Classification of Tensor Network State : Symmetry and Topology

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Outline

I. Introduction

- Topological Phases in Condensed Matter Physics
- Tensor Network and Tensor Network States : MPS, PEPS

II. Classification of TNS

- Matrix Product States : SPT
- Projected Entangled Pair States : Z2 Topological Order

III. Summary & Outlook

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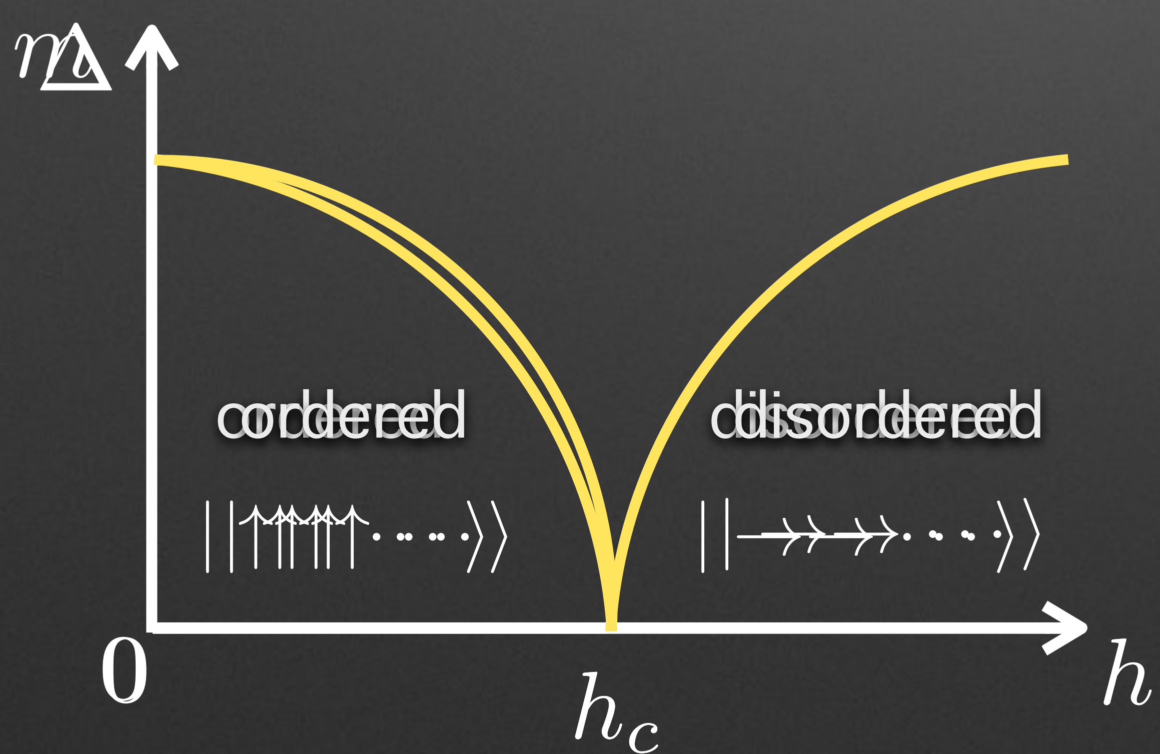
❖ Quantum Phases and Quantum Phase Transitions

- Classification of QPs is one of main object in Condensed Matter Physics
  - Why? Many different models & materials show essentially the same behavior
  - Example : Transverse Field Ising Model (TFIM)

➤ $$H = -g \sum_{i=1}^{N} \sigma_i^x - \sum_{i=1}^{N} \sigma_i^z \sigma_{i+1}^z$$

Fluctuation Ordering

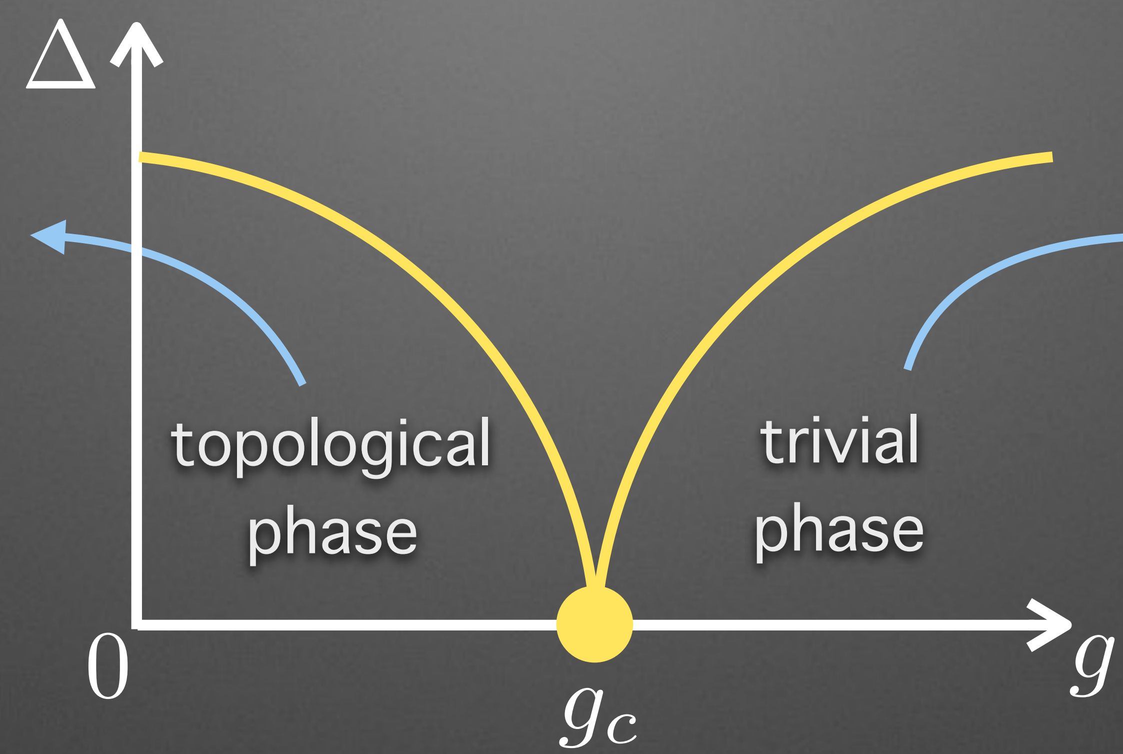


- continuous QPT between spontaneously symmetry broken (SSB) ⇔ disordered phases
- the most characteristic case of the quantum phase transition
  - → so-called “conventional QPT”
  - → Ginzburg-Landau theory
- one can classify the QPs with a pair of mathematical objects (G_H, G_Ψ)
  - ex.) TFIM $(Z_2, \{1\})$

◆ Topological Phases

➤ People found some examples of QPs and QPTs which cannot be understood by the conventional QPT : no SSB of any symmetry

- No SSB, no local order, but still distinguishable!
- cannot be adiabatically connected to a product state



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  ex) $|\psi\rangle = |000\cdots\rangle$

➤ Topological phases are divided into 2 categories

① Symmetry Protected Topological phase (SPT)

→ Distinct only in the presence of symmetry
→ Gapped but gapless mode at edge. e.g.) Topological Insulator, Haldane phase, etc

② Topologically Ordered phase

❖ Quantum Phases and Quantum Phase Transitions

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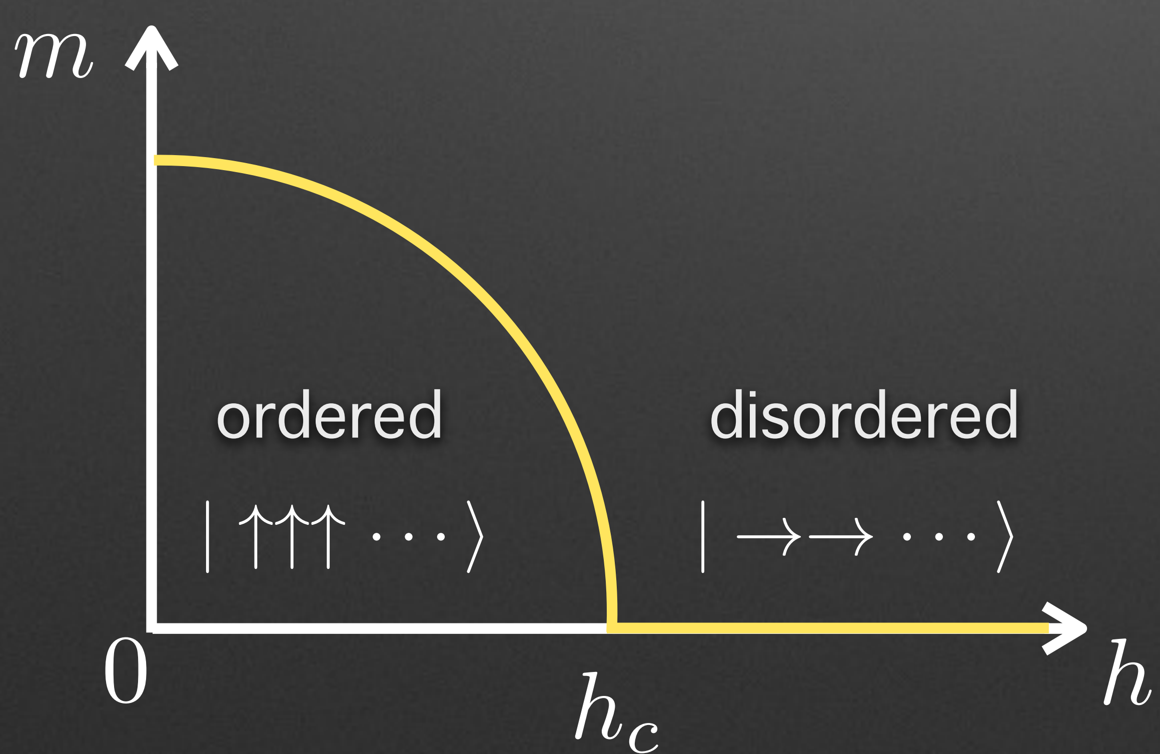

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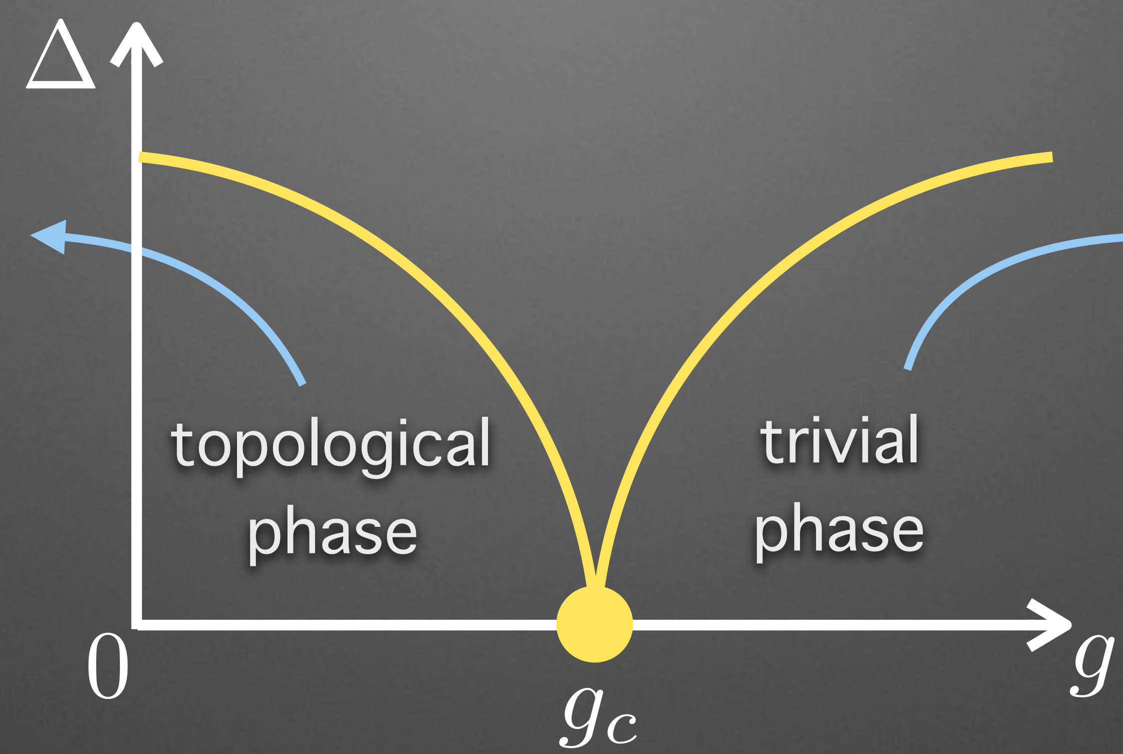

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② Topologically Ordered phase

→ Non-trivial without symmetry
→ Ground state (topological) degeneracy. e.g.) Z2 spin liquid, FQH, etc

- 1 Dimensional System

  - There is no topologically ordered phase in 1D
    : any ground state can be adiabatically connected to a product state, if no symmetry is imposed

  - SPT phases do exist in 1D
    : we want to classify and construct them!

  - Any gapped G.S. in 1D can be exactly described by a Matrix Product State (MPS)
    : classification of SPT in 1D → classification of MPS in the presence of symmetry

- 2 Dimensional System

  - Topologically ordered states do exist
    : we want to construct them using Projected Entangled Pair States (PEPS)!

  - SPT phase do exist in 2D
    : The way how we describe SPT with 2D TNS has been recently developed

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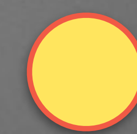
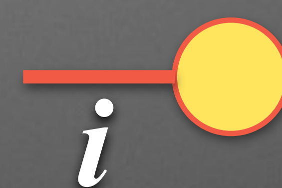
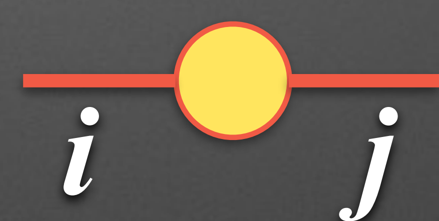
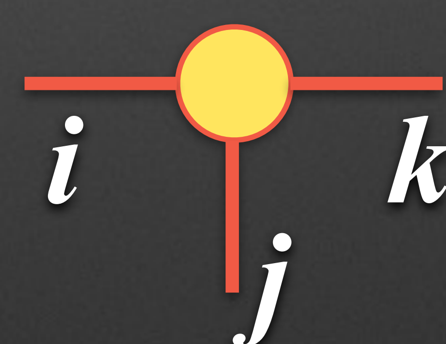
◆ Graphical representation

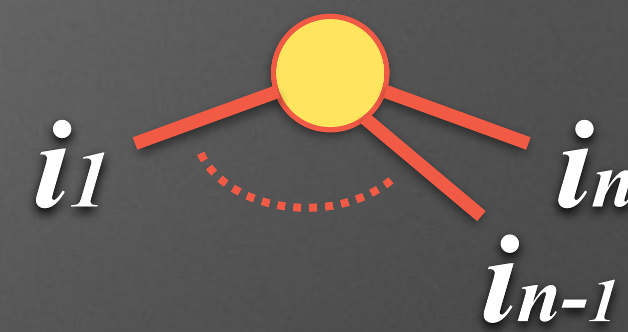
- rank 0 (scalar) : $C =$ 

- rank 1 (vector) : $v_i =$ 

- rank 2 (matrix) : $M_{ij} =$ 

- rank 3 : $T_{ijk} =$ 

rank *n*



◆ Basic Operation (1) - Contraction

- Matrix product : $C = AB \longrightarrow C_{ij} = \sum_k A_{ik} B_{kj}$

TN

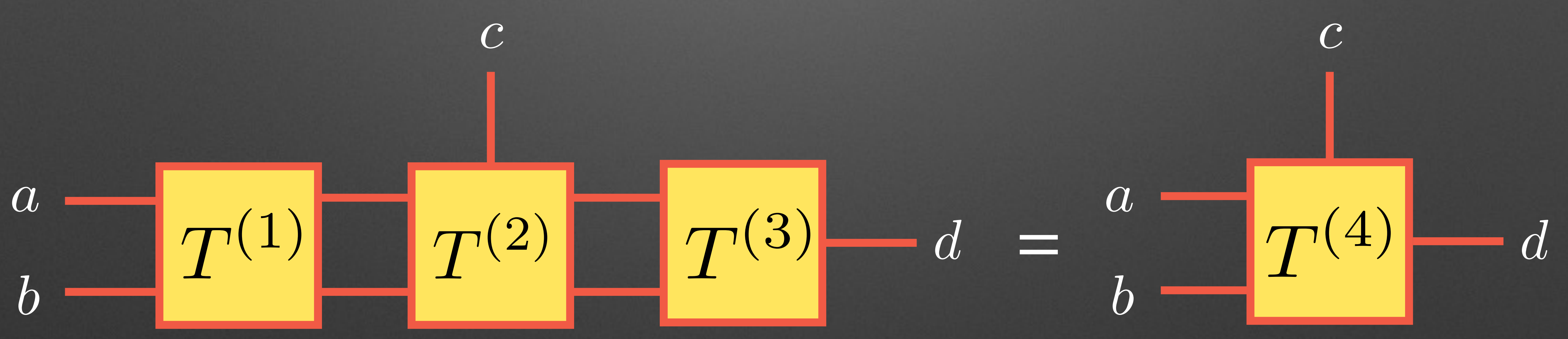
- Contraction : Sum over connected legs

e.g.)

Basic Operation (1) - Contraction

- Products of multiple high rank tensors :

e.g.) $$\sum_{i,j,k,l} T^{(1)}_{abij} T^{(2)}_{ijckl} T^{(3)}_{dkl} = T^{(4)}_{abcd}$$ Already stressful…

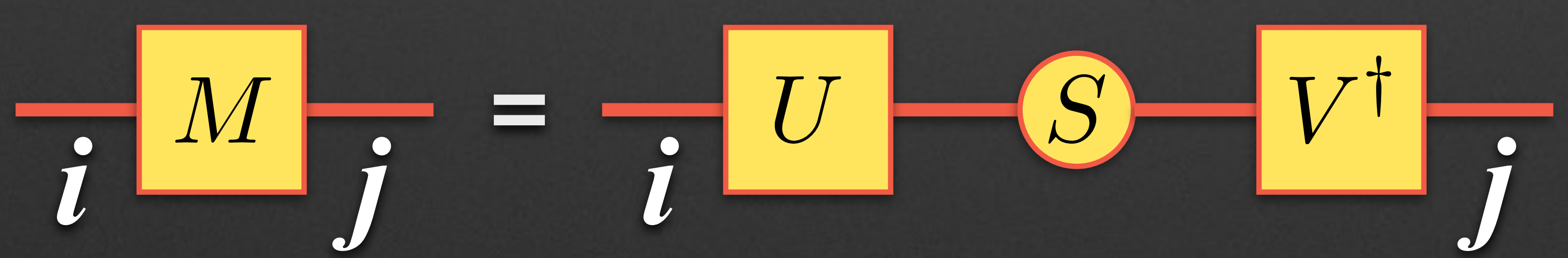


After getting used to graphical representation,
complex network of tensors are intuitively understandable!

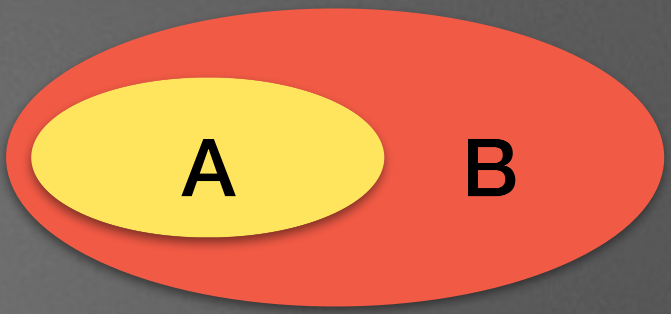
◆ Basic Operation (2) - Singular Value Decomposition

- $\underset{(m \times n)}{M_{ij}} = U_{ik} S_k V^{\dagger}_{kj}$ $\quad U, V$: unitary matrix

$$= \underset{(m \times m)}{\begin{bmatrix} & & \\ & U & \\ & & \end{bmatrix}} \underset{(m \times n)}{\left[\begin{array}{ccc|c} \sigma_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \sigma_\chi & \\ & & 0 \; \ddots & \end{array}\right]} \underset{(n \times n)}{\begin{bmatrix} & & \\ & V^{\dagger} & \\ & & \end{bmatrix}}$$

- 

Quantum Entanglement



- $$|\psi\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}|i^A j^B\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{\chi} U_{ik} S_k V_{kj}^{\dagger} |i^A\rangle \otimes |j^B\rangle$$

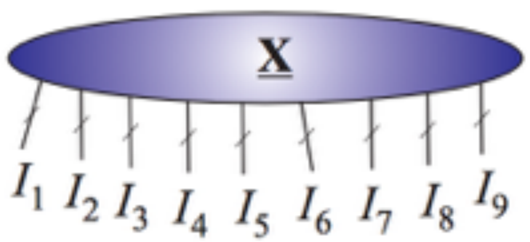
$$= \sum_{k=1}^{\chi} S_k \left(\sum_{i=1}^{m} U_{ik} |i^A\rangle \right) \otimes \left(\sum_{i=1}^{m} V_{kj}^{\dagger} |j^B\rangle \right)$$

$$= \sum_{k=1}^{\chi} S_k |\phi_k^A\rangle \otimes |\phi_k^B\rangle$$ Schmidt decomposition!

- $\begin{cases} (S_k)^2 & \text{: Entanglement Spectrum} \\ S_{\mathrm{vN}} \equiv -\sum_{k=1}^{\chi} (S_k)^2 \log (S_k)^2 & \text{: Entanglement Entropy} \end{cases}$

Tensor Network

- What is TN?

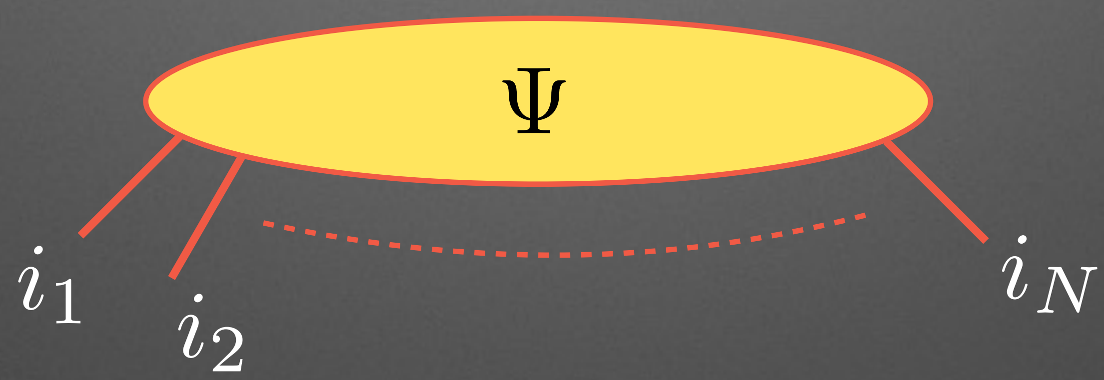


Specific form of TN depends on property of original tensor!

◆ Connection between TN & Many-body state

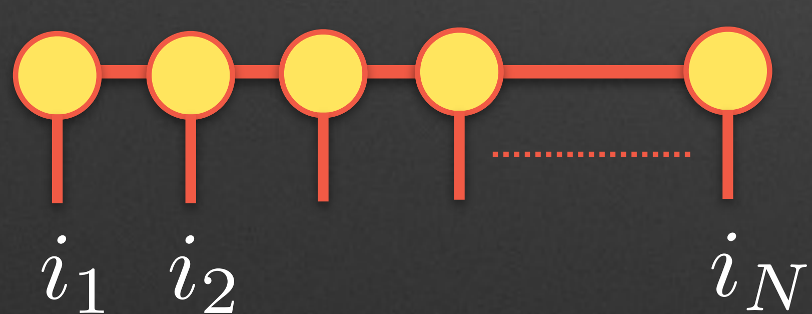
- $|\psi\rangle = \sum_{i_1,\cdots,i_N} \Psi_{i_1,i_2,\cdots,i_N} |i_1, i_2, \cdots, i_N\rangle$

- $\Psi_{i_1,\cdots,i_N} =$

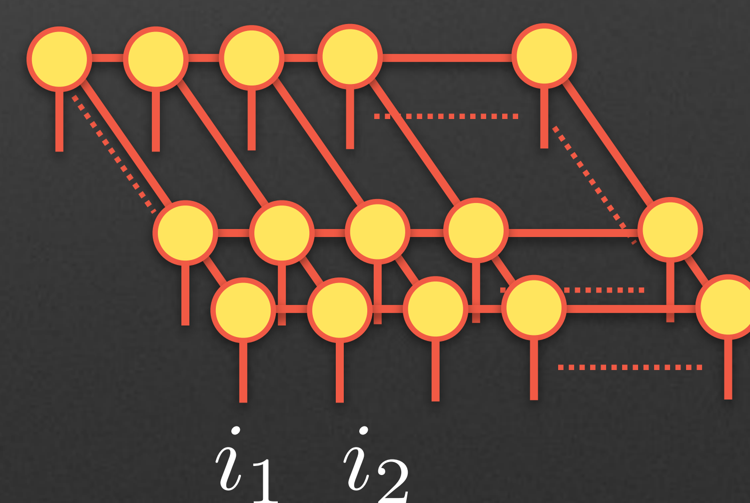


Huge tensor!
$\sim O(d^N)$

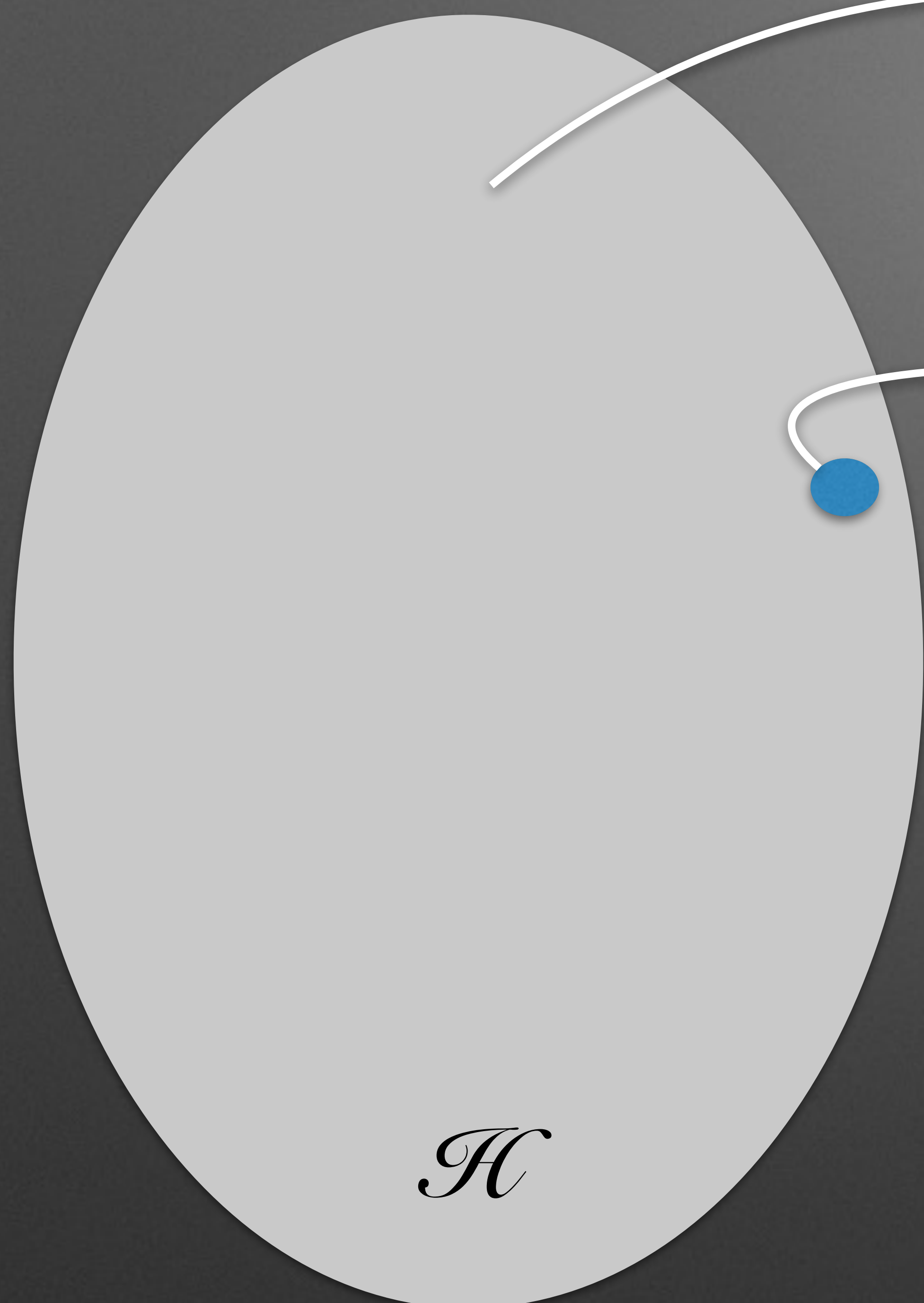
MPS (1D)



PEPS (2D)



- Area Law of Entanglement Entropy



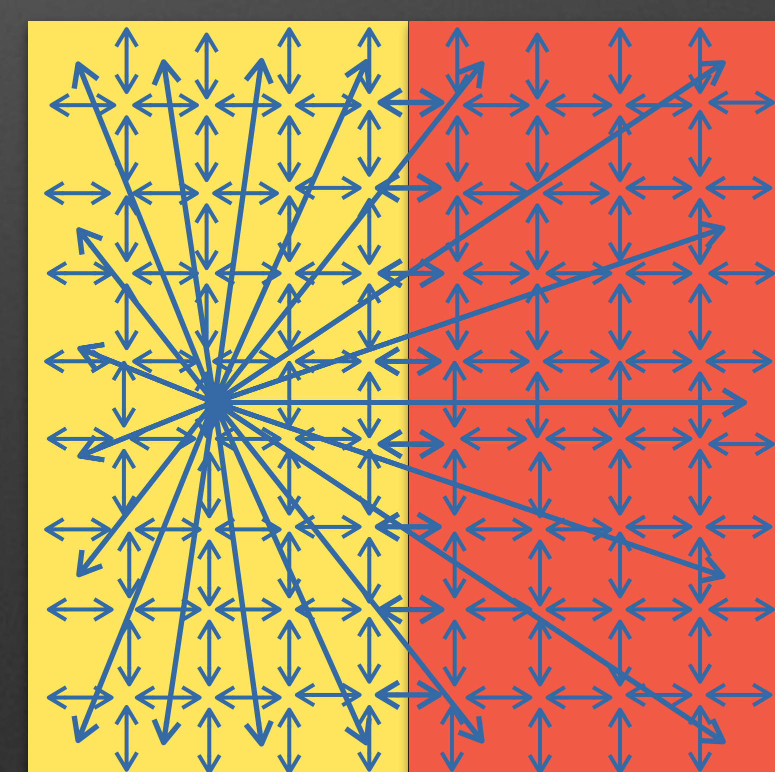
Hilbert space of N-body system

Generic quantum states satisfy

$$S_{\mathrm{vN}} \sim L^{D}(V)$$: volume law

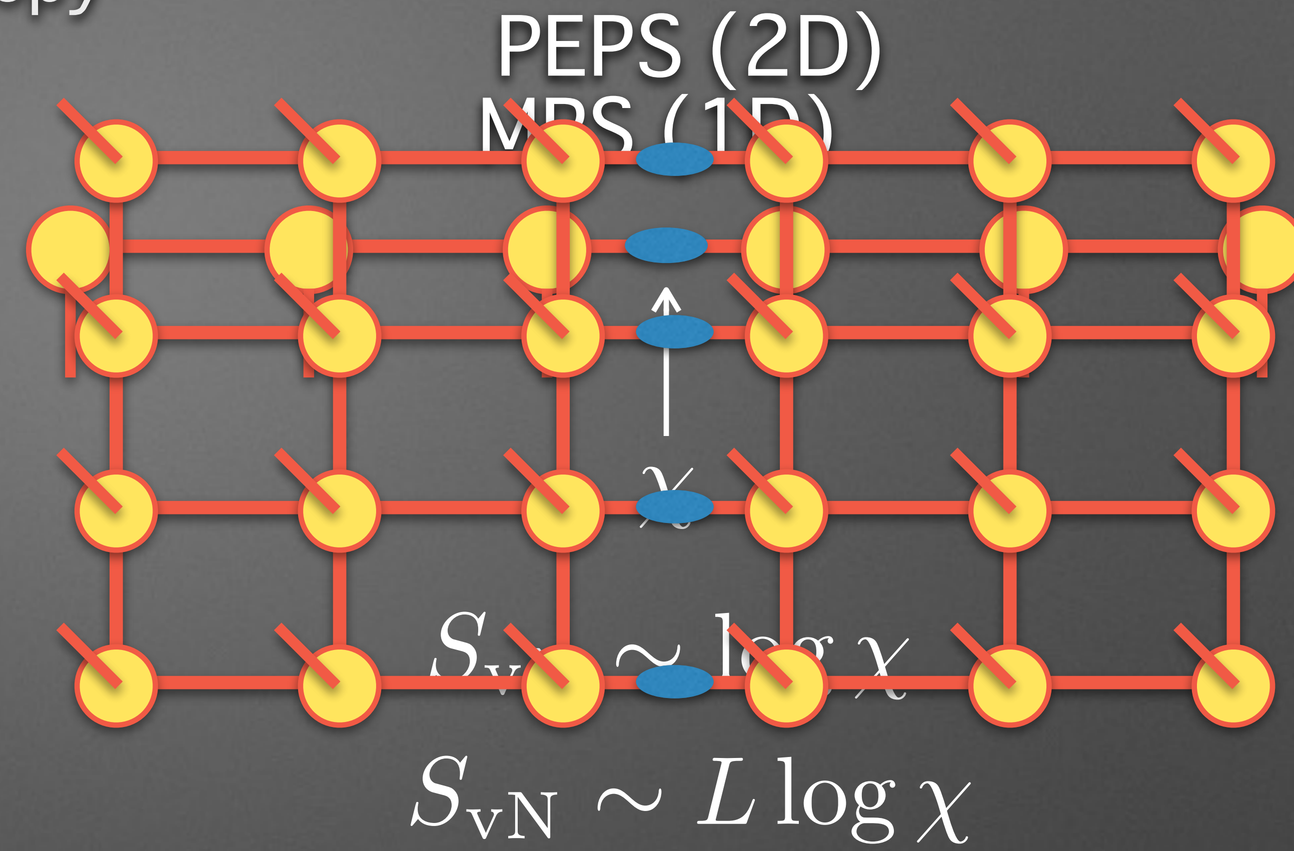
Set of Relevant states or
ground states of local Hamiltonian

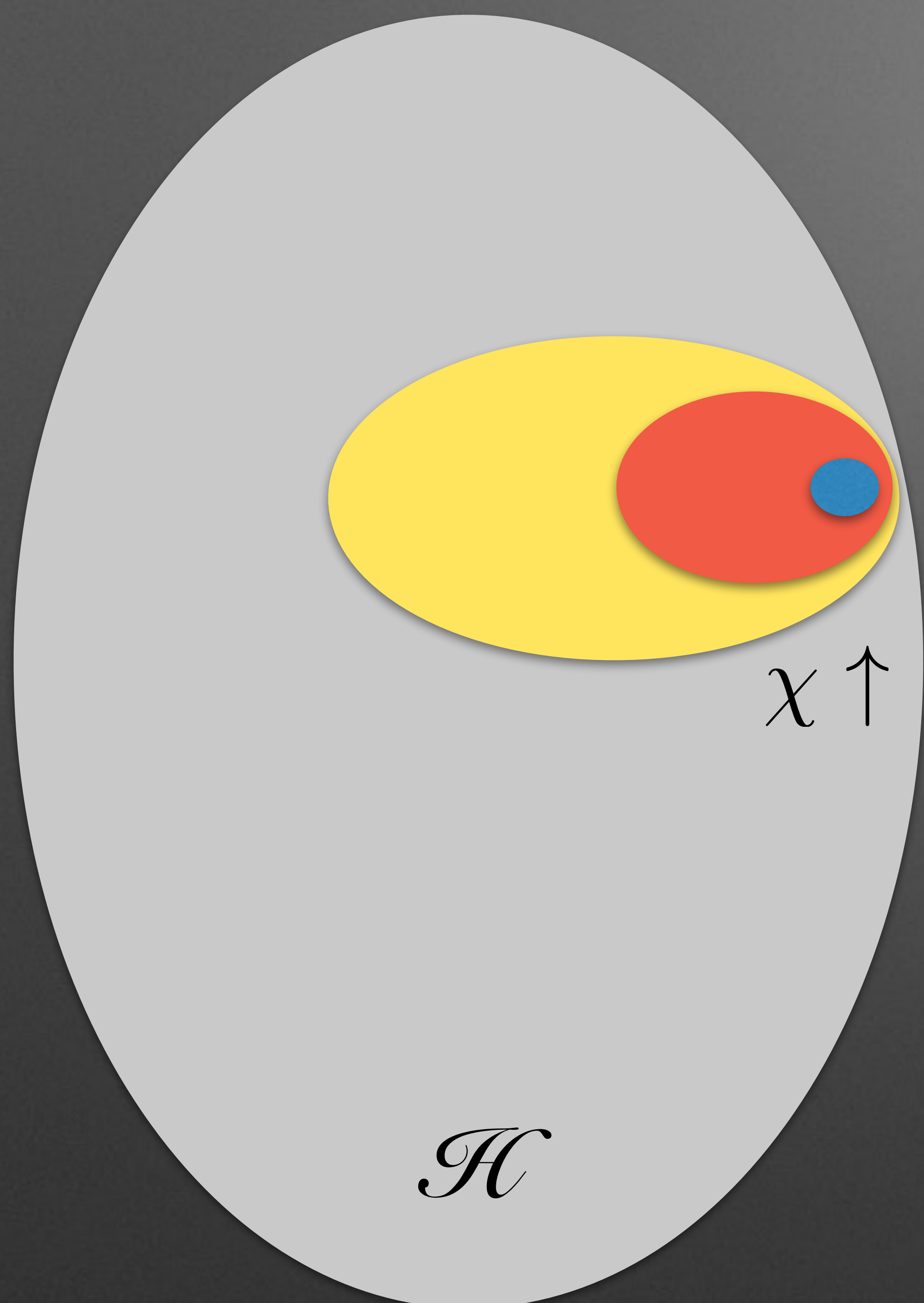
$$S_{\mathrm{vN}} \sim L^{D-1}(\partial V)$$: area law



N-body system

- Area Law of Entanglement Entropy





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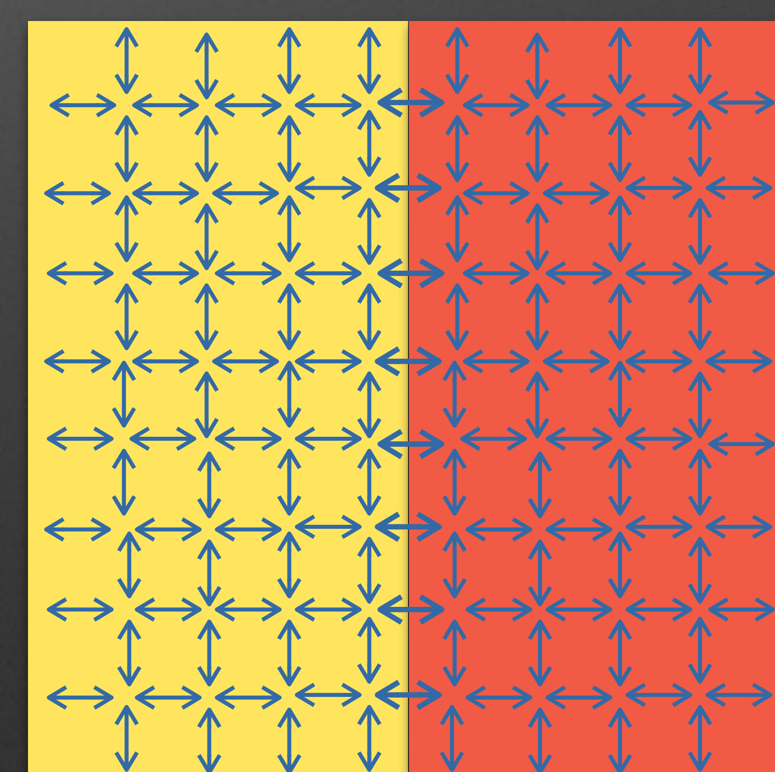

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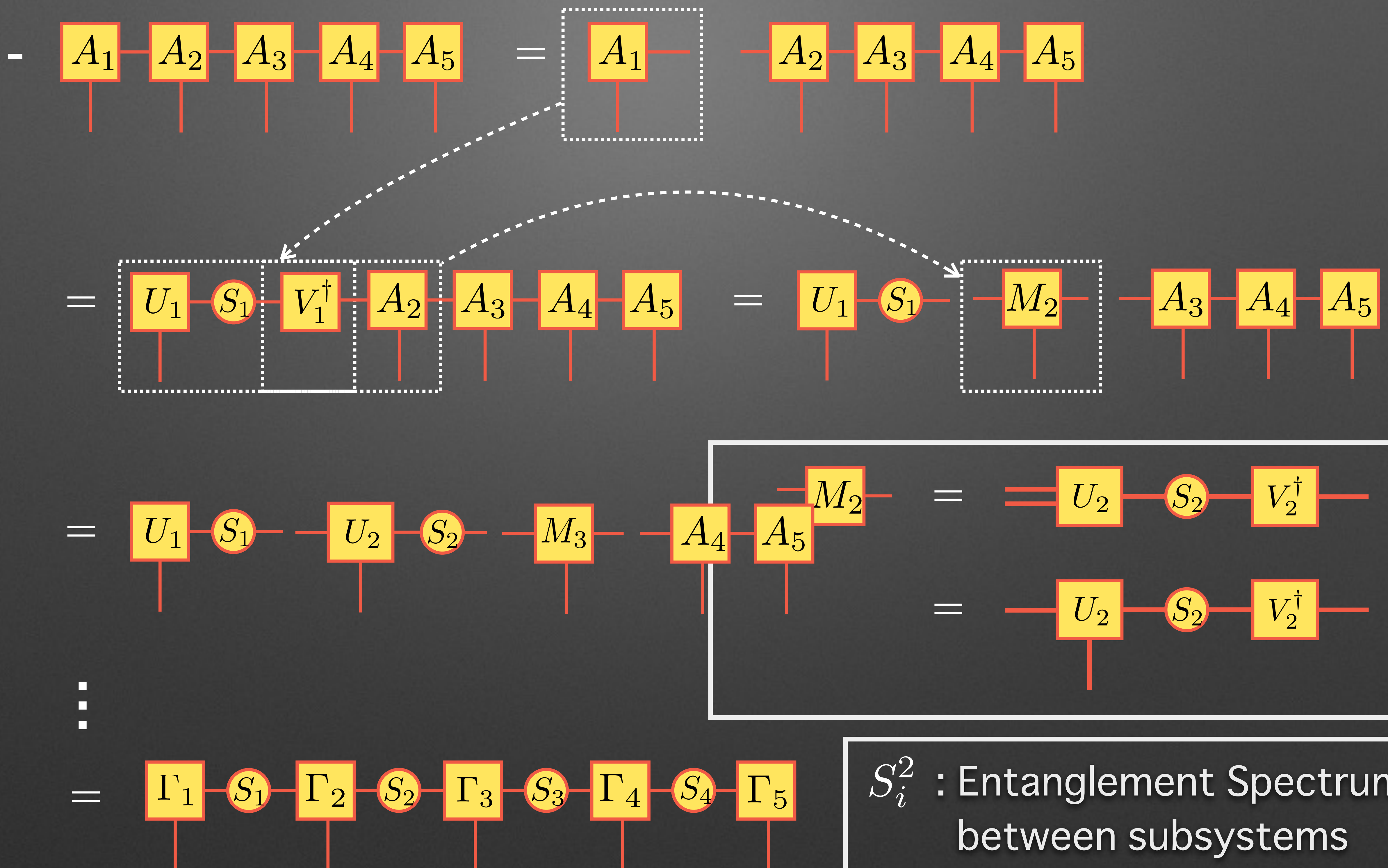
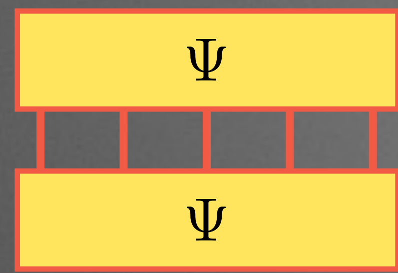
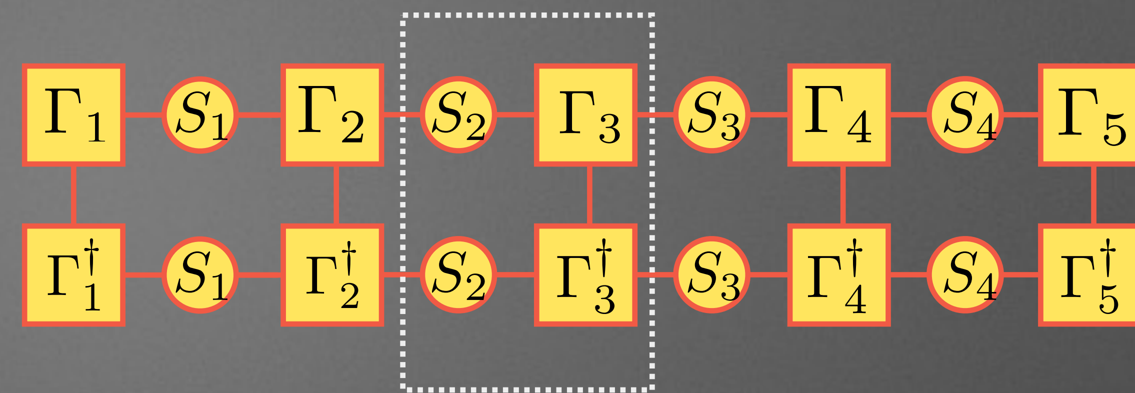
❖ Canonical Form of MPS

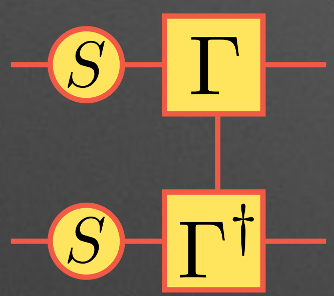


S_i^2 : Entanglement Spectrum between subsystems $x \leq i$ and $x > i$

- Transfer Matrix in Canonical (Pure) MPS

- $\langle \Psi | \Psi \rangle =$ 

$=$ 

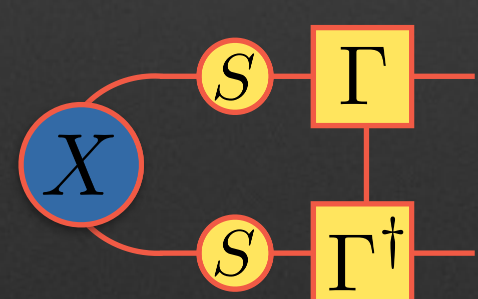
- 

: $(\chi^2 \times \chi^2)$ - transfer matrix (TM)

- TM : Linear mapping from matrix to matrix

  - $\epsilon(X) := \sum_m \Gamma_m S X S \Gamma_m^\dagger$

  - $\epsilon\Big($ 

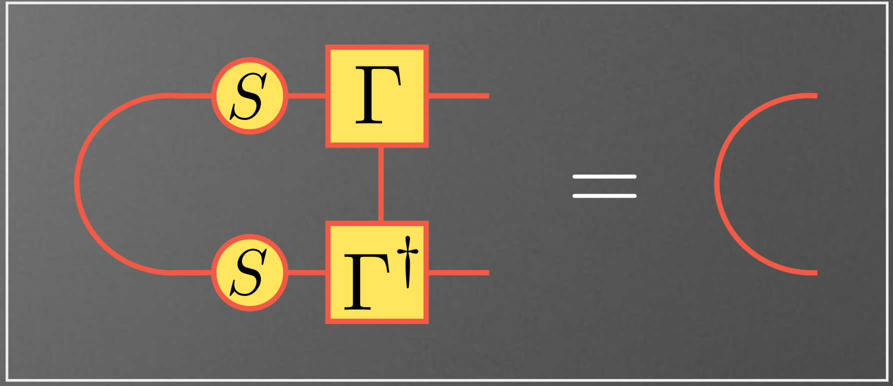
$\Big) =$ 

$=$ 

- Transfer Matrix in Canonical (Pure) MPS

  - One can show following two properties

    ① $\epsilon(\mathbb{I}) = \sum_m \Gamma_m S^2 \Gamma_m^\dagger = \mathbb{I} \longleftrightarrow$

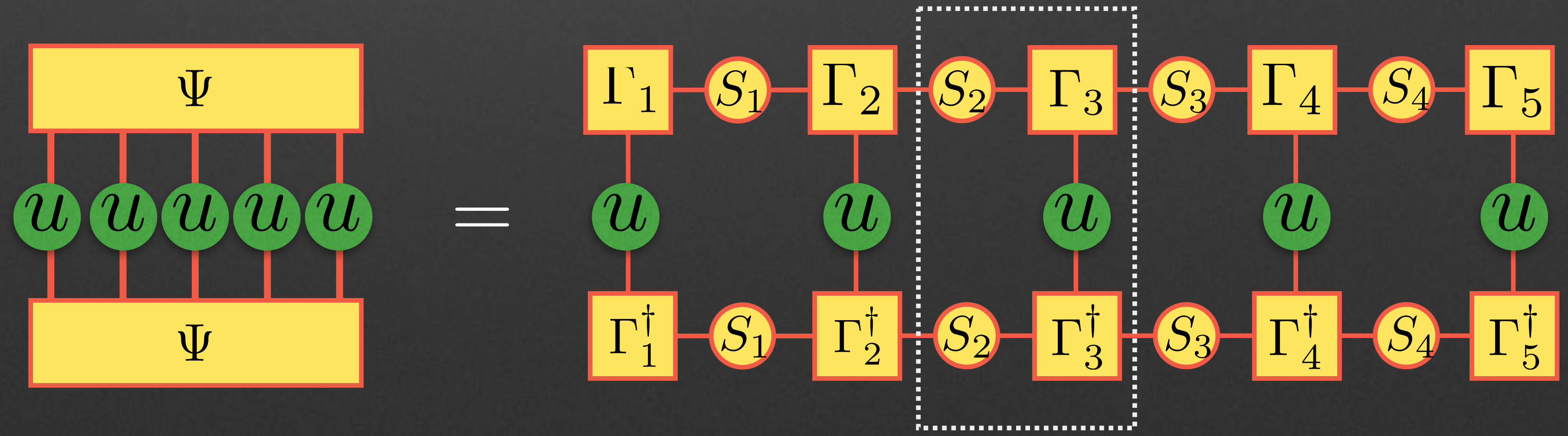


    ② $\epsilon(X) = \lambda X$: eigenvalues $|\lambda| \leq 1$ & Largest $|\lambda|$ is unique

- Global Symmetry Operation

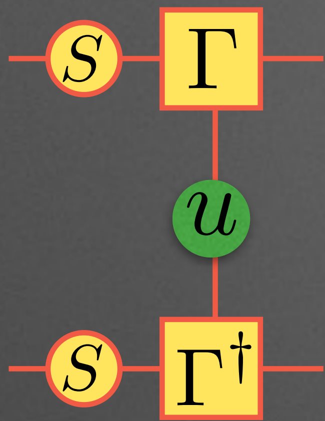
  ➤ $U = \otimes_{i=1}^N u_i$ (ex) Spin rotation, Time-Reversal, etc

  ➤ $\langle \Psi | U | \Psi \rangle =$



- New Transfer Matrix

$$u_{mn} = e^{i\theta_m}\delta_{mn}$$

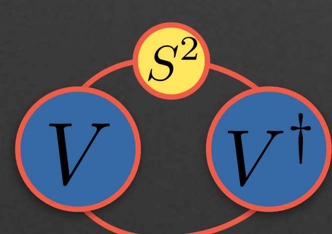
\- 

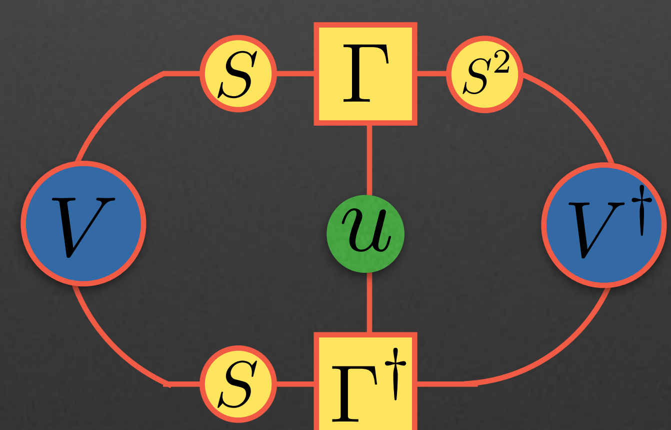
$$\longleftrightarrow \quad \epsilon_u(X) = \sum_m e^{i\theta_m}\Gamma_m S X S \Gamma_m^\dagger$$

We want to know the eigenvalues of New TM!

\- $$\epsilon_u(V) = \lambda V = \sum_m e^{i\theta_m}\Gamma_m S V S \Gamma_m^\dagger$$

➤ $$|\lambda|\,\mathrm{Tr}\left[V S^2 V^\dagger\right] = \left|\sum_m e^{i\theta_m}\,\mathrm{Tr}\left[\Gamma_m S V S \Gamma_m^\dagger S^2 V^\dagger\right]\right|$$

$|\lambda|$ 

$=$ | 

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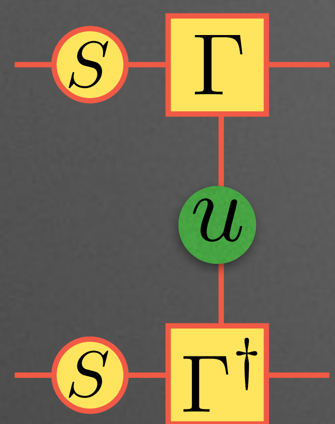
- 

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➤ $$|\lambda|\mathrm{Tr}\left[V S^2 V^\dagger\right] = \left|\sum_m e^{i\theta_m}\,\mathrm{Tr}\left[\Gamma_m S V S \Gamma_m^\dagger S^2 V^\dagger\right]\right|$$

$$= \left|\sum_m \mathrm{Tr}\left(X_m^\dagger Y_m\right)\right|$$

$$\boxed{\begin{aligned} X_m &= e^{-\theta_m} S\Gamma_m S V^\dagger \\ Y_m &= S V^\dagger \Gamma_m S \end{aligned}}$$

Cauchy-Schwarz inequality

$$|\vec{v}_1 \cdot \vec{v}_2| \leq |\vec{v}_1||\vec{v}_2|$$

$$\leq \left|\sum_m \mathrm{Tr}\left(X_m^\dagger X_m\right)\right|^{\frac{1}{2}} \left|\sum_m \mathrm{Tr}\left(Y_m^\dagger Y_m\right)\right|^{\frac{1}{2}}$$

- New Transfer Matrix

➤ $$|\lambda|\mathrm{Tr}\left[VS^2V^\dagger\right] \leq \left|\underbrace{\sum_m \mathrm{Tr}\left(S\Gamma_m S V^\dagger V S\Gamma_m^\dagger S\right)}_{=\,\epsilon\left(V^\dagger V\right)}\right|^{\frac{1}{2}} \left|\underbrace{\sum_m \mathrm{Tr}\left(SV^\dagger\Gamma_m S^2\Gamma_m^\dagger V S\right)}_{=\,\epsilon\left(\mathbb{I}\right)\,=\,\mathbb{I}}\right|^{\frac{1}{2}}$$

$$\epsilon(X) := \sum_m \Gamma_m S X S \Gamma_m^\dagger$$

- New Transfer Matrix

➤ $$|\lambda| \mathrm{Tr}\left[V S^2 V^\dagger\right] \leq \left|\sum_m \mathrm{Tr}\left(S\Gamma_m S V^\dagger V S \Gamma_m^\dagger S\right)\right|^{\frac{1}{2}} \left|\sum_m \mathrm{Tr}\left(S V^\dagger \Gamma_m S^2 \Gamma_m^\dagger V S\right)\right|^{\frac{1}{2}}$$

$\because\ \epsilon\left(V^\dagger V\right)$ is bounded by $V^\dagger V$

$$= \left|\mathrm{Tr}\left[\epsilon\left(V^\dagger V\right) S^2\right]\right|^{\frac{1}{2}} \left|\mathrm{Tr}\left(V S^2 V^\dagger\right)\right|^{\frac{1}{2}}$$

$$\leq \left|\mathrm{Tr}\left(V^\dagger V S^2\right)\right|^{\frac{1}{2}} \left|\mathrm{Tr}\left(V S^2 V^\dagger\right)\right|^{\frac{1}{2}}$$

$$= \mathrm{Tr}\left(V S^2 V^\dagger\right)$$

➤ $|\lambda| \leq 1$: Physically, it is obvious! Otherwise $\langle\Psi|U|\Psi\rangle \cancel{>} \langle\Psi|\Psi\rangle$

➤ Requiring the symmetry : $U|\Psi\rangle = e^{i\Theta}|\Psi\rangle$

↳ $\langle\Psi|U|\Psi\rangle = e^{i\Theta}\langle\Psi|\Psi\rangle$

↳ $|\lambda| = 1$

- Symmetry Imposed MPS
  - Symmetry or $|\lambda| = 1$ gives us 2 constraints

① $\epsilon\left(V^\dagger V\right) = V^\dagger V$: $V^\dagger V = \mathbb{I}$ or V is unitary

② $X_m \parallel Y_m$: $\alpha X_m = Y_m$

$$\hookrightarrow \alpha e^{i\theta_m} S\Gamma_m S V^\dagger = S V^\dagger \Gamma_m S$$

$$\hookrightarrow |\alpha|^2 \sum_m S\Gamma_m S V^\dagger V S \Gamma_m^\dagger S = \sum_m S V^\dagger \Gamma_m S^2 \Gamma_m^\dagger V S$$

$$= \epsilon\left(V^\dagger V\right) = \mathbb{I} \qquad = \epsilon\left(\mathbb{I}\right) = \mathbb{I}$$

- Symmetry Imposed MPS
  - Symmetry or $|\lambda| = 1$ gives us 2 constraints

① $\epsilon\left(V^\dagger V\right) = V^\dagger V$: $V^\dagger V = \mathbb{I}$ or V is unitary

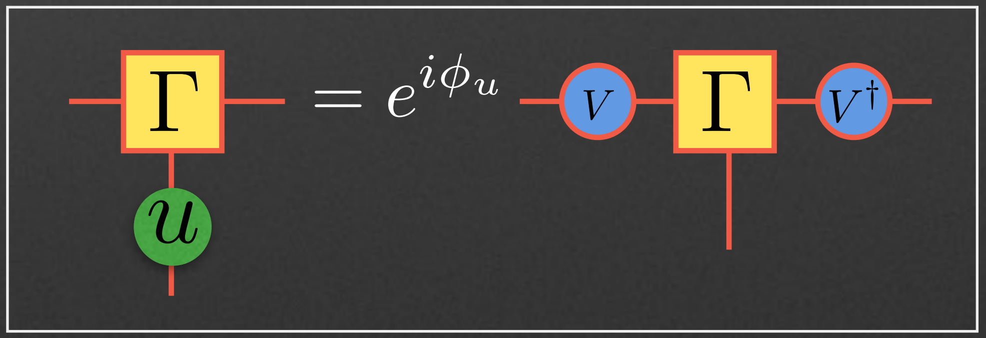
② $X_m \parallel Y_m$: $\alpha X_m = Y_m$

↳ $\alpha e^{i\theta_m} S\Gamma_m S V^\dagger = S V^\dagger \Gamma_m S$

↳ $|\alpha|^2 \sum_m S\Gamma_m S V^\dagger V S \Gamma_m^\dagger S = \sum_m S V^\dagger \Gamma_m S^2 \Gamma_m^\dagger V S$

↳ $|\alpha|^2 S^2 = S^2 \qquad \therefore \ \alpha = e^{-i\phi_u}$

$$\boxed{e^{i\theta_m}\Gamma_m = e^{i\phi_u} V \Gamma_m V^\dagger} \leftrightarrow$$



If satisfied, the MPS is symmetric under U

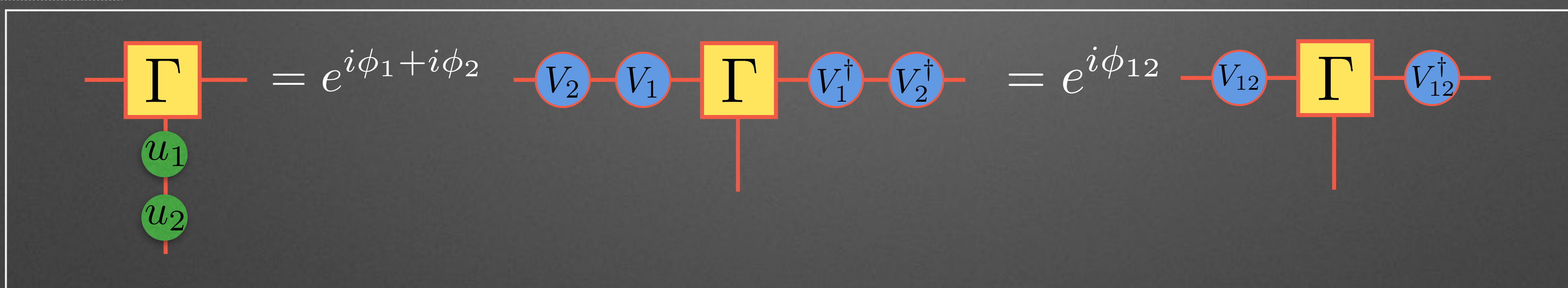
- Projective Representation

  - Acting u twice,

$$: \quad u_2 u_1 \Gamma = e^{i\phi_{u_1}+i\phi_{u_2}} V_{u_2} V_{u_1} \; \Gamma \; V_{u_1}^{\dagger} V_{u_2}^{\dagger}$$

$$\doteq e^{i\phi_{[u_1 u_2]}} V_{[u_1 u_2]} \; \Gamma \; V_{[u_1 u_2]}^{\dagger}$$

$u_1 u_2 \in G_u$



  - $\begin{cases} \phi_{[u_1 u_2]} = \phi_{u_1} + \phi_{u_2} \\ V_{u_1} V_{u_2} = e^{i\theta} V_{[u_1 u_2]} \end{cases}$: V_u is a projective representation of G_u

  - non-trivial projective representation (or symmetry fractionalization) leads non-trivial degeneracies in Entanglement Spectrum (ES) characterizing topological phase from trivial phase

[M. Oshikawa et al. PRB (2010)]

❖ Example : Haldane Phase (S=integer)

- Model : $H = J\sum_i \vec{S}_i\cdot\vec{S}_{i+1} + U_{zz}\sum_i (S_i^z)^2$

- $Z_2\times Z_2$ symmetry : spin flipping along x and z axes

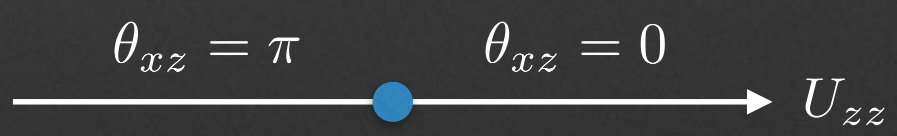
- Acting each Z_2^α twice,

$$\begin{cases} V_{Z_2^x}V_{Z_2^x} = e^{i\theta_x}V_{(Z_2^x)^2} = e^{i\theta_x} \\ V_{Z_2^z}V_{Z_2^z} = e^{i\theta_z}V_{(Z_2^z)^2} = e^{i\theta_z} \end{cases} \xrightarrow[V_{Z_2^\alpha}]{\text{redefine}} \begin{cases} (V_{Z_2^x})^2 = \mathbb{I} \\ (V_{Z_2^z})^2 = \mathbb{I} \end{cases}$$

- Acting combined $Z_2^x Z_2^z$ twice,

$$: \ (V_{Z_2^x}V_{Z_2^z})(V_{Z_2^x}V_{Z_2^z}) = e^{i\theta_{xz}}V_{(Z_2^y)^2} = e^{i\theta_{xz}} \longrightarrow \boxed{V_xV_z = e^{i\theta_{xz}}V_zV_x}$$

$$: \ e^{i2\theta_{xz}} = 1 \qquad \therefore\ \theta_{xz} = 0 \ \text{or}\ \pi$$

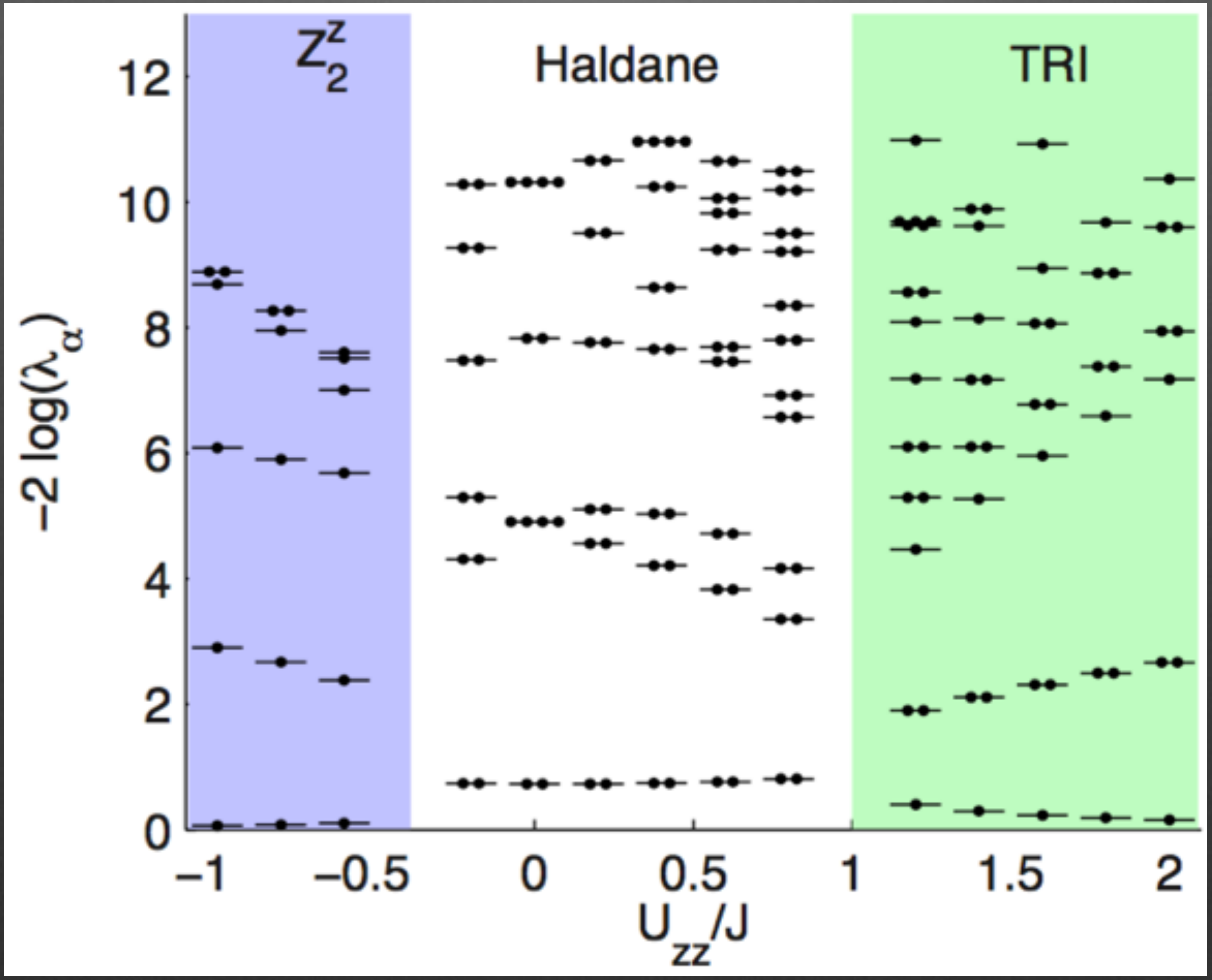


We should pass through a phase transition!

❖ Example : Haldane Phase (S=integer)

➤ One can show $[S, V_\alpha] = 0$. Therefore, when $\theta_{xz} = \pi$ or $V_x V_z = -V_z V_x$

Entanglement Spectrum S is at least double degenerate !



[M. Oshikawa et al. PRB (2010)]

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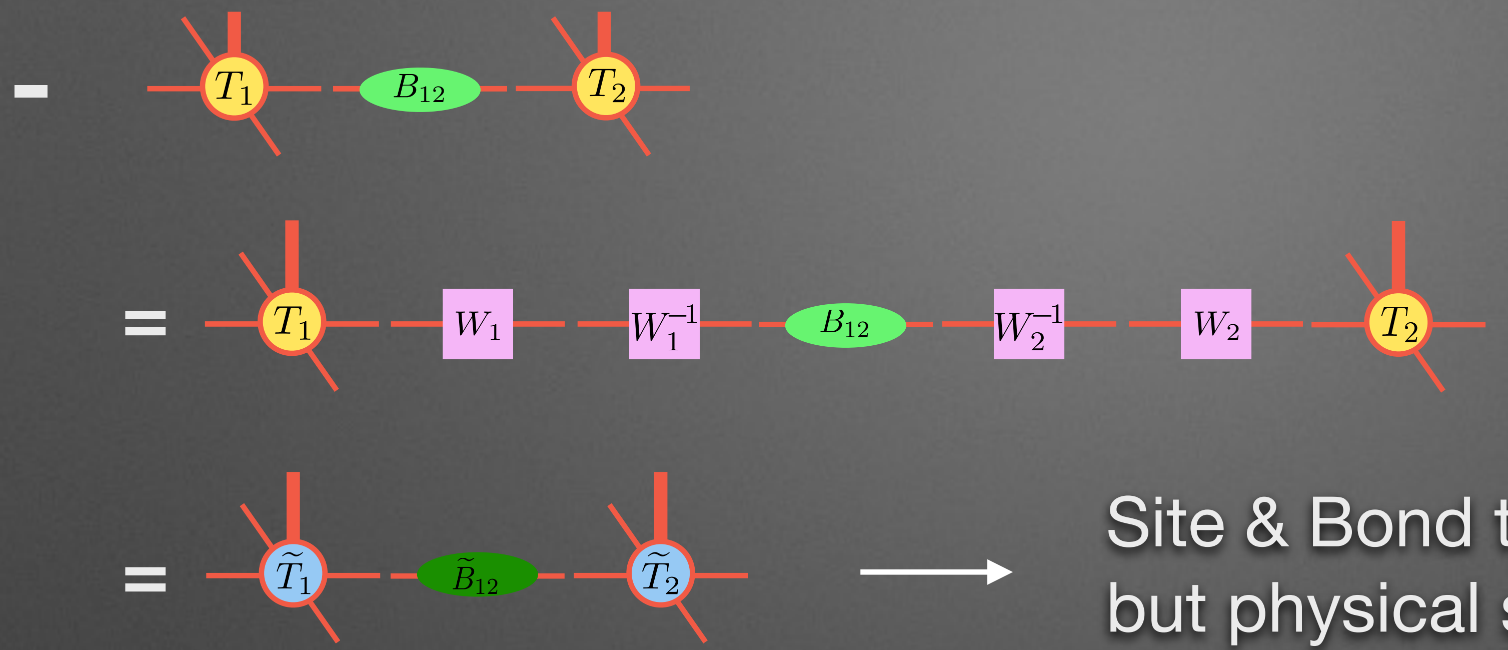
❖ Gauge Transformation on TNS



Gauge redundancy
GL(D,ℂ)

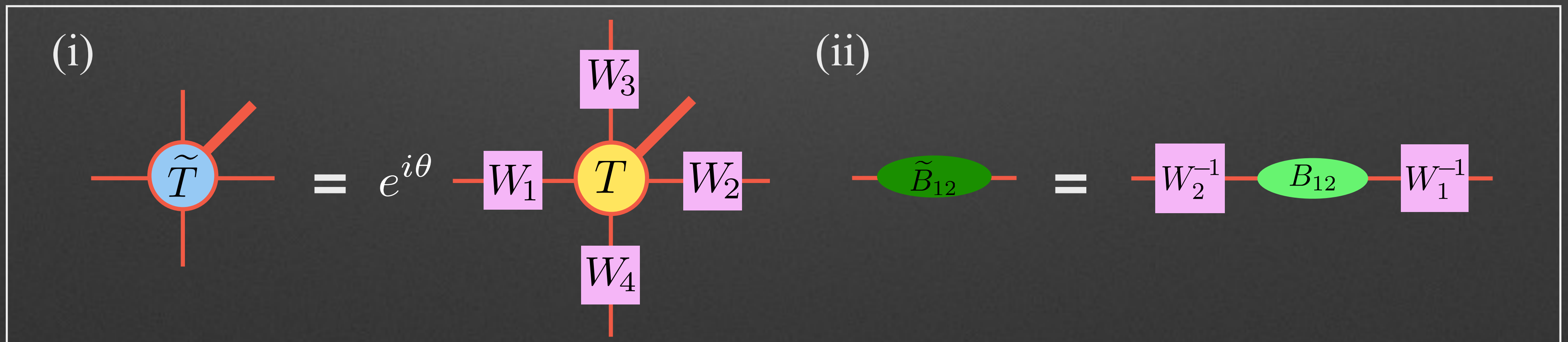
↑

Many-to-one
correspondence

↑

Site & Bond tensors are changed,
but physical state is not changed!

- $$\begin{cases} [\widetilde{\mathbf{T}}(\vec{x})]_{abcd} = e^{i\theta(\vec{x})}[W(\vec{x},1)]_{aa'}[W(\vec{x},2)]_{bb'}[W(\vec{x},3)]_{cc'}[W(\vec{x},4)]_{dd'}[\mathbf{T}(\vec{x})]_{a'b'c'd'} \\ [\widetilde{\mathbf{B}}(\vec{x},\vec{y})]_{ab} = [W^{-1}(\vec{x},1)]_{a'a}[W^{-1}(\vec{y},2)]_{b'b}[\mathbf{B}(\vec{x},\vec{y})]_{a'b'} \end{cases}$$



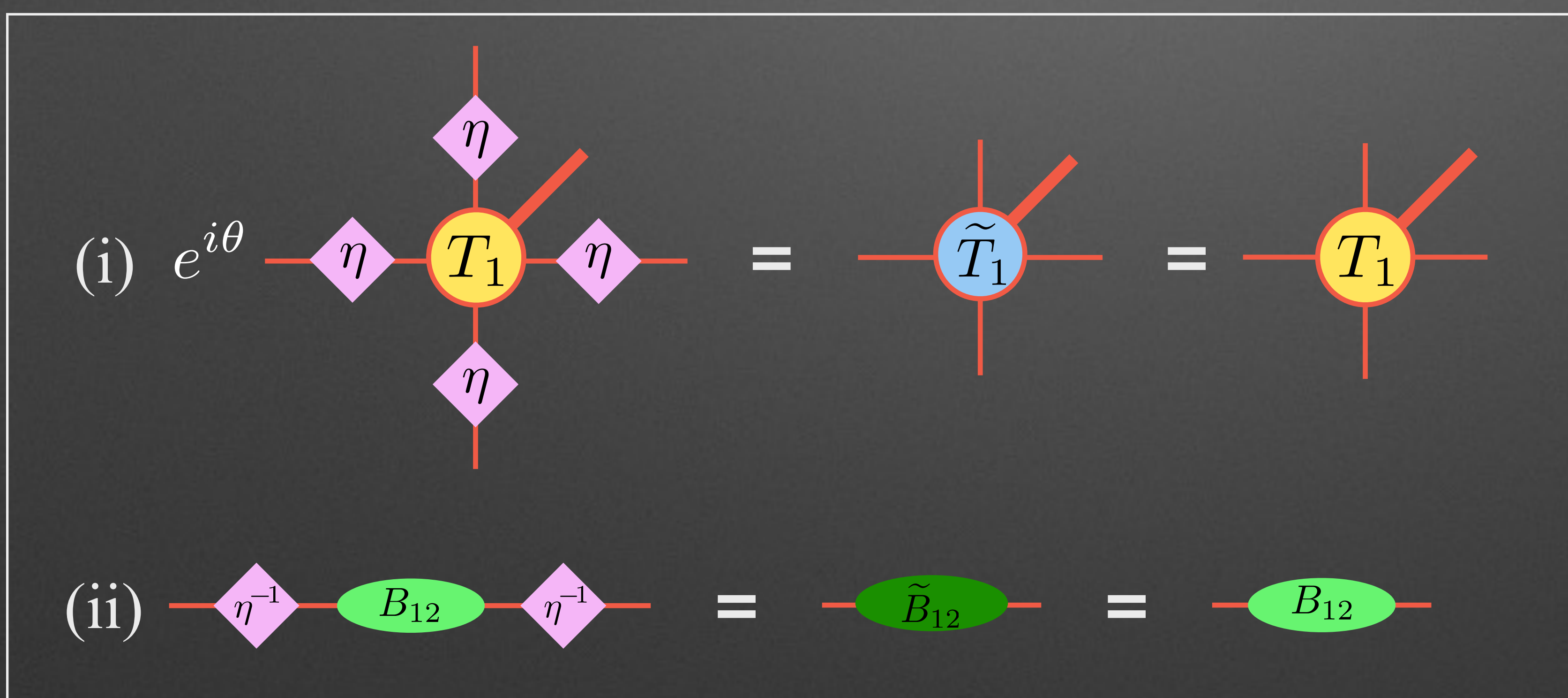
- $$\{\mathbf{T}(\vec{x}),\mathbf{B}(\vec{x},\vec{x}')\} \xleftrightarrow{\{e^{i\theta(\vec{x})},W(\vec{x})\}} \{\widetilde{\mathbf{T}}(\vec{x}),\widetilde{\mathbf{B}}(\vec{x},\vec{x}')\}$$

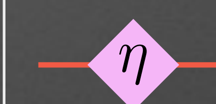
❖ Gauge Transformation on TNS

- Invariant Gauge Group (IGG):

[X. G. Wen, 2001]

$$\begin{cases} \text{(i) } [\mathbf{T}(\vec{x})]_{abcd} = e^{i\theta(\vec{x})}\eta_{aa'}\eta_{bb'}\eta_{cc'}\eta_{dd'}[\mathbf{T}(\vec{x})]_{a'b'c'd'} \\ \text{(ii) } [\mathbf{B}(\vec{x},\vec{y})]_{ab} = \eta^{-1}_{a'a}\eta^{-1}_{b'b}[\mathbf{B}(\vec{x},\vec{y})]_{a'b'} \end{cases}$$





$\rightarrow \{I\}$: trivial IGG

$\rightarrow \{I, g\}$: Z_2 IGG

$\vdots$

IGG is directly related with the gauge dynamics!

[X. Wen, PRB 82, 165119(2010)] [B. Swingle and X. Wen, arXiv:1001.4517]

[N. Schuch et. al, Ann. Phys. 325, 2153 (2010)] [S. Jian and Y. Ran, PRB 92, 104414(2015)]

◆ Z_2 topological order (1)

ex) Deconfining phase of Z_2 gauge theory

◆ Low-energy quasi-particles
$$\begin{cases} \textit{1 : trivial particle} \\ e \textit{ : chargon} \\ m \textit{ : fluxon} \\ f \textit{ : bound state of } e \textit{ and } m \end{cases}$$
[Kitaev, 2006]

◆ Anyonic statistics :

	Self-braiding	Mutual braiding
e	bosonic	fermionic
m	bosonic	
f	fermionic	

◆ Fusion rule

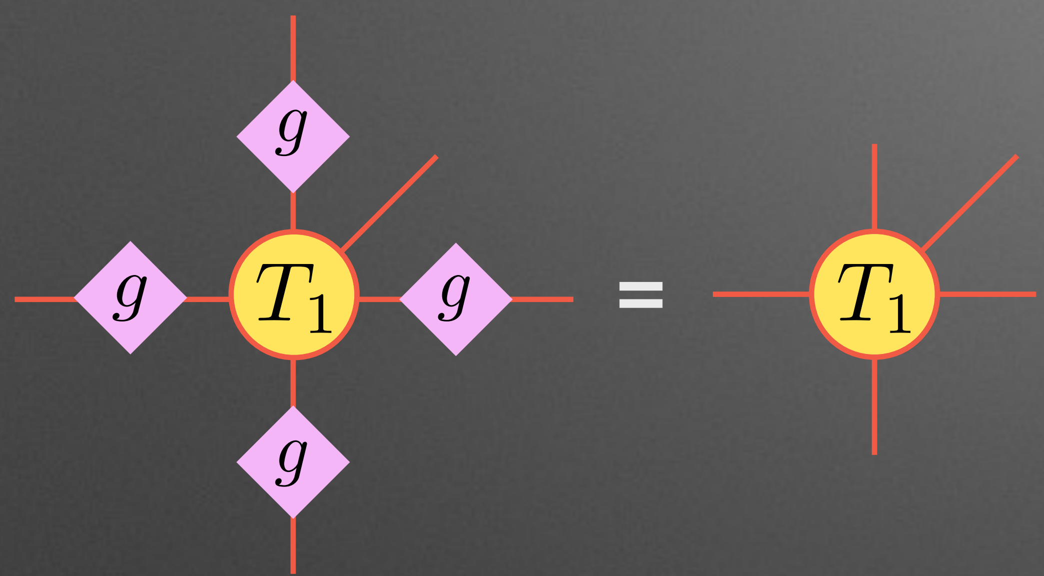
$$e \times e = 1$$

$$m \times m = 1$$

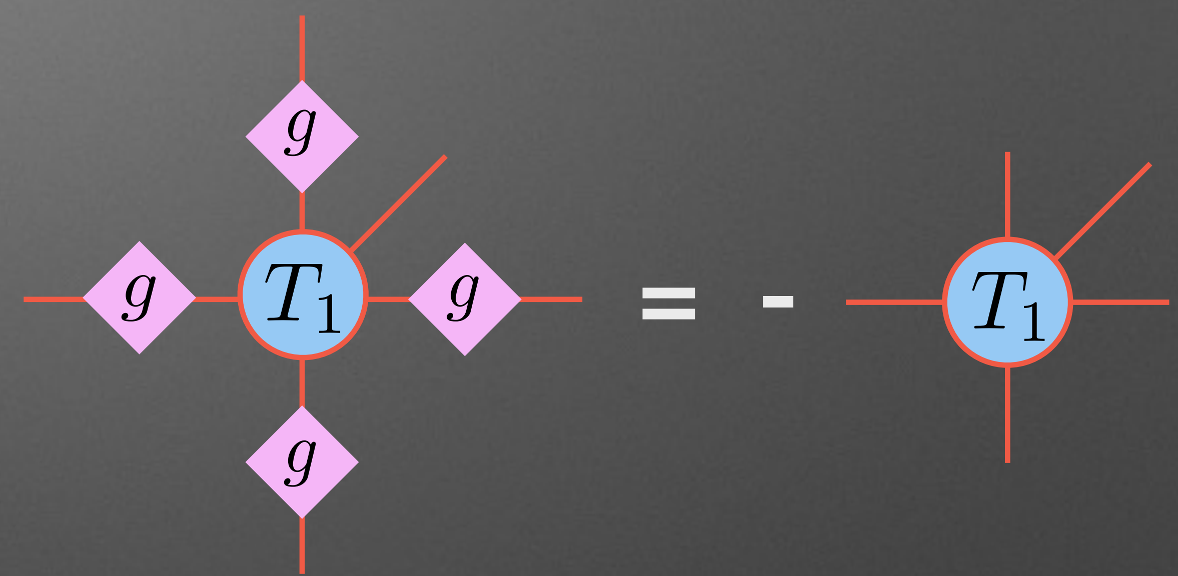
$$e \times m = f$$

Z2 topological order (2)

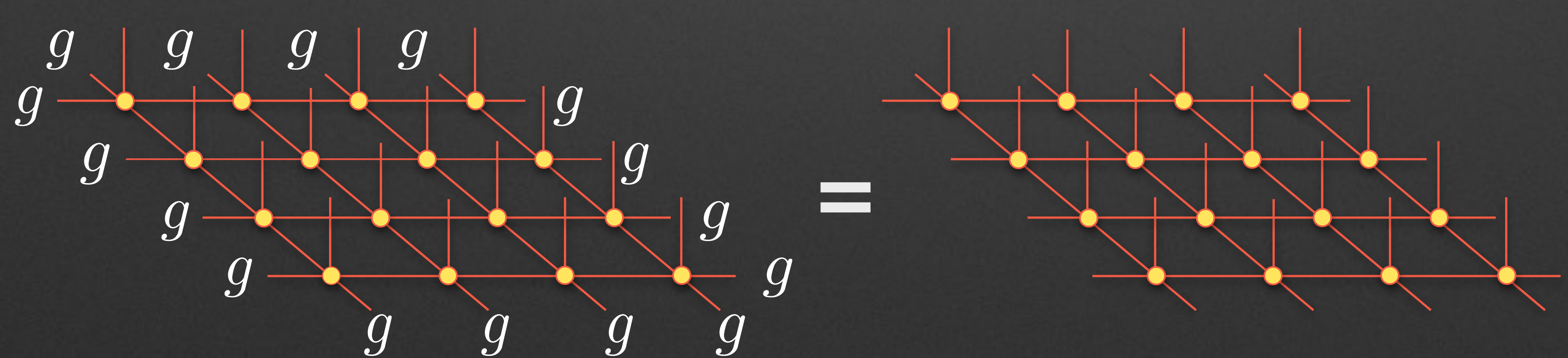
(a) Z2 even Tensor



(b) Z2 odd Tensor



(c) TNS with Z2 even Tensor

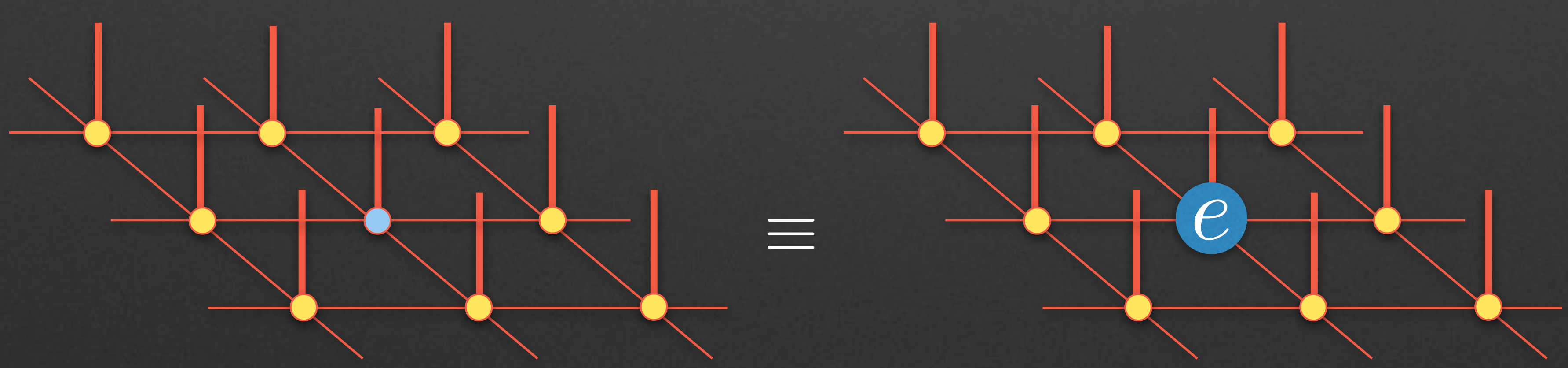


◆ Z_2 topological order (3)

(d) Fluxon

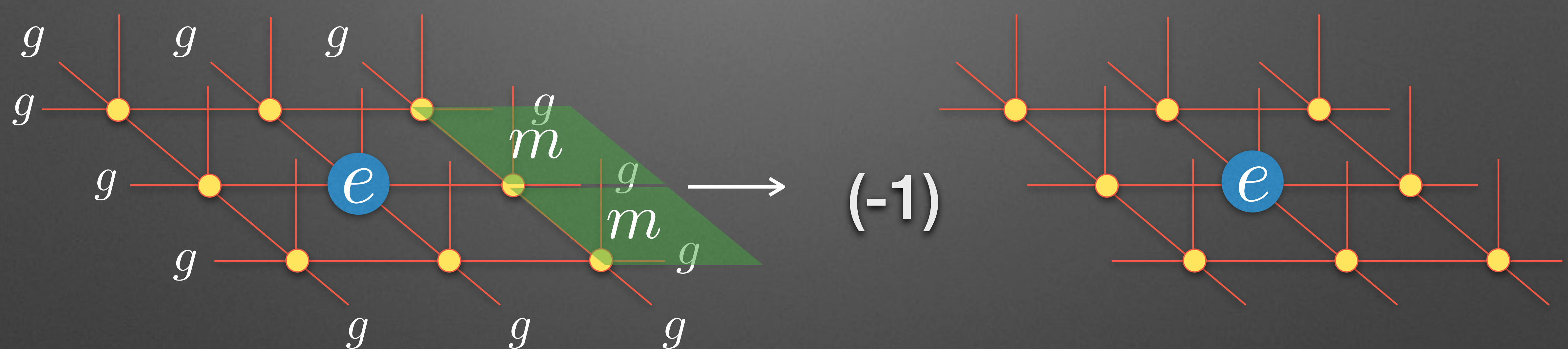
Moving fluxon

(f) Chargon



Z_2 topological order (4)

- Braiding statistics between Fluxon and Chargon



Z_2 IGG invariant TN ⟶ Z2 topologically ordered state

[S. Jian and Y. Ran(2015)]

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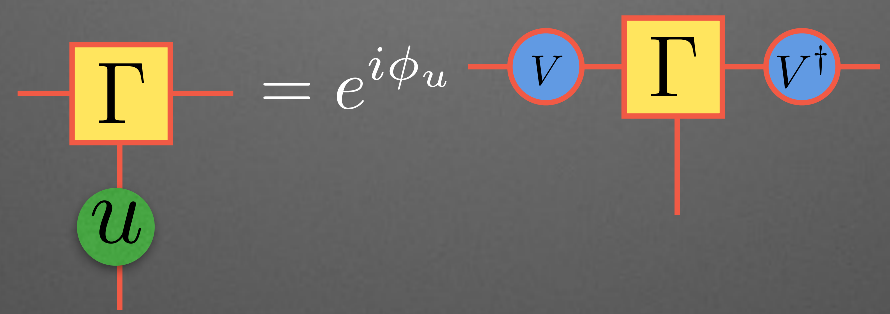
Summary

1. By imposing symmetry on MPS, one can derive the constraint



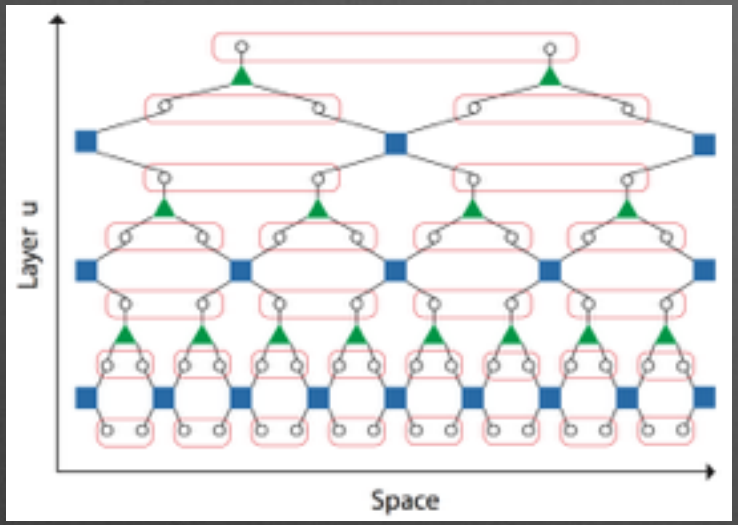
2. For a given symmetry, one can classify SPT phases using above eq.

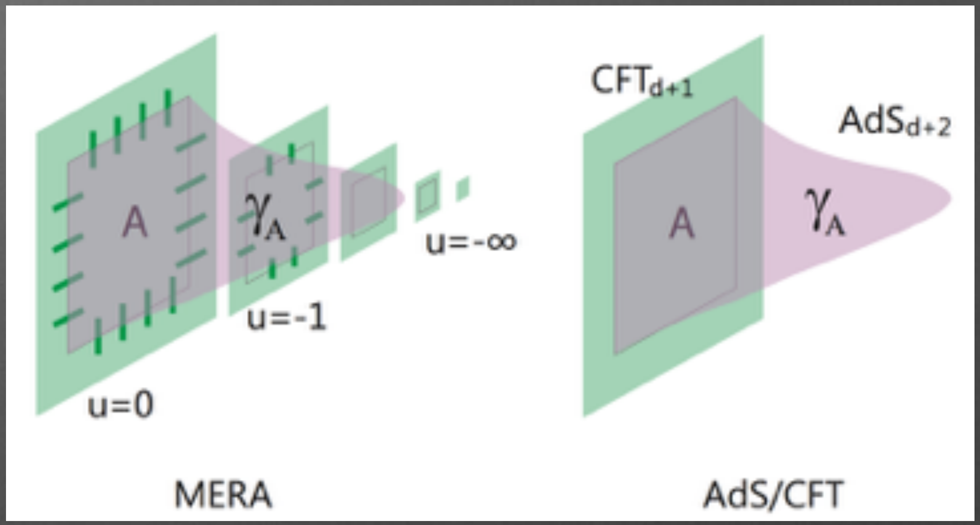
3. Z_2 topologically ordered state is systematically constructed by PEPS

4. One can impose symmetries on PEPS to classify the quantum states in terms of the projective symmetry group

Outlook

1. Multi-scale Entanglement Renormalisation Ansatz (MERA)





2. Classification SPT in 2D

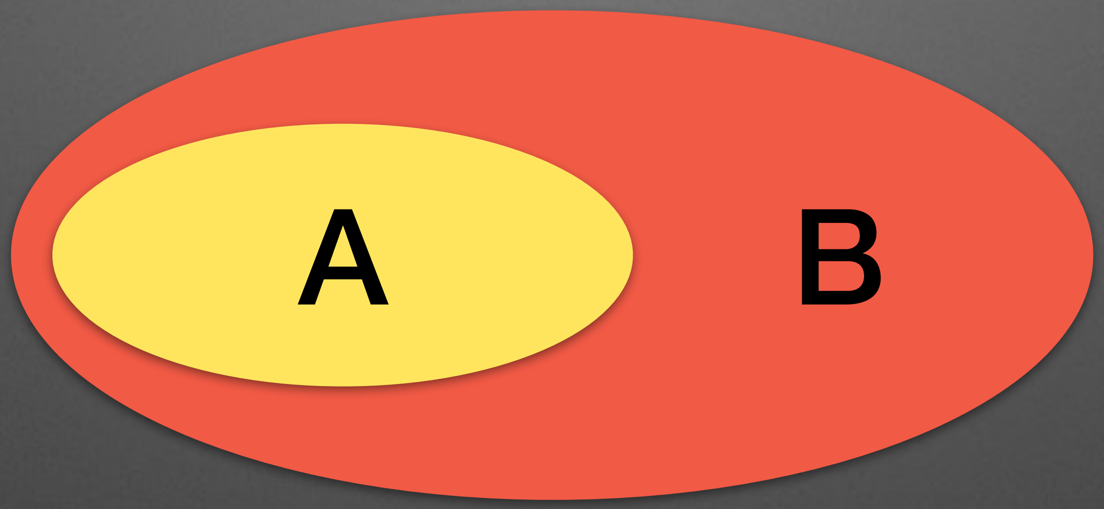
3. Numerical tools to find the ground states and thermally excited states and etc….

Thank you very much!

❖ When TNSs work very well?

- Quantum Entanglement

– Physical phenomenon that occurs when groups of particles are generated in ways such that the quantum state of each particle cannot be described independently [wiki]



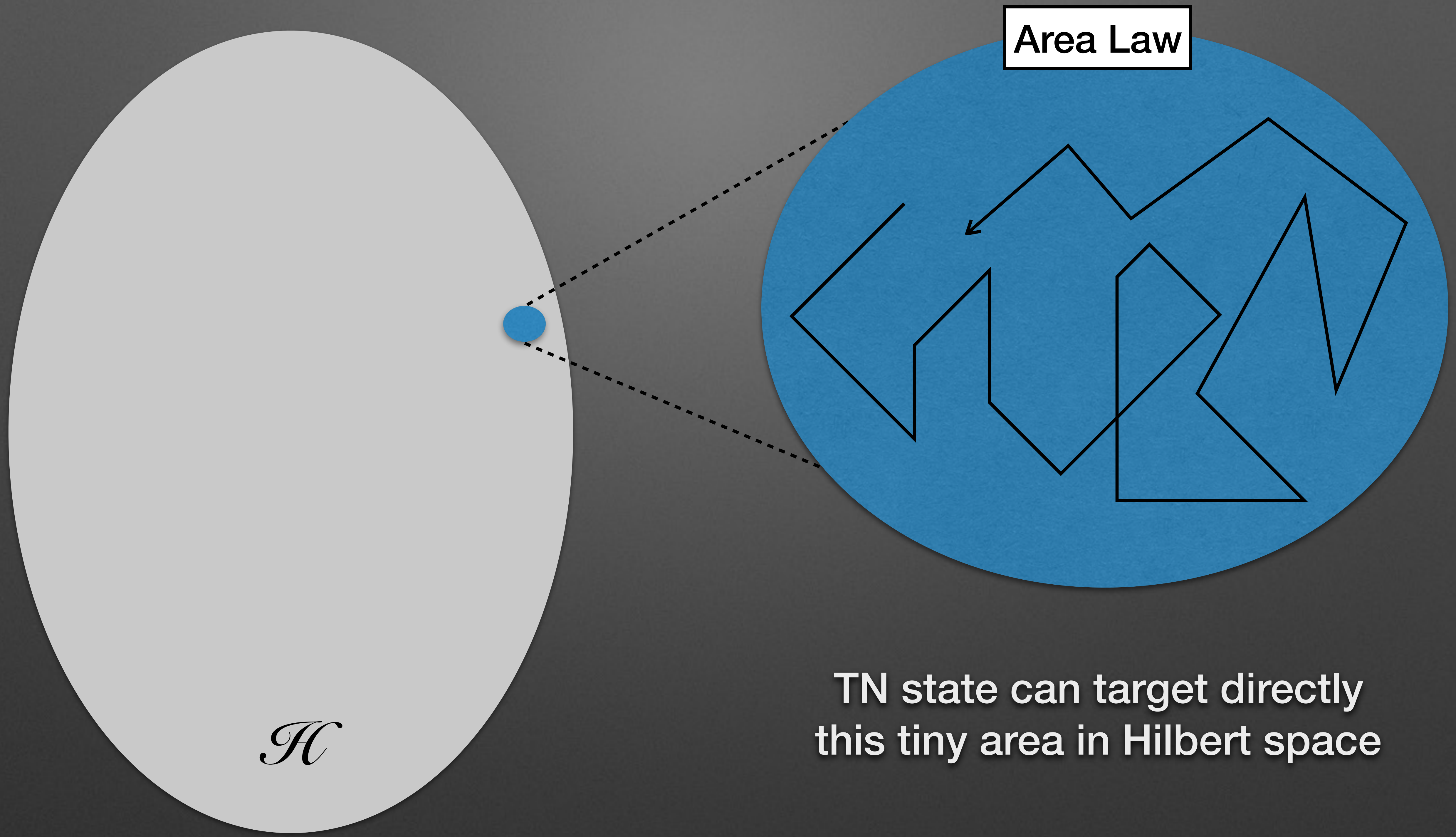
– Measured by von Neumann (entanglement) entropy:

$$S_{\mathrm{vN}} = -\mathrm{Tr}[\rho_A \log \rho_A] = -\mathrm{Tr}[\rho_B \log \rho_B]$$

e.g.) $|\uparrow_A \uparrow_B\rangle \longrightarrow S_{\mathrm{vN}} = 0$

$\frac{1}{\sqrt{2}}(|\uparrow_A \downarrow_B\rangle - |\downarrow_A \uparrow_B\rangle) \longrightarrow S_{\mathrm{vN}} = \log 2$

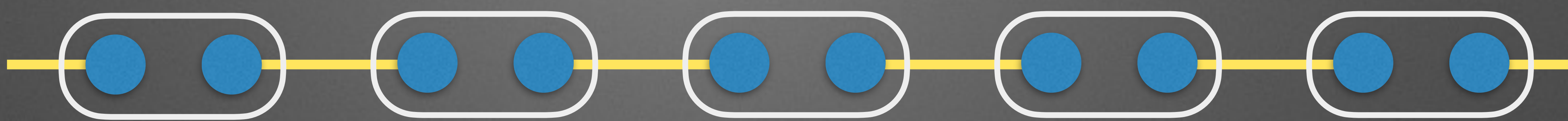
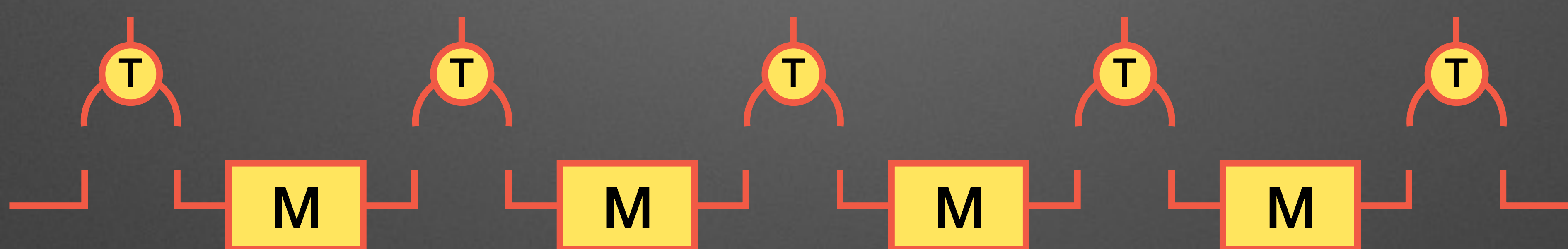
- Area Law of Entanglement Entropy

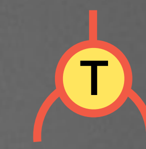


TN state can target directly this tiny area in Hilbert space

Hilbert space of N-body system

❖ Exemplary TNSs

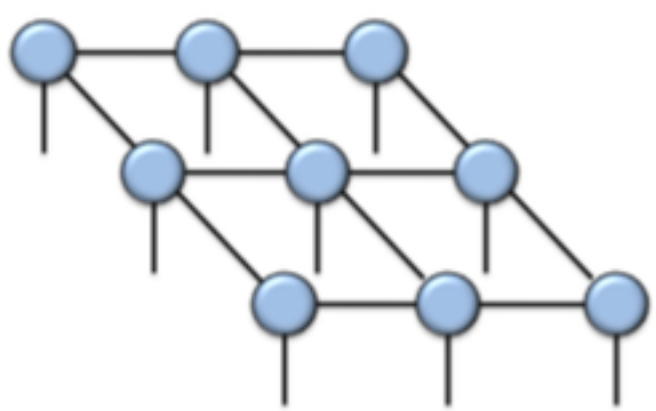
- Affeck-Lieb-Kennedy-Tasaki (AKLT) state
  - $H = \sum_i S_i \cdot S_{i+1} + \frac{1}{3}(S_i \cdot S_{i+1})^2$
  - 
  - 



$= C^{1,m_k}_{\frac{1}{2},m_i;\frac{1}{2},m_j}$

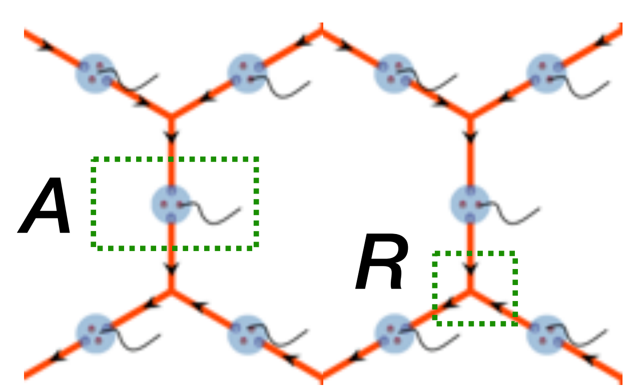
$= C^{0,0}_{\frac{1}{2},m_i;\frac{1}{2},m_j}$

- Nearest Neighbour Resonating Valence Bond (NN-RVB) state

$|\Psi\rangle =$ 

PEPS

ex) NN RVB on Kagame

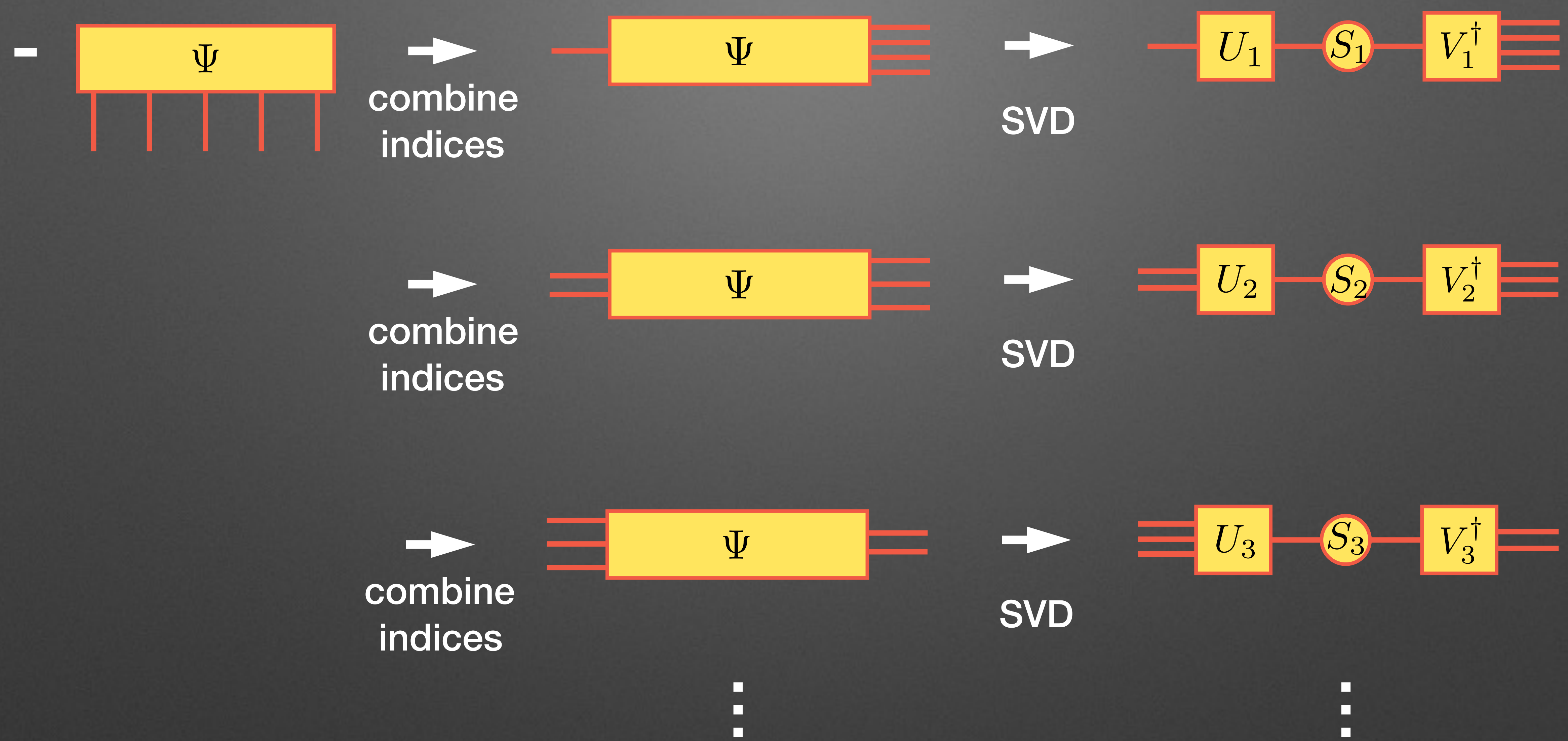


$$\begin{cases} A^s_{2,s} = A^s_{s,2} = 1, \\ \qquad\qquad (s = 0, 1) \\ \text{otherwise} = 0 \end{cases}$$

$$\begin{cases} R_{abc} = \varepsilon_{abc} \\ R_{222} = 1 \end{cases}$$

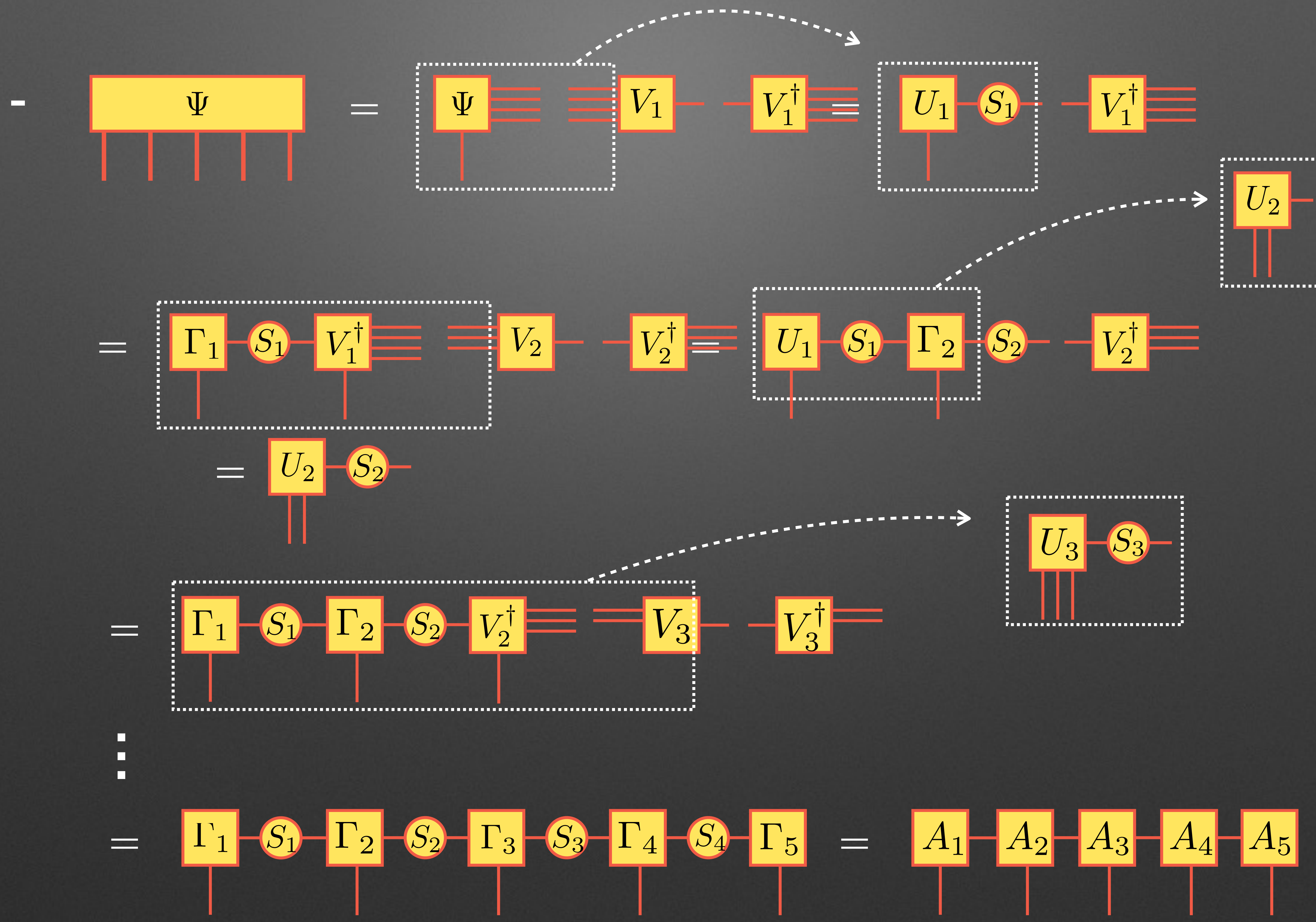
[Cirac et. al., 2013]

❖ Canonical Form of MPS

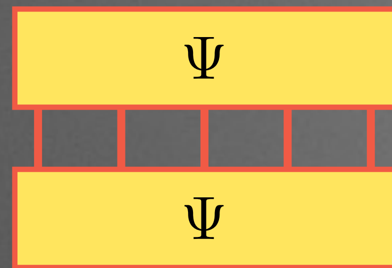


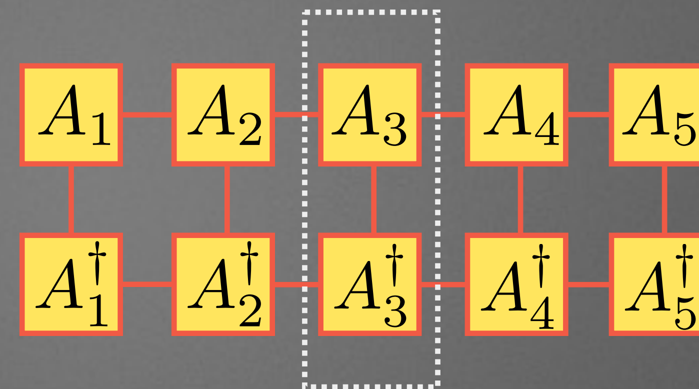
Resolution of Identity :

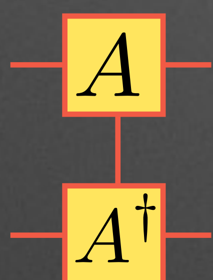
Canonical Form of MPS



- Transfer Matrix

- $\langle \Psi | \Psi \rangle =$ 

$=$ 

- 

: $(\chi^2 \times \chi^2)$ - transfer matrix (TM)