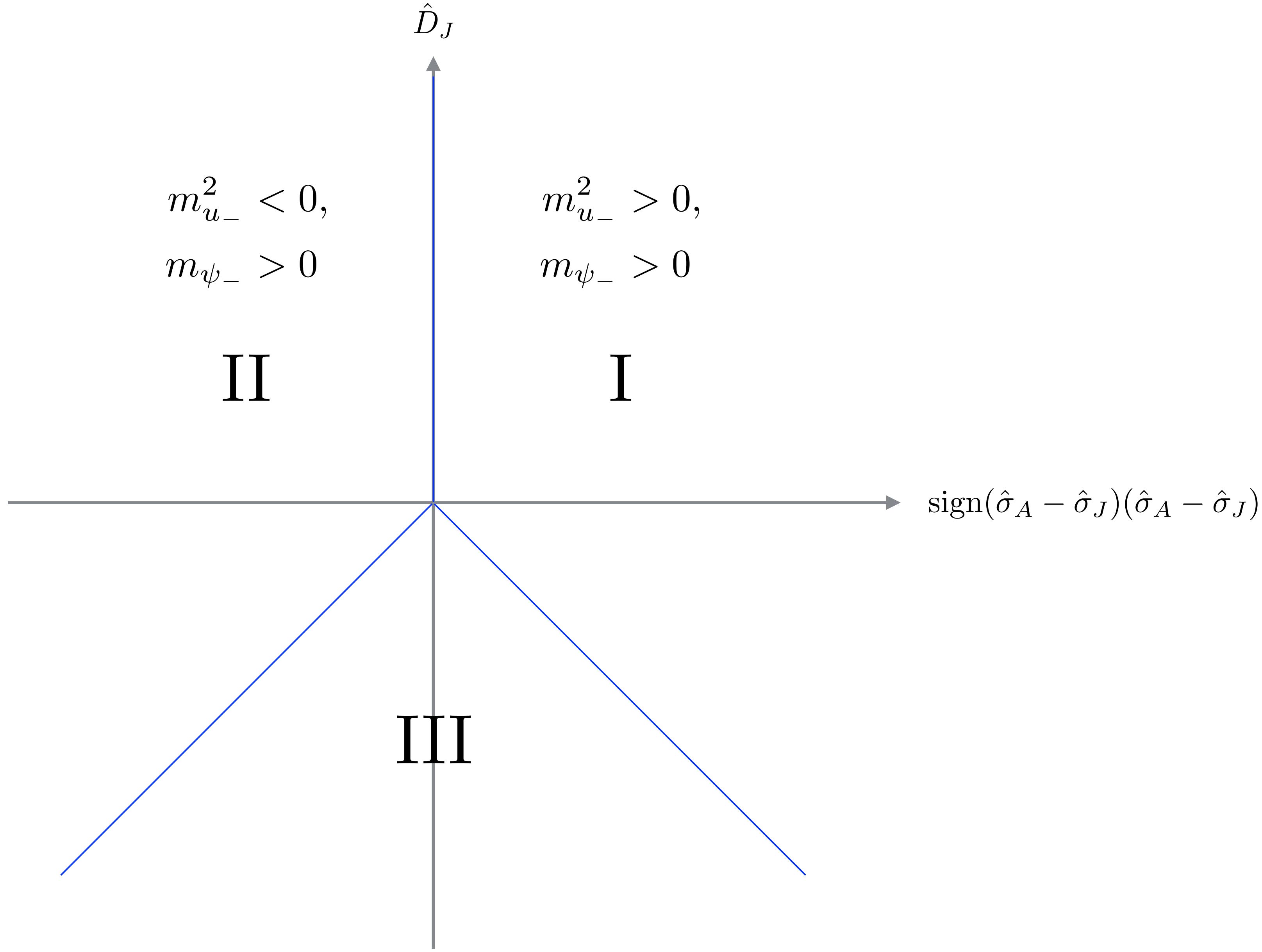
* Duality implies that there is a single critical point as we vary $\hat{\sigma}_A - \hat{\sigma}_J$ in the small range we consider.

* A simple calculation tells one that the k-matrix is:

$$\mathcal{L}_{\rm CS}^{(B)} = \frac{1}{8\pi}{\rm sgn}(m_{\psi_-})(-a+\hat{A}_A-\hat{A}_R)d(-a+\hat{A}_A-\hat{A}_R) - \Theta(-m_{u_-}^2)(-a+\hat{A}_A)^2$$
$$+\frac{1}{8\pi}\Big[(a+\hat{A}_A-\hat{A}_R)d(a+\hat{A}_A-\hat{A}_R) - 4\hat{A}_A d\hat{A}_A + 4\hat{A}_A d\hat{A}_R - 2\hat{A}_R d\hat{A}_R - 4\hat{A}_J da\Big]$$

* Dual theories must have matching k-matrices!

The only reasonably parsimonious phase diagram for theory B is then



The resulting critical theory has:

$$\mathcal{L}^{(B)}_{\mathrm{sQED3}} = |D_{-a+A_A} u_-|^2 - m^2_{u_-}|u_-|^2 - \lambda_{u_-}|u_-|^4 + \frac{1}{4\pi} ada - \frac{1}{2\pi}\hat{A}_J da - \frac{1}{4\pi}\hat{A}_A d\hat{A}_A.$$

Setting some of the background fields

$$\hat{A}_A = \hat{A}_R = 0$$

and doing some re-naming, we have shown:

$$\bar{\Psi} i \not{D}_{\hat{A}} \Psi - \frac{1}{8\pi}\hat{A}d\hat{A} \leftrightarrow |D_{-a}\varphi|^2 - |\varphi|^4 + \frac{1}{4\pi} ada - \frac{1}{2\pi}\hat{A}da$$

This is the promised duality between a free Fermi theory and scalar QED3.

As argued by Senthil, Seiberg, Witten and Wang, starting from this duality, and using SL(2,Z) arguments, one can derive several further dualities:

$$|D_{\hat{A}}\phi|^2 - |\phi|^4 + \frac{1}{4\pi}\hat{A}d\hat{A} \leftrightarrow \bar{\psi}i\not{D}_a\psi - \frac{1}{8\pi}ada - \frac{1}{2\pi}\hat{A}da.$$

$$|D_{\hat{A}}\phi|^2 - |\phi|^4 \leftrightarrow |D_a\varphi|^2 - |\varphi|^4 - \frac{1}{2\pi}\hat{A}da,$$

$$\bar{\Psi}i\not{D}_{\hat{A}}\Psi - \frac{1}{8\pi}\hat{A}d\hat{A} \leftrightarrow \bar{\Psi}i\not{D}_{-a}\Psi + \frac{1}{8\pi}ada + \frac{1}{2\pi}bda + \frac{2}{4\pi}bdb - \frac{1}{2\pi}\hat{A}db\,.$$

Understanding the derivation of any one is therefore sufficient.