Dirac fermions in condensed matters

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Outline

1. Dirac fermions in relativistic wave equations
2. How do Dirac fermions appear in condensed matters?
3. Band crossing, symmetry, and dimensionality
4. Dirac fermions and topology
5. Interacting Dirac fermions
6. New low energy excitations and their classification
7. Summary

What is a Dirac fermion?

"Dirac equation"

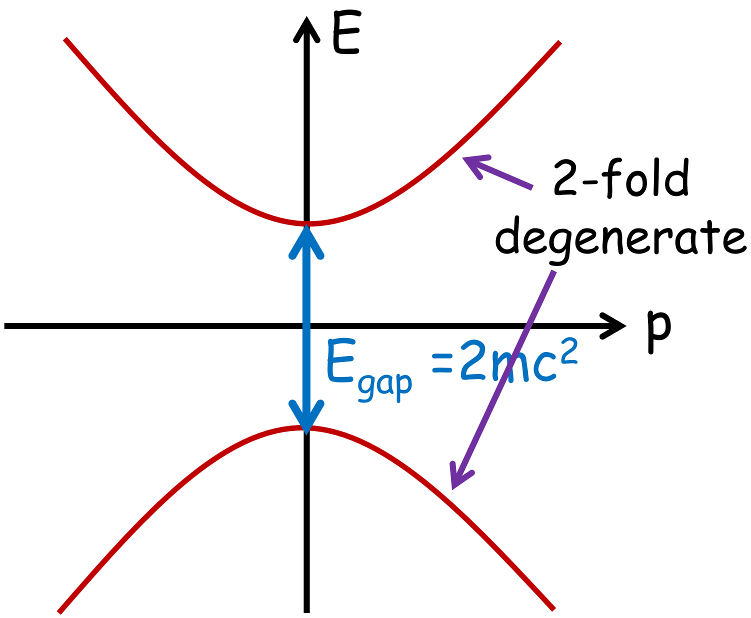
- Relativistic wave equation describing charged massive spin-1/2 fermions

$$i\frac{\partial}{\partial t}\psi = [\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m]\,\psi = [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z + \beta m]\,\psi$$

- To obtain $E^2 = p^2c^2 + m^2c^4$

$$\alpha_i^2 = \beta^2 = 1, \quad \{\alpha_i, \beta\} = 0, \quad \{\alpha_i, \alpha_j\} = 2\delta_{ij},$$

$$\alpha_i = \sigma_i \otimes \tau_3 \quad \beta = \sigma_0 \otimes \tau_1$$



$$i\frac{\partial}{\partial t}\psi = \begin{pmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{p} & m \\ m & -\boldsymbol{\sigma}\cdot\boldsymbol{p} \end{pmatrix}\psi$$

ψ describes a four-component Dirac fermion.

Weyl fermions

- Massless limit of Dirac equation

$$i\hbar\frac{\partial}{\partial t}\psi = \begin{pmatrix} c\boldsymbol{\sigma}\cdot\boldsymbol{p} & 0 \\ 0 & -c\boldsymbol{\sigma}\cdot\boldsymbol{p} \end{pmatrix}\psi$$

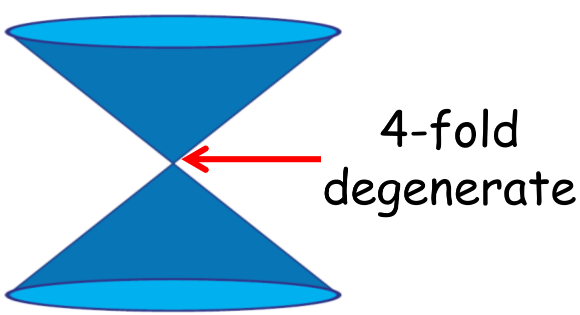
$$\psi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$$

$$i\hbar\frac{\partial}{\partial t}\chi_\pm = \pm c\boldsymbol{\alpha}\cdot\boldsymbol{p}\chi_\pm$$

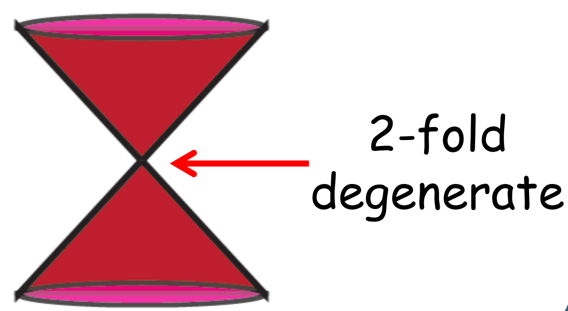
"Weyl equations"

$\chi_\pm$ describe two-component Weyl fermions.

A massless Dirac fermion



A massless Weyl fermion



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“Dirac fermions” in condensed matters

“Emergent” low energy excitations

“More is different“ P.W.Anderson

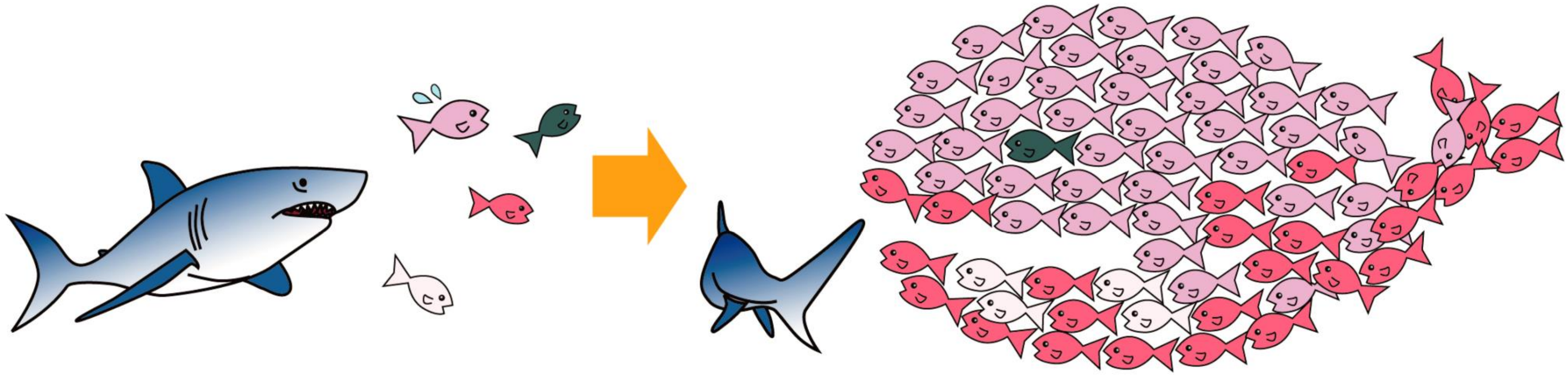
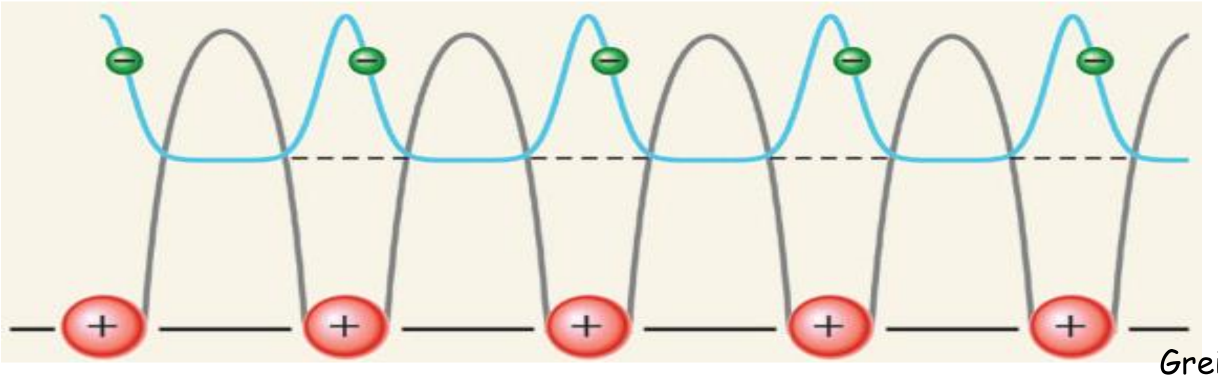


Fig. from C. Terakura

Novel phenomena/effects/functions of solids
which can never be expected for individual constituent elements

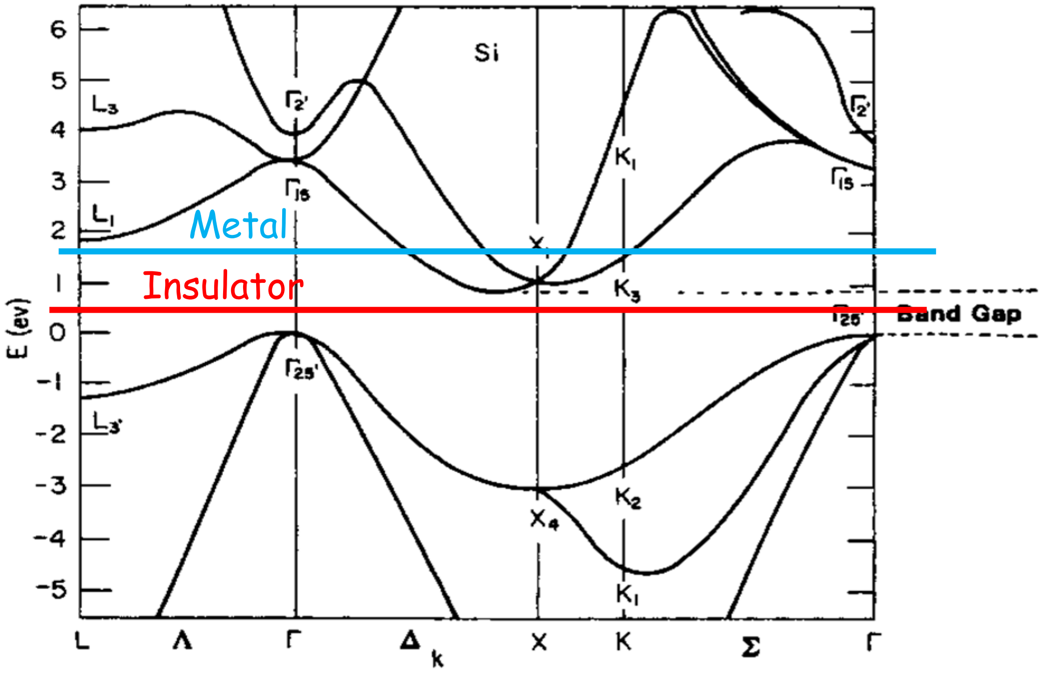
Electronic band structure in periodic solids



Greiner and Folling (2008)

Bloch's theorem: $\Psi_{n,k}(\boldsymbol{r}+\boldsymbol{R}) = e^{i\boldsymbol{k}\cdot\boldsymbol{R}}\Psi_{n,k}(\boldsymbol{r})$

$$E_n(\boldsymbol{k}+\boldsymbol{G}) = E_n(\boldsymbol{k})$$

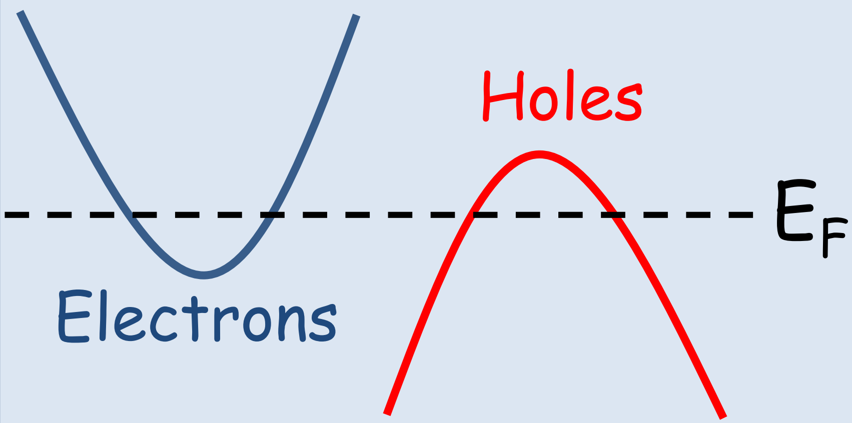


Low energy excitations in metals

- Electrons on a lattice form a band structure
- Low energy excitations are described by emergent particles

Schrodinger particles

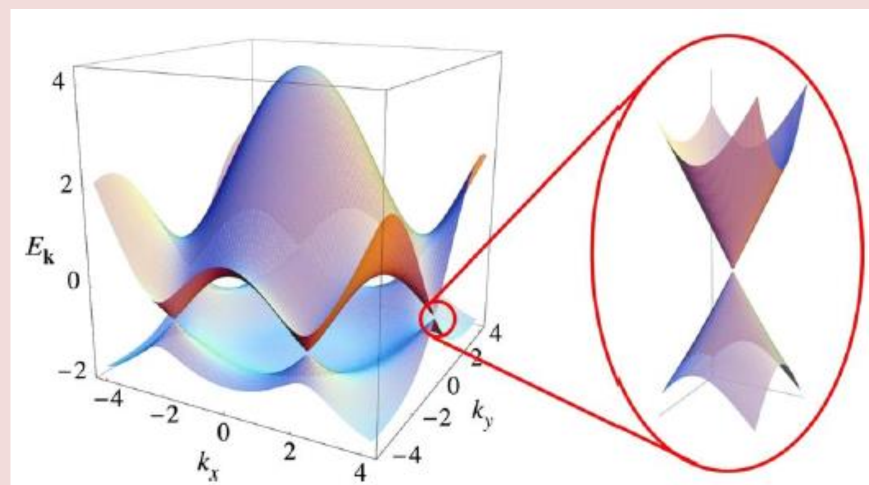
e.g.) Ordinary metals



$$H = -\frac{1}{2m}\frac{\partial^2}{\partial \mathbf{r}^2},$$

Dirac particles

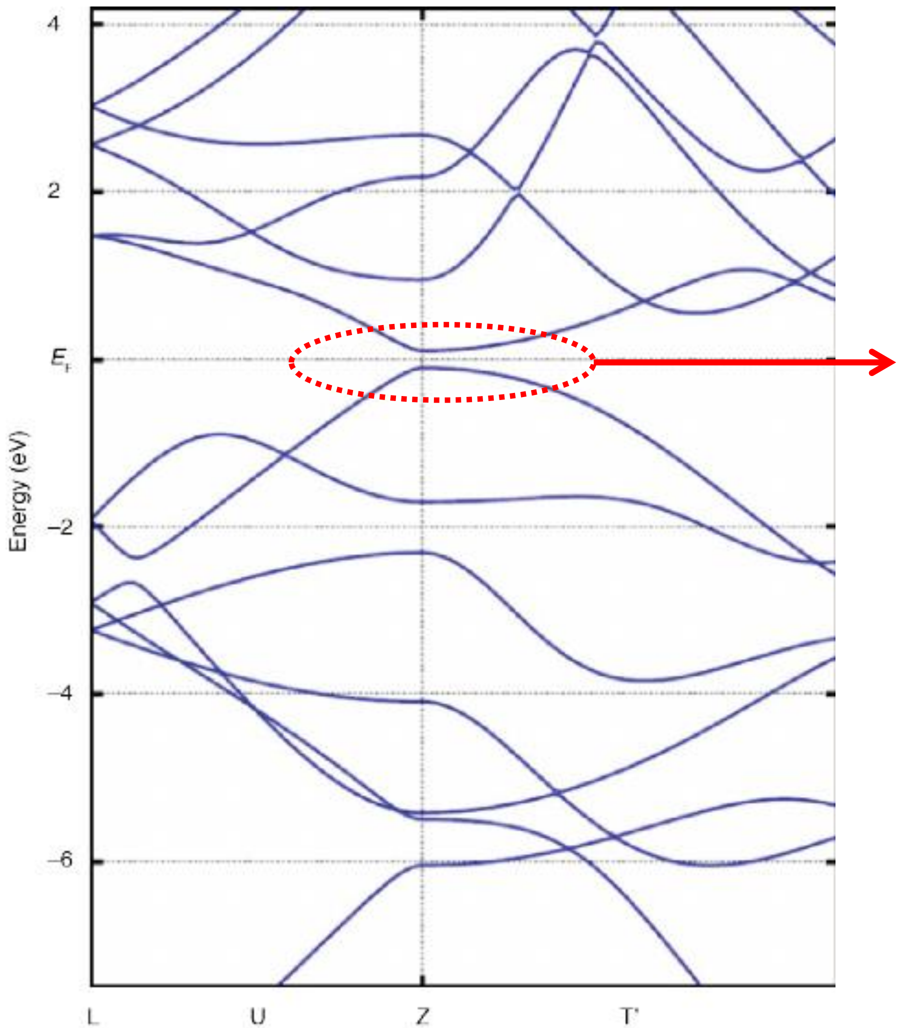
e.g.) 2D graphene



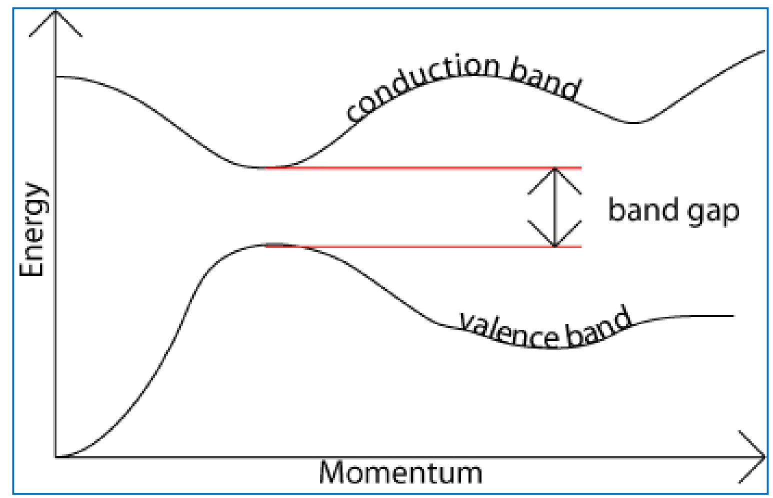
$$H = -iv\frac{\partial}{\partial \mathbf{r}},$$

Massless Dirac fermions: when and why?

- Narrow –gap semiconductors



Bulk black phosphorus
(J. Kim et al, Science, 2015)



"Effective two-band model"

$$H(k)= f_0(k)+ f_1(k)\sigma_1 + f_2(k)\sigma_2 + f_3(k)\sigma_3$$

$$\Delta E_{gap} = 2\underline{(f_1^2 + f_2^2 + f_3^2)^{1/2}}$$

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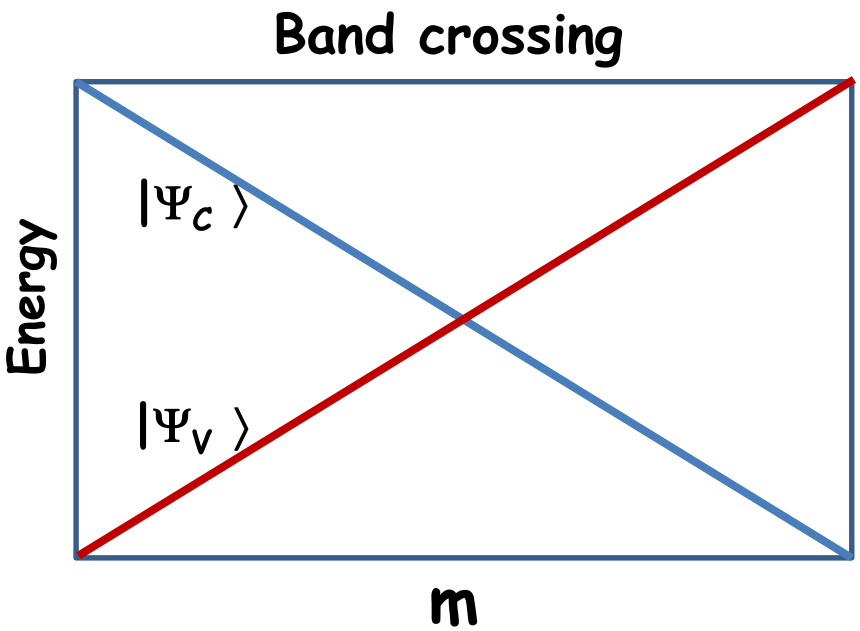
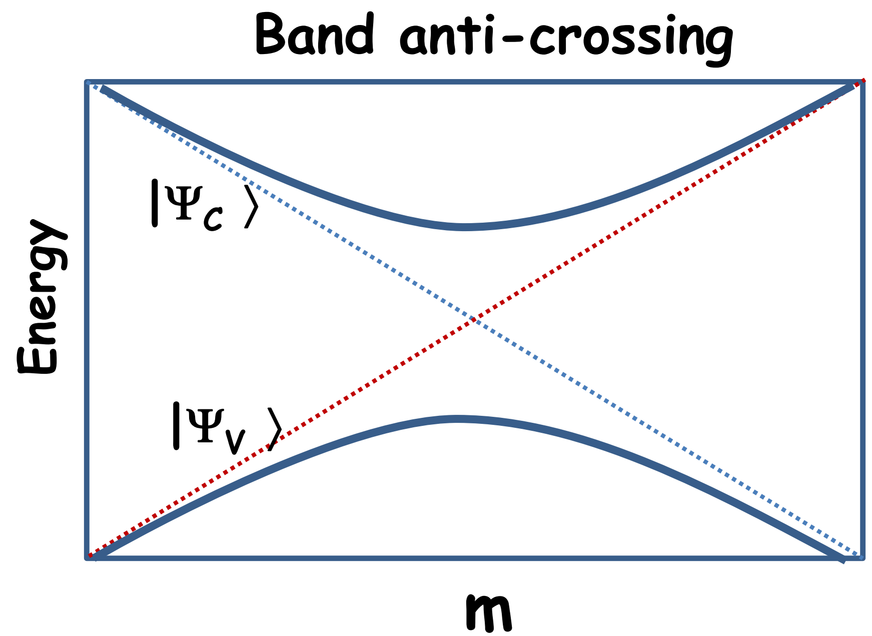
Accidental band crossing

- A Dirac particle can be created by band crossing!



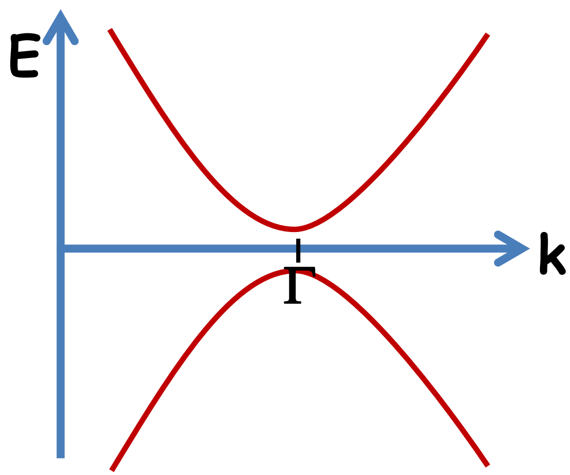
v.s.



(**m**: tuning parameter)

Accidental band crossing is not easy to achieve !

Band crossing in generic systems

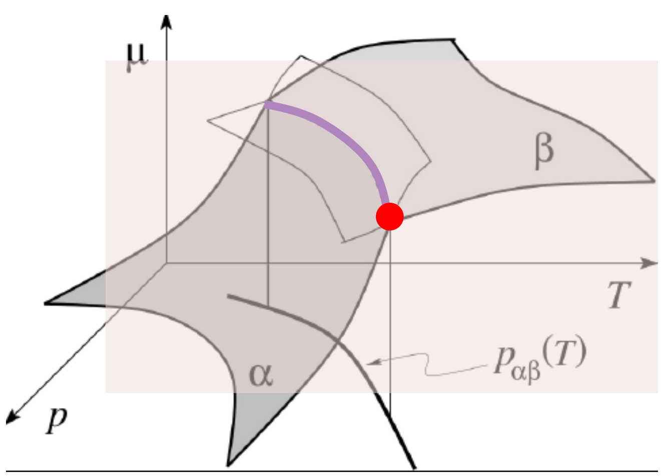


$$H = f_0 + f_1\sigma_1 + f_2\sigma_2 + f_3\sigma_3$$

$$\Delta E = 2\underline{(f_1^2 + f_2^2 + f_3^2)^{1/2}}$$

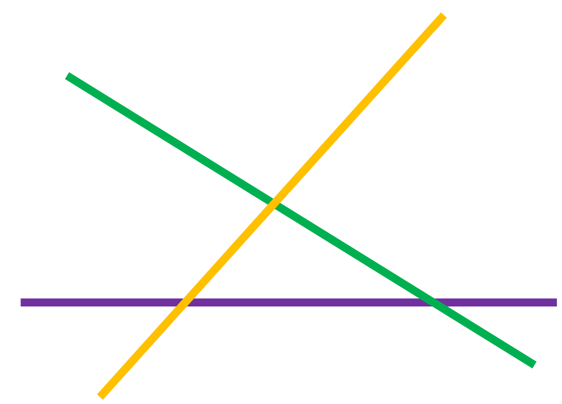
Gap closing :
$$\begin{cases} f_1(k_x, k_y, k_z) = 0 \\ f_2(k_x, k_y, k_z) = 0 \\ f_3(k_x, k_y, k_z) = 0 \end{cases}$$

In 3D : crossing of three planes



Weyl points can exist!

In 2D : crossing of three lines

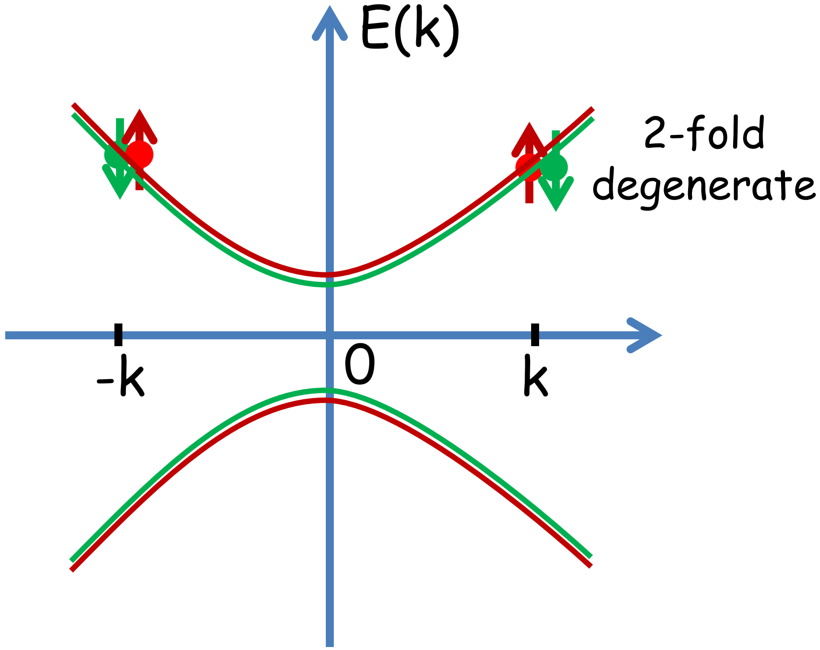


No solution!

Symmetry and band degeneracy

- Time-reversal(T): $E_{n,\uparrow}(k)=E_{n,\downarrow}(-k)$
- Inversion(P): $E_{n,\sigma}(k)=E_{n,\sigma}(-k)$

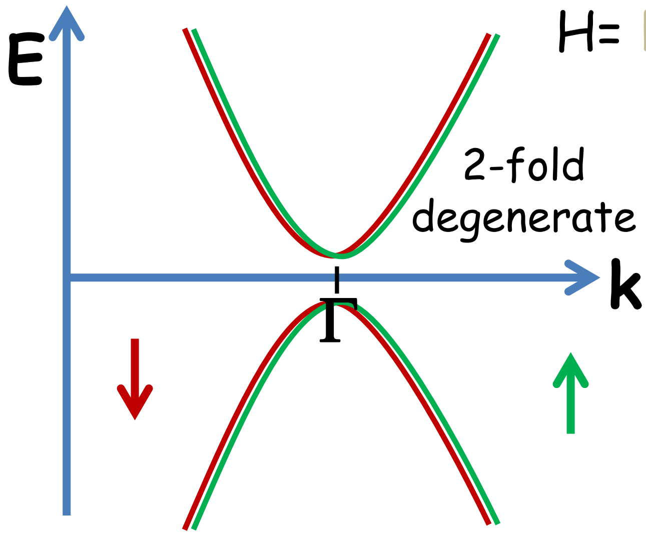
$\Rightarrow$ $E_{n,\uparrow}(k)=E_{n,\downarrow}(k)$



Four bands should cross to generate a Dirac point!

Band crossing and T, P symmetries

- The effective Hamiltonian should be a 4 by 4 matrix!



$H = b_0(k) + b_1(k)\Gamma_1 + b_2(k)\Gamma_2 + b_3(k)\Gamma_3 + b_4(k)\Gamma_4 + b_5(k)\Gamma_5$

where Γ_i is 4 by 4 matrix satisfying $\{\Gamma_i, \Gamma_j\} = 2\delta_{ij}$

e.g.) $\Gamma_1 = \tau_z\sigma_x$, $\Gamma_2 = \tau_z\sigma_y$, $\Gamma_3 = \tau_z\sigma_z$, $\Gamma_4 = \tau_x$, $\Gamma_5 = \tau_y$

$$H^2 = b_1(k)^2 + b_2(k)^2 + b_3(k)^2 + b_4(k)^2 + b_5(k)^2 = E^2$$

$$E_{CB} - E_{VB} = 2\sqrt{b_1^{\ 2} + b_2^{\ 2} + b_3^{\ 2} + b_4^{\ 2} + b_5^{\ 2}}$$

Gap-closing condition : $b_1(k) = b_2(k) = b_3(k) = b_4(k) = b_5(k) = 0$

In general, a band crossing is impossible unless additional symmetries other than T, P

SU(2) symmetry and graphene

- Graphene has both T and P symmetries

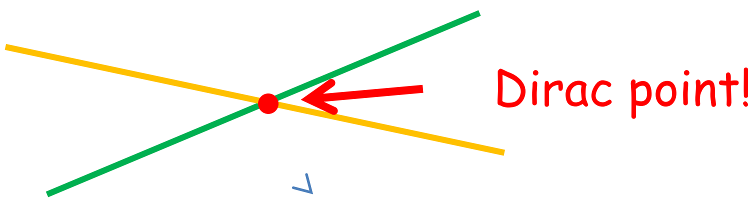
$$H = b_1(k)\,\Gamma_1 + b_2(k)\,\Gamma_2 + b_3(k)\,\Gamma_3 + b_4(k)\,\Gamma_4 + b_5(k)\,\Gamma_5$$
$$= b_1(k)\,\tau_z\sigma_x + b_2(k)\,\tau_z\sigma_y + b_3(k)\,\tau_z\sigma_z + b_4(k)\,\tau_x + b_5(k)\,\tau_y$$

$\sigma_{x,y,z}$ describe spin degrees of freedom

Spin SU(2) symmetry requires $b_1(k) = b_2(k) = b_3(k) = 0$

Gap closing condition:

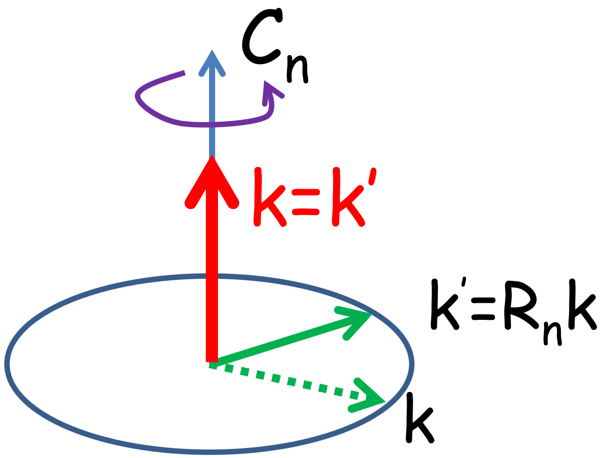
$$\begin{cases} b_4(k_x, k_y) = 0 \\ b_5(k_x, k_y) = 0 \end{cases}$$



In the presence of spin-orbit coupling, graphene is a gapped quantum spin Hall insulator!

Massless fermion and rotation symmetry

Wang, Dai, Fang (2012, 2013);

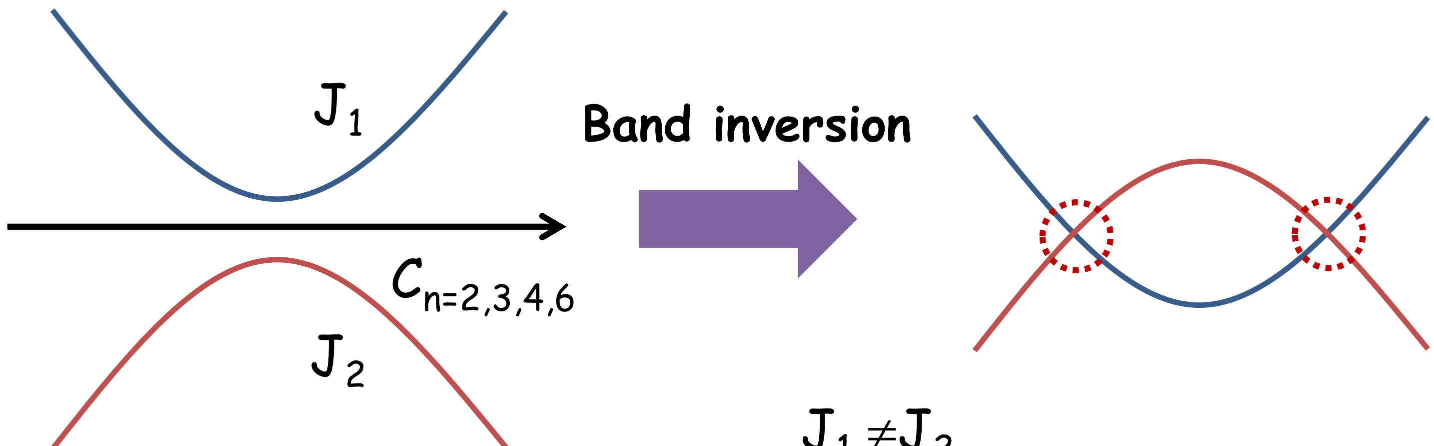


- C_n : rotation by $2\pi/n$ about an axis

$$C_n H(\mathbf{k}) C_n^{-1} = H(R_n \mathbf{k})$$

$[H(k), C_n]=0$ with k on the rotation axis

- Bands carry quantized rotation eigenvalues

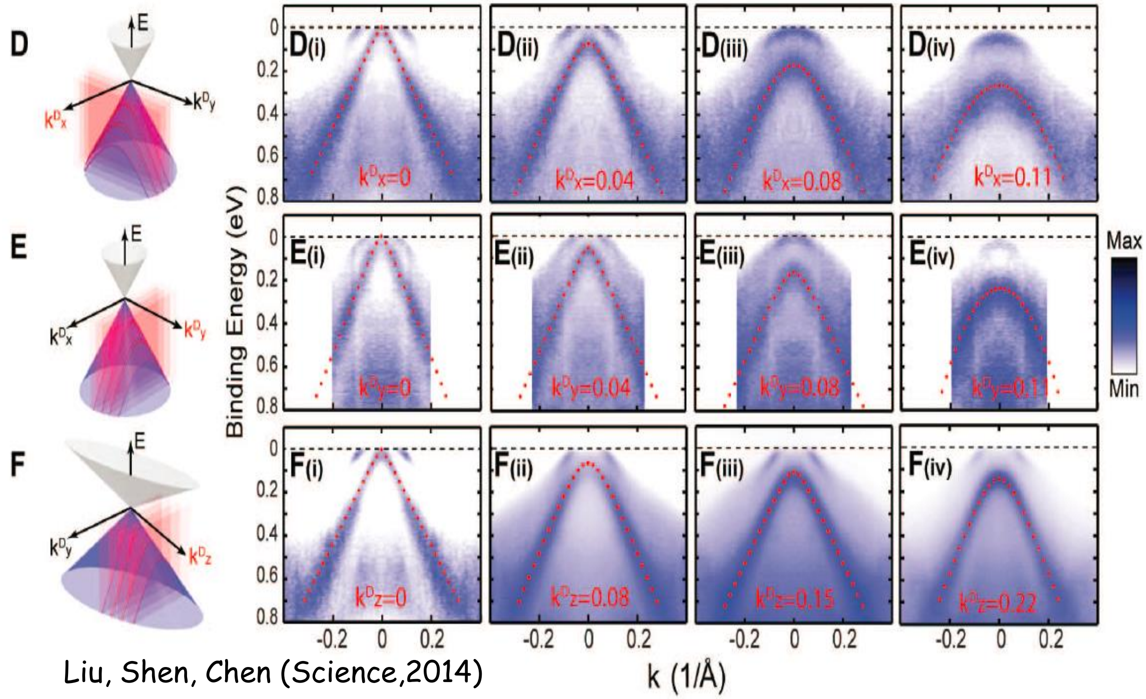


$J_1 \neq J_2$

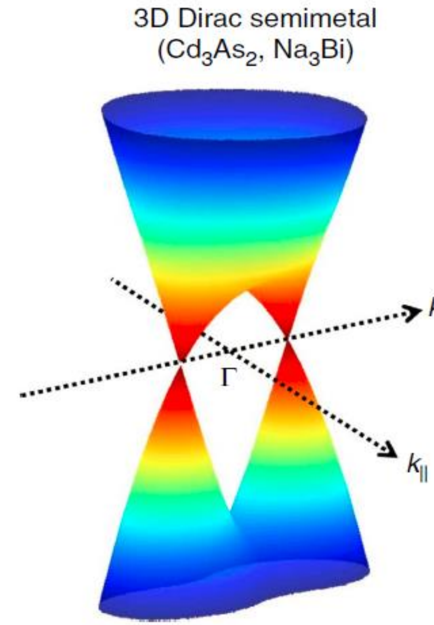
$$H = \begin{pmatrix} H_{CB} & V_{mix} \\ V_{mix} & H_{VB} \end{pmatrix} = \begin{pmatrix} H_{J1} & 0 \\ 0 & H_{J2} \end{pmatrix}$$

Observation of 3D Dirac semimetals

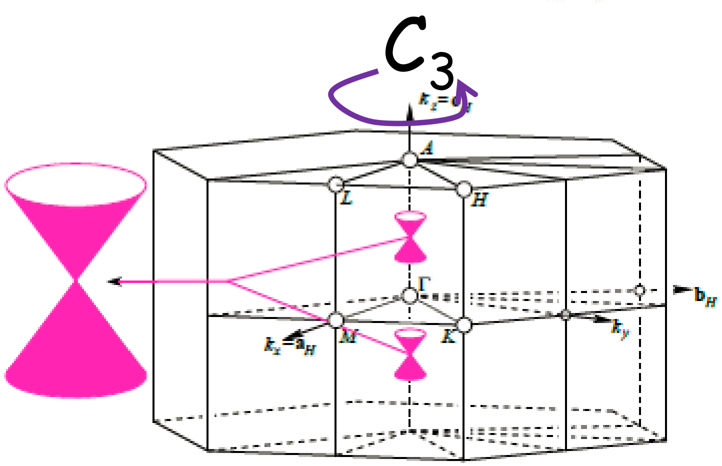
Cd_3As_2, Na_3Bi are confirmed as a 3D Dirac SM!

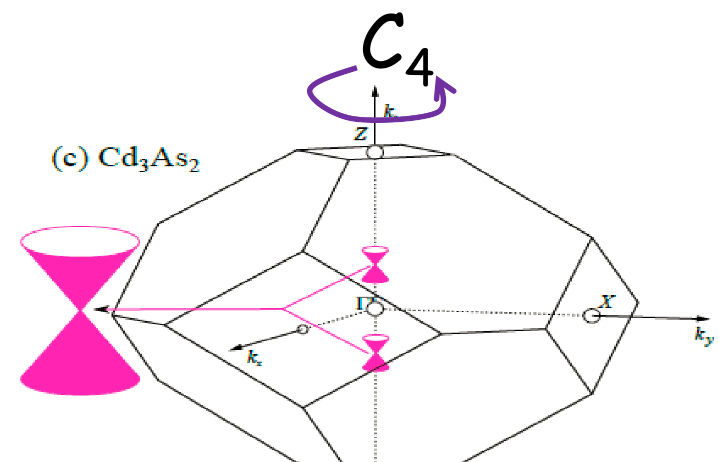


Liu, Shen, Chen (Science,2014)



Neupane, Hasan (Nat. Comm. 2014)





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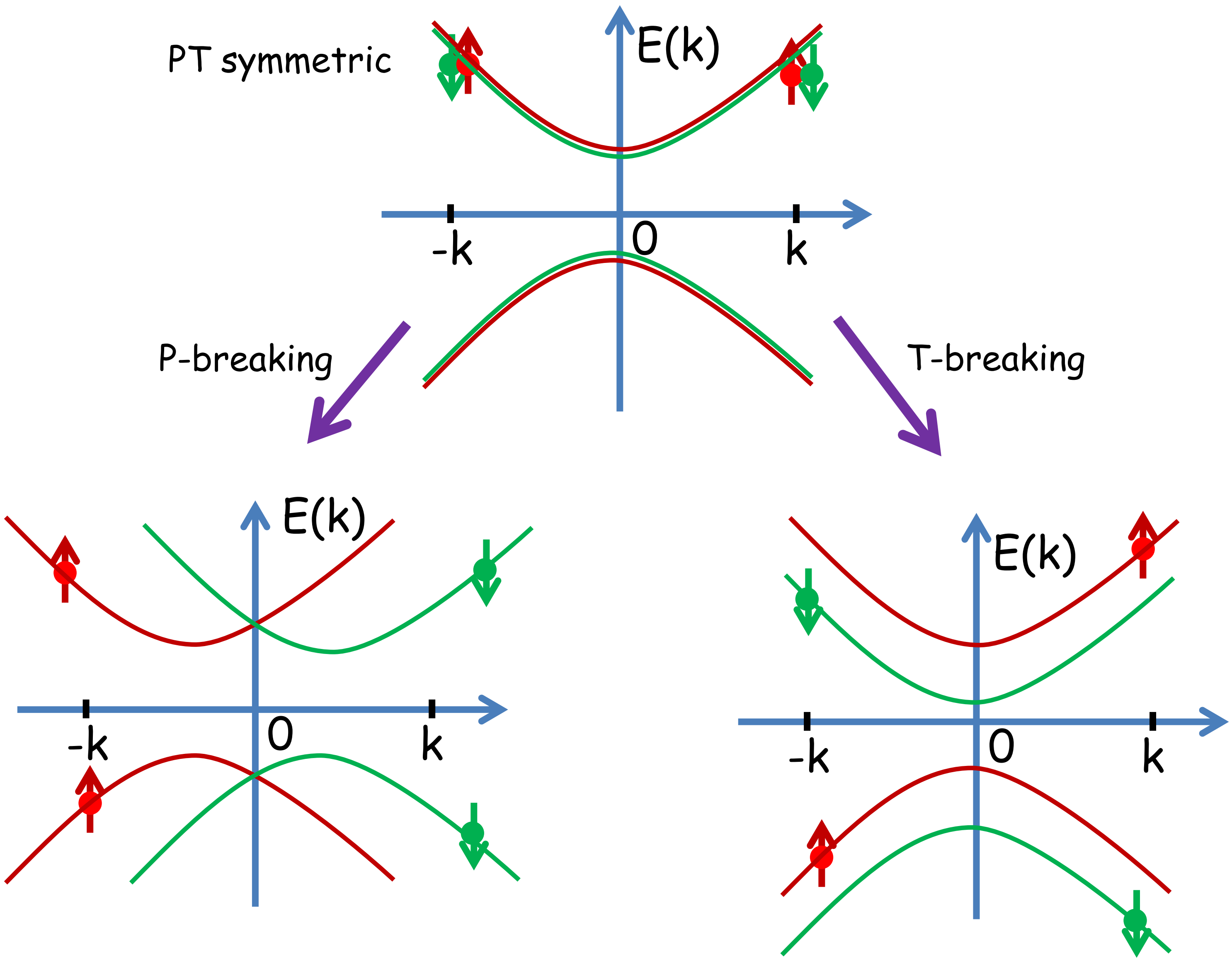
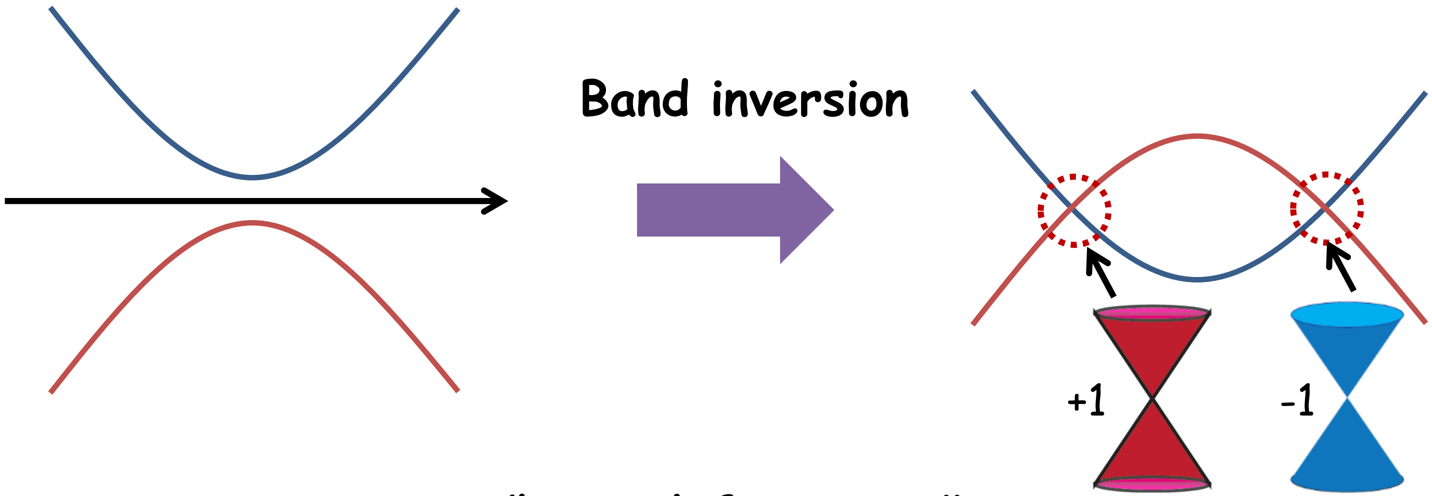
Let's break P or T symmetry



2 by 2 matrix description is possible!

Emergence of Weyl fermions

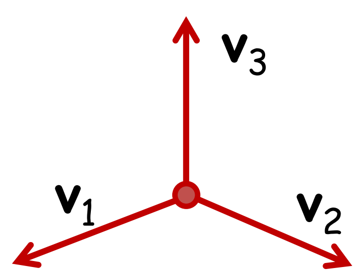


"Weyl fermion"

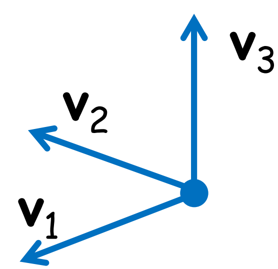
$$H(\mathbf{k}) = (v_1 \cdot k)\sigma_x + (v_2 \cdot k)\,\sigma_y + (v_3 \cdot k)\,\sigma_z$$

"Handedness (or chirality)"

$$\text{Chirality} = \frac{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}{|\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)|}$$



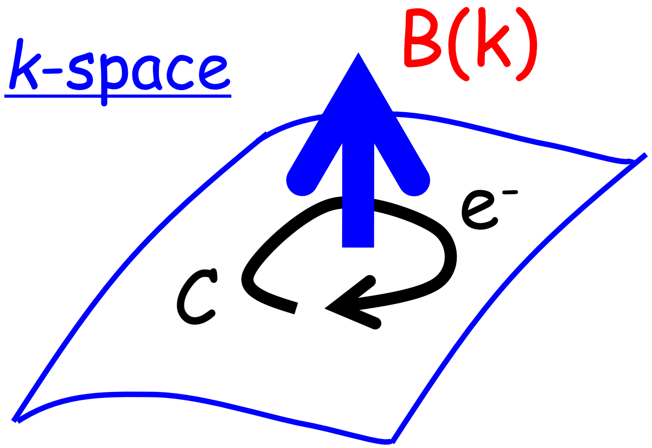
Right-handed (+1)



Left-handed (-1)

A Weyl point is a k-space magnetic monopole

- Berry phase and adiabatic evolution



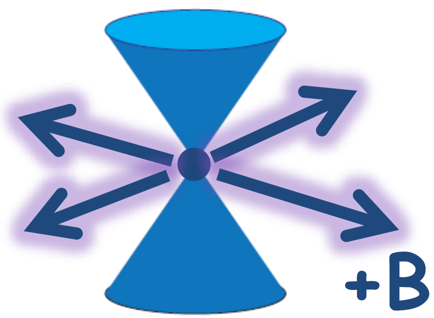
"Berry phase" $\Psi(\mathbf{k}) = e^{i\gamma_C}\Psi_0(\mathbf{k})$

$$\gamma_C = \sum_m \oint_C \mathbf{dk} \cdot \mathbf{A}_m(\mathbf{k}) = \sum_m \int d\boldsymbol{S} \cdot [\nabla \times \boldsymbol{A}_m(\boldsymbol{k})]$$

k-space gauge field

$$\mathbf{A}_n(\mathbf{k}) = i\langle u_n(\mathbf{k})|\nabla_{\mathbf{k}}|u_n(\mathbf{k})\rangle$$

- A Weyl point as a k-space magnetic monopole

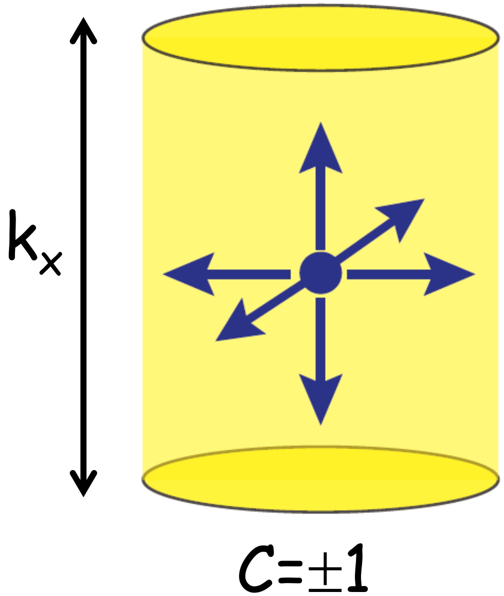


$$\mathrm{B} = \nabla \times \mathbf{A}(\mathbf{k}) \qquad \mathbf{A}(\mathbf{k}) = i\langle u_{\mathbf{k}}|\nabla_{\mathbf{k}}|u_{\mathbf{k}}\rangle$$

$$\frac{1}{2\pi}\nabla_{\mathbf{k}} \cdot \mathrm{B}(\mathbf{k}) = \pm\delta(\mathbf{k}) \qquad \left(\quad \mathrm{B}(\mathbf{k}) \propto \frac{1}{k^2}\hat{k} \right)$$

A Weyl point has a quantized topological (chiral) charge (±1) !

Quantum Hall effect in Weyl semimetals

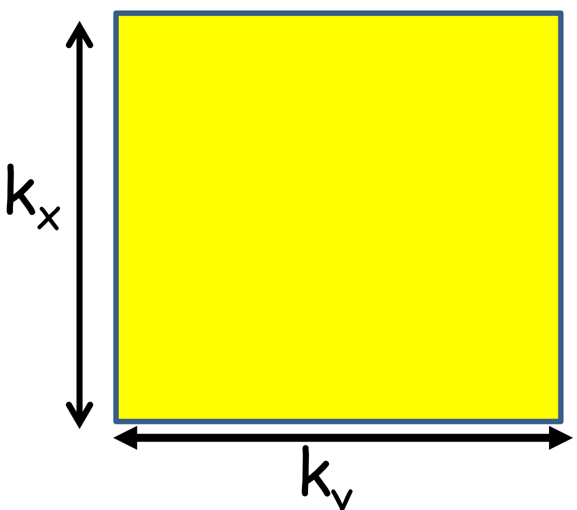


Quantum Hall effect?

$$C = \frac{1}{2\pi} \oint d\mathbf{S}_k \cdot \mathbf{B}(\mathbf{k}) \quad \left(\because \mathbf{B}(\mathbf{k}) \propto \frac{1}{k^2}\hat{k} \right)$$

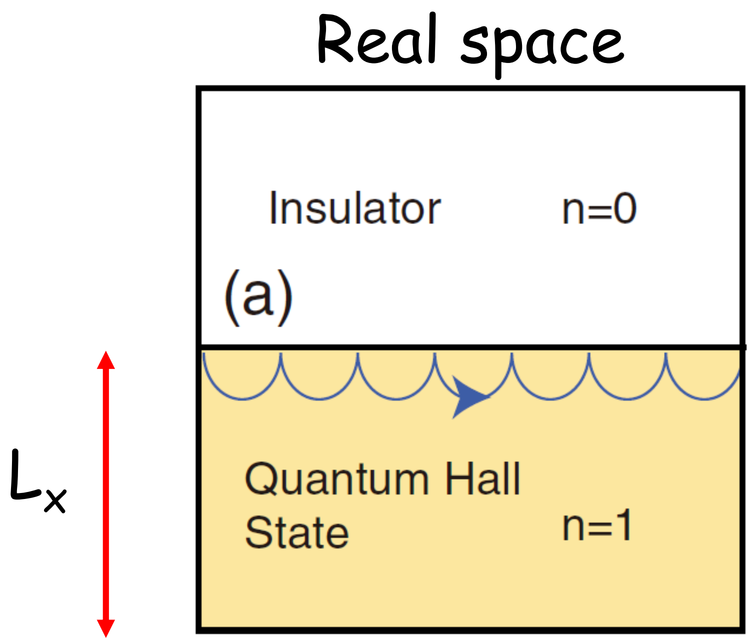
= Chern number of the torus

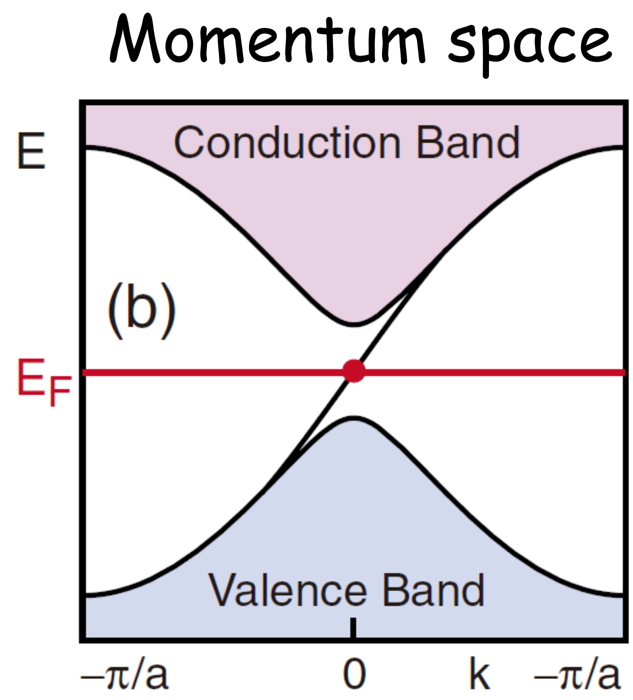
= Total monopole charge in the torus



$$\sigma_{xy} = \frac{e^2}{h} N_C = \frac{e^2}{h} \sum_{m \in \text{occupied}} \int \frac{d^2k}{2\pi} \left[\nabla_k \times \mathbf{A}_m(\mathbf{k}) \right]_z$$

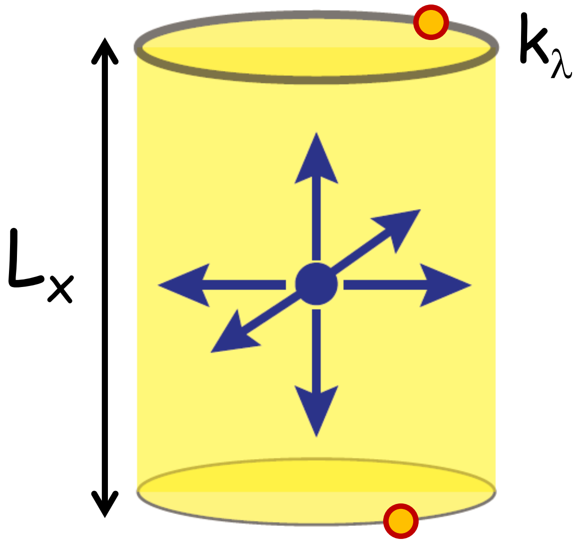
Quantum Hall Effect and chiral edge states

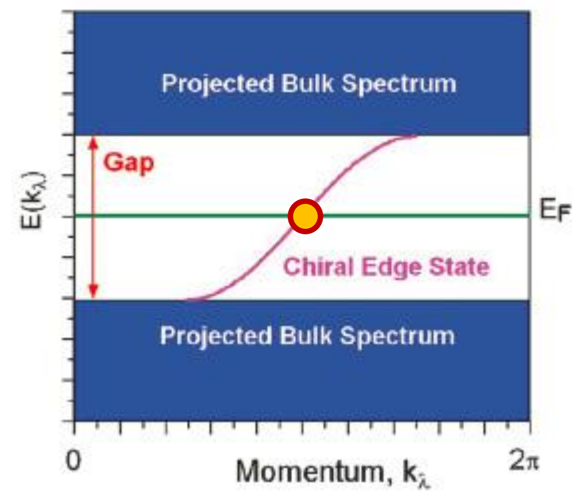




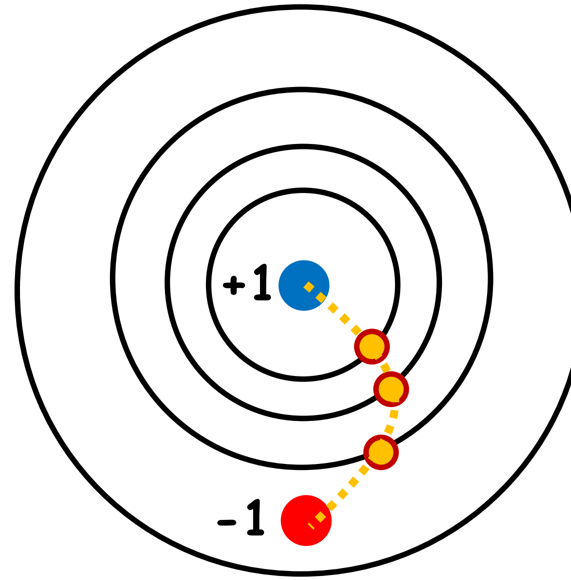
Surface Fermi arc of 3D Weyl SM

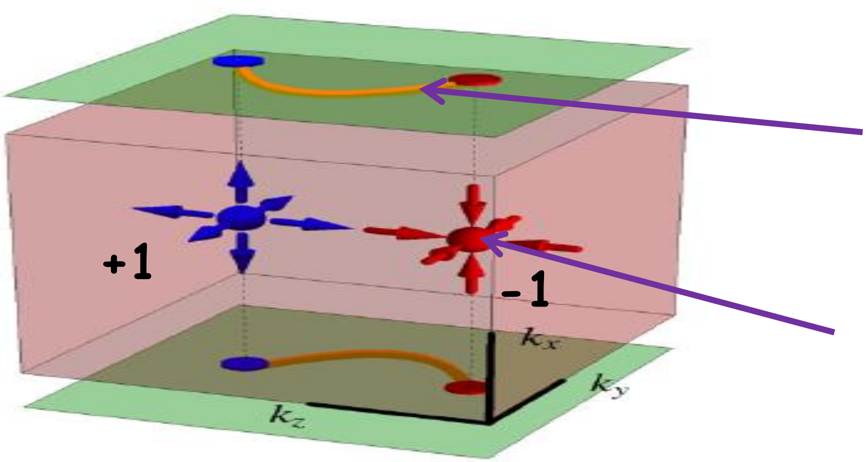
- A sample with finite length along z direction





Top surface



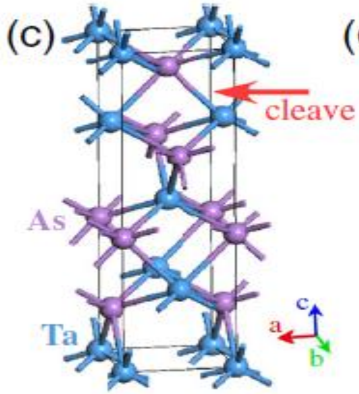


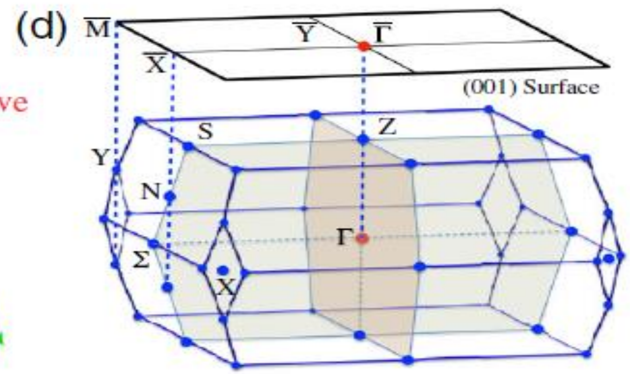
- Surface states : Fermi Arc
- Topological invariant
  : chiral charge

W.Witczak-Krempa, G.Chen, Y.B.Kim, L.Balents

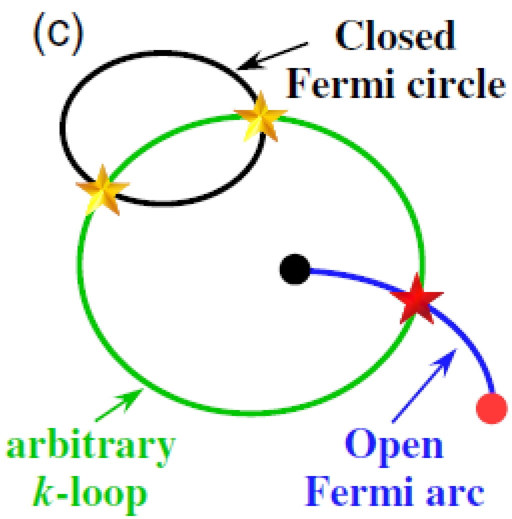
Observation of Fermi arcs

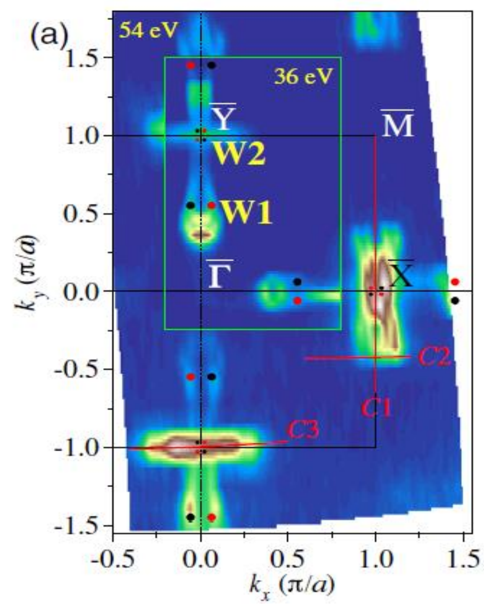
- TaAs, NbP, NbAs, TaP

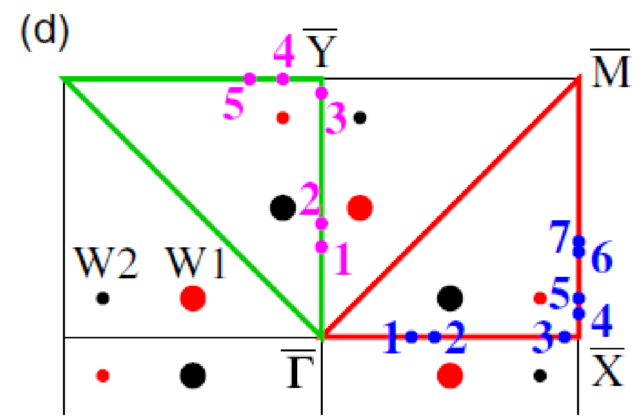




Band calculation predicts
12 Weyl points







Lv,Ding (PRX, 2015)

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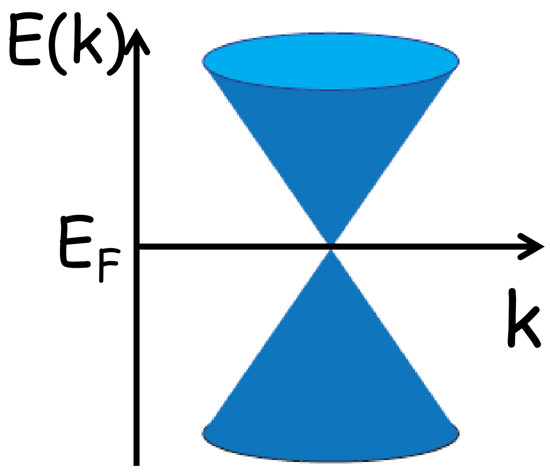
Quantum critical point of topological PT

"Criticality of interacting Weyl/Dirac fermions"

$$H_{QCP} = v_1 k_1 \sigma_1 + v_2 k_2 \sigma_2 + v_3 k_3 \sigma_3$$



Vanishing density of states

$$V_{sc}(r) = \frac{e^2}{\epsilon_0}\frac{1}{r}e^{-q_{TF}r} \quad , q_{TF}{}^2 \propto \mathrm{D}(E_F) = 0$$

Long-range Coulomb potential!

Fermi points (similar to 1D system)

Non-Fermi liquid (Luttinger liquid)

Coulomb interaction in Dirac systems

- Coupling constant

$$E_{\mathrm{kin}} = v|\vec{k}| \qquad E_{\mathrm{C}} = \frac{e^2}{\varepsilon r} \qquad \frac{E_{\mathrm{C}}}{E_{\mathrm{kin}}} \sim \frac{e^2}{\varepsilon v} \equiv \alpha$$

"There is a single dimensionless coupling constant α"

v=c/300, ε=1~100, α=0.1~1

- Effective Lagrangian: "Quantum electrodynamics"

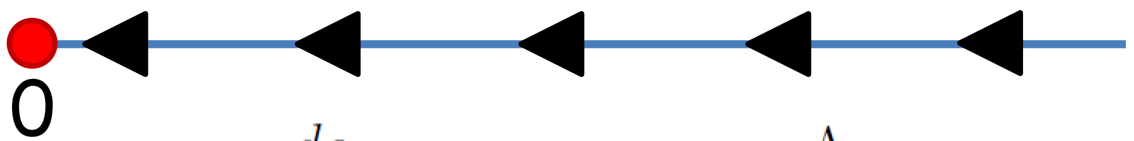
$$\mathcal{L} = \bar{\psi}\left(\gamma^0 p_0 - v\vec{\gamma}\cdot\vec{p} - m\right)\psi + \frac{1}{2}\left(\varepsilon\vec{E}^2 - \frac{1}{\mu}\vec{B}^2\right) - e\bar{\psi}\gamma^0\psi A_0 - e\frac{v}{c}\bar{\psi}\gamma^i\psi A_i$$

Fermi velocity << Light velocity (No Lorentz invariance)

⇒ Instantaneous Coulomb potential

Marginal interaction and log-corrections

- Coulomb interaction is marginally irrelevant



$$\alpha = \frac{e^2}{\varepsilon v}$$

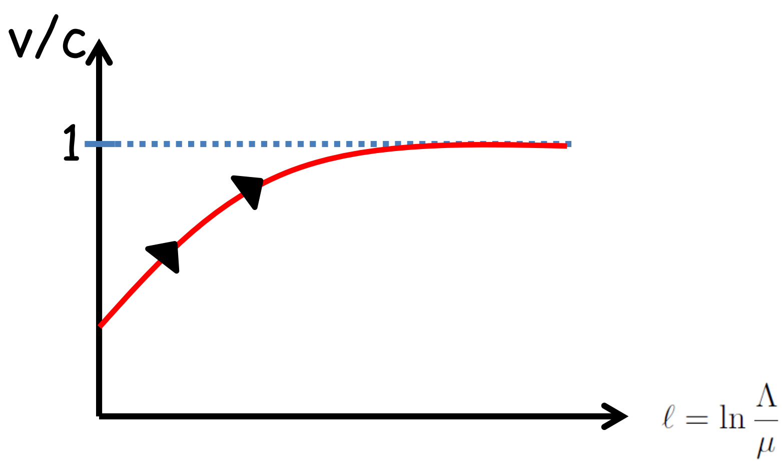
$$\frac{d\alpha}{d\ell} \approx -\alpha^2 \qquad \ell = \ln\frac{\Lambda}{\mu}$$

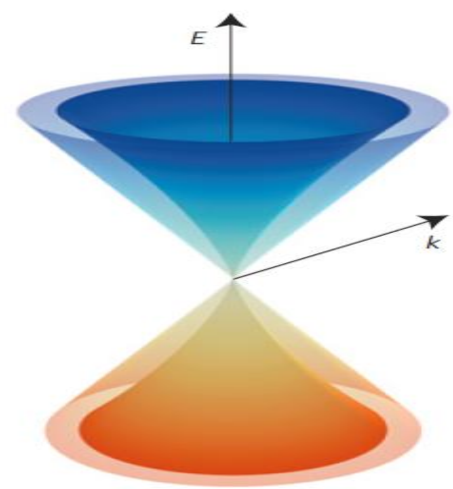
"Logarithmic enhancement of various physical quantities"

(V. N. Kotov et al., RMP (2012);D.T.Son ,PRB (2007); Gonzalez et al., Nucl.Phys.B (1994))

- Velocity renormalization and emergent Lorentz invariance

$$v(k) = v\left[1 + \frac{\alpha}{4}\ln\left(\frac{\Lambda}{k}\right)\right]$$



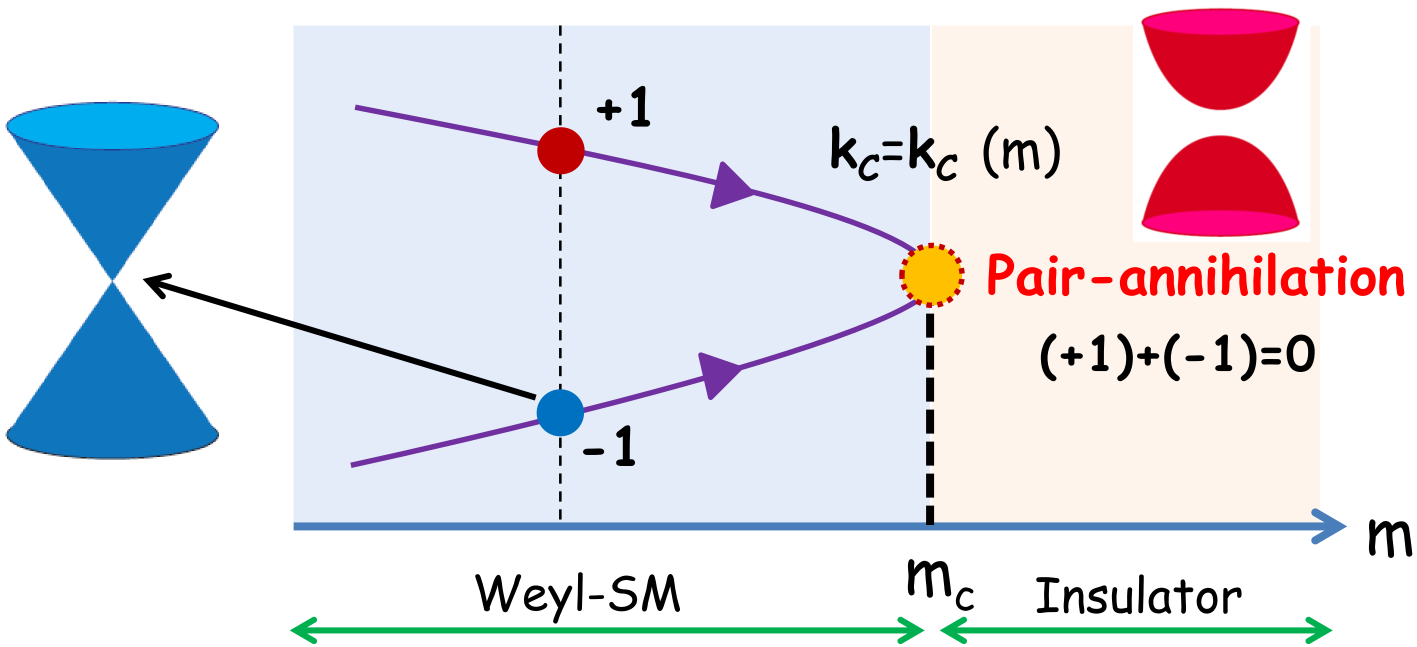


Elias, Geim et al.(2011)

QCP between a Weyl SM to an insulator

The chiral charge of a Weyl point guarantees its stability

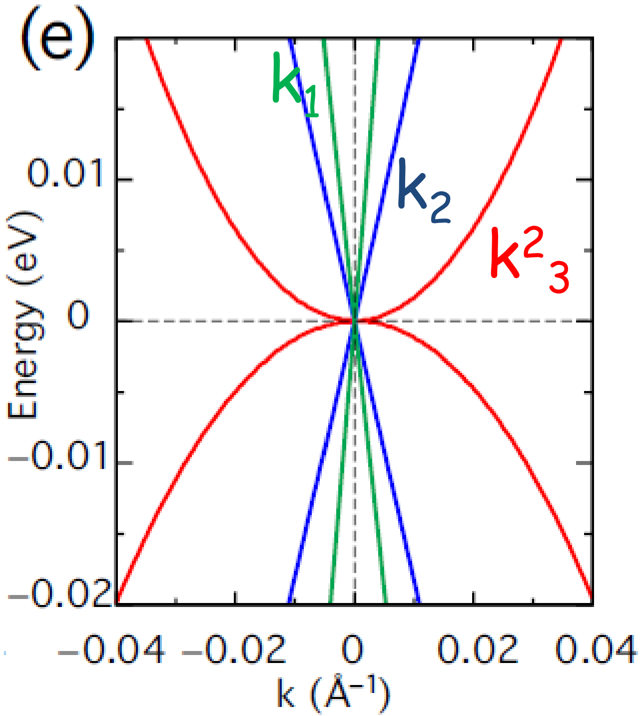
→ A pair-annihilation is required



$$H = k_1 \sigma_1 + k_2 \sigma_2 + [(m-m_c) + k_3^2]\sigma_3$$

Anisotropic Weyl fermions at QCP

$$H_{QCP} = v\, k_1\, \sigma_1 + v\, k_2\, \sigma_2 + A\, k^2{}_3\, \sigma_3$$



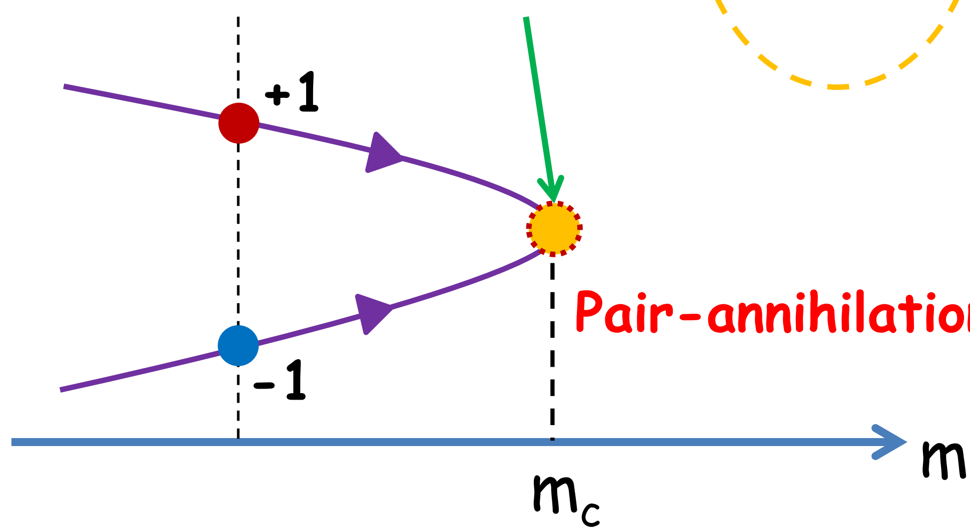
Anisotropic dispersion at QCP

: direct consequence of zero chiral charge !

$$H_{+Weyl} = v\, k_1\, \sigma_1 + v\, k_2\, \sigma_2 + v\, k_3\, \sigma_3$$

+

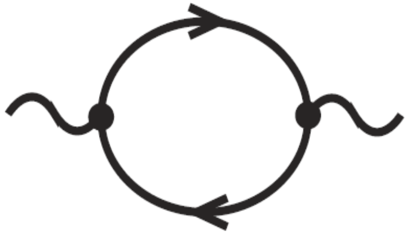
$$H_{-Weyl} = v\, k_1\, \sigma_1 + v\, k_2\, \sigma_2 - v\, k_3\, \sigma_3$$



Correlation effect at QCP

- Polarization

$$\Pi(\mathbf{q}) =$$



$$= -B_\perp q_\perp^{3/2} - B_3 q_3^2$$

(A.A. Abrikosov)

- Screened Coulomb interaction

$$V_C(\mathbf{q}) \sim \frac{1}{q_\perp^{3/2} + \eta q_3^2}$$

"Anisotropic partial screening!"

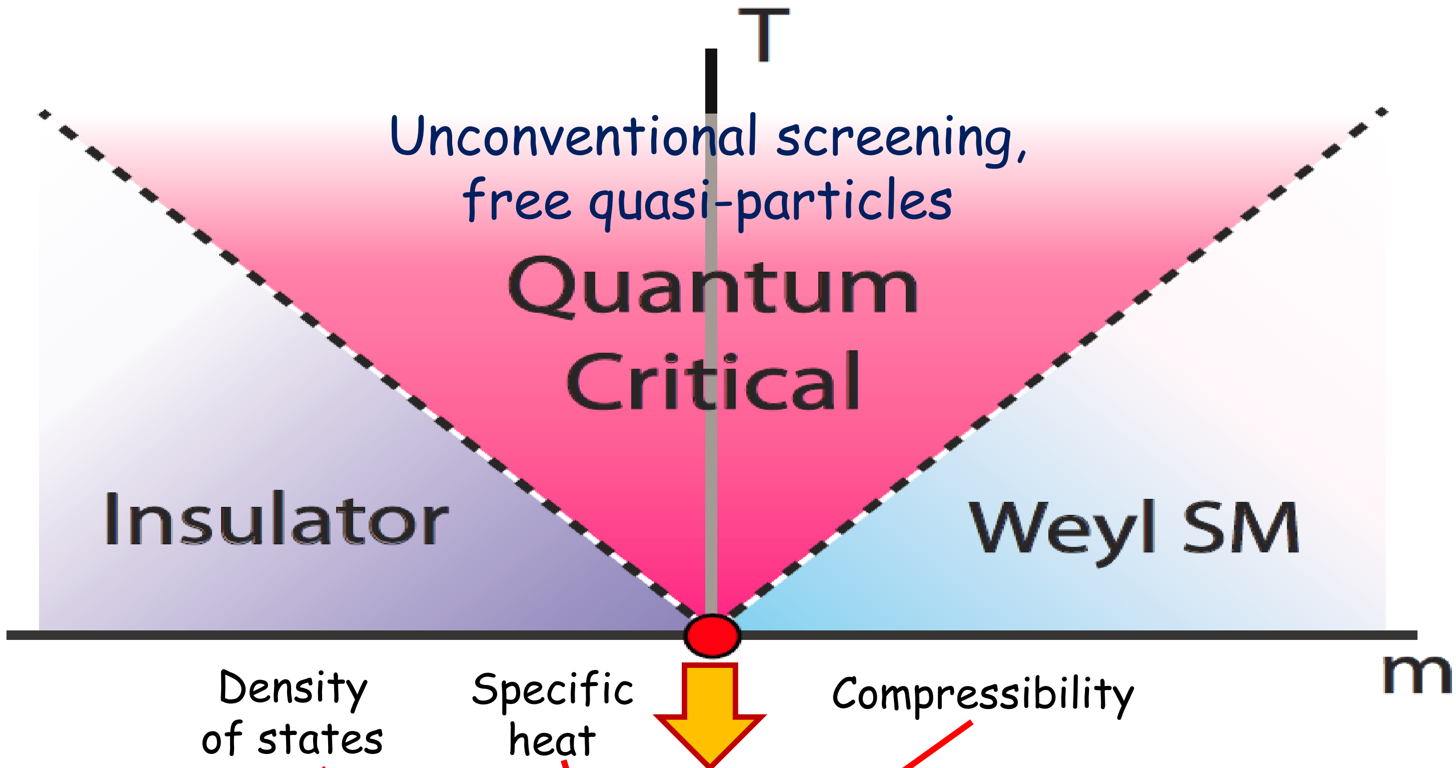
In real space : $V_{\mathrm{C}}(r_\perp, z=0) \sim \frac{1}{r_\perp^{5/4}}, \quad V_{\mathrm{C}}(r_\perp = 0, z) \sim \frac{1}{|z|^{5/3}},$

"Effective interaction between fermions became weaker!"

Electrons behaves like free particles!

Interacting anisotropic Dirac fermions

"Unconventional quantum criticality"



Density of states — Specific heat — Compressibility — Diamagnetic susceptibility

	$D(\varepsilon)$	$C_V(T)$	$\kappa(T)$	$\chi_D(T)$
Weyl semi-metal	ε^2	$T^3(\ln T)^3$	$T^2(\ln T)^3$	$(\ln T)^2$
At the QCP	$\varepsilon^{3/2}$	$T^{5/2}$	$T^{3/2}$	$T^{-1/2}$

(**B.-J. Yang**, N. Nagaosa et al., Nature Physics, 2014)

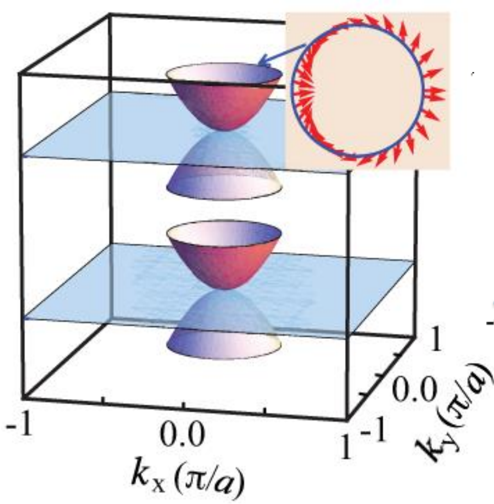
Outline

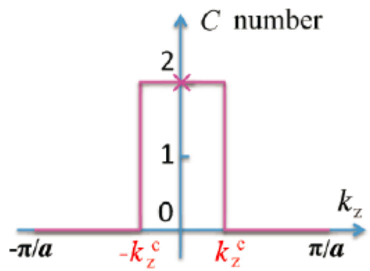
1. Dirac fermions in relativistic wave equations
2. How do Dirac fermions appear in condensed matters?
3. Band crossing, symmetry, and dimensionality
4. Dirac fermions and topology
5. Interacting Dirac fermions
6. New low energy excitations and their classification
7. Summary

New Weyl/Dirac fermions

"Double Weyl SM"

- Chiral charge=±2
- $H=(k_x^2 - k_y^2)\sigma_1+2k_xk_y\sigma_2+k_z\sigma_3$





- C_4 or C_6 rotation symmetry is required
- $HgCr_2Se_4$, $SrSi_2$

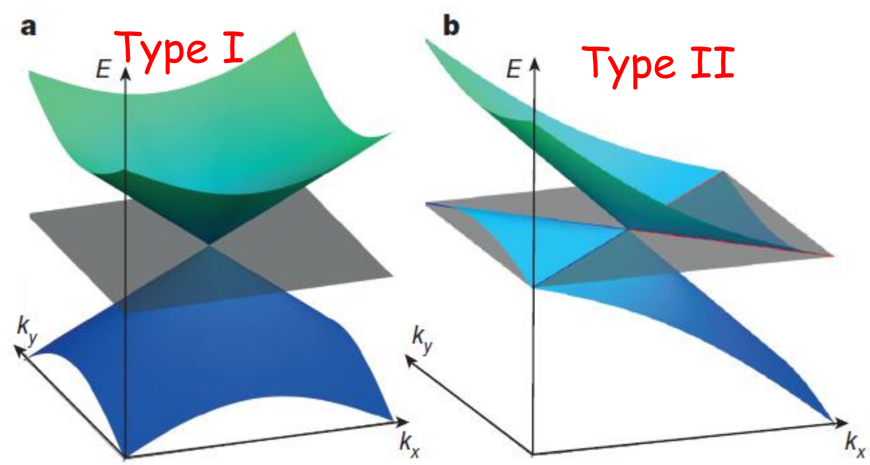
X. Gu, X. Dai, Z. Fang, PRL (2012)

"Type-II Weyl SM"

- Tilted Dirac cone

$$H(\boldsymbol{k}) = \sum_{i=x,y,z} k_i A_{i0} + \sum_{i,j=x,y,z} k_x A_{ij}\sigma_j$$

$$\varepsilon_\pm(\boldsymbol{k}) = \sum_{i=x,y,z} k_i A_{i0} \pm \sqrt{\sum_{j=x,y,z}\left(\sum_{i=x,y,z} k_i A_{ij}\right)^2}$$

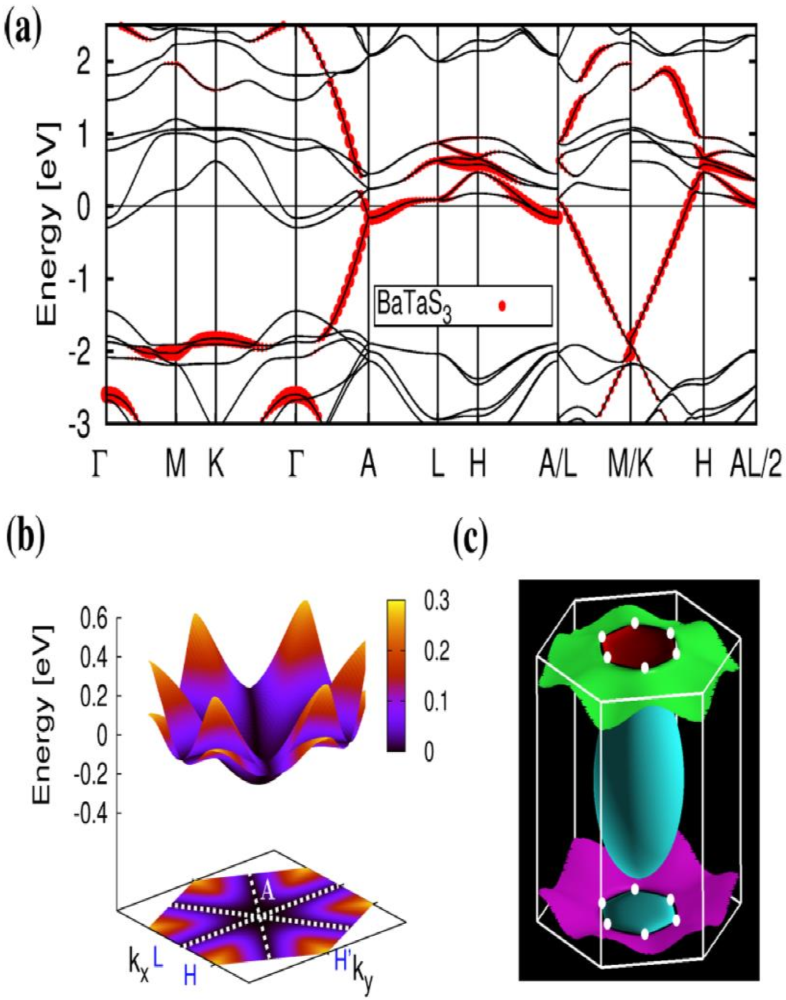


- WTe_2

Soluyanov, Dai, Bernevig, Nature (2015)

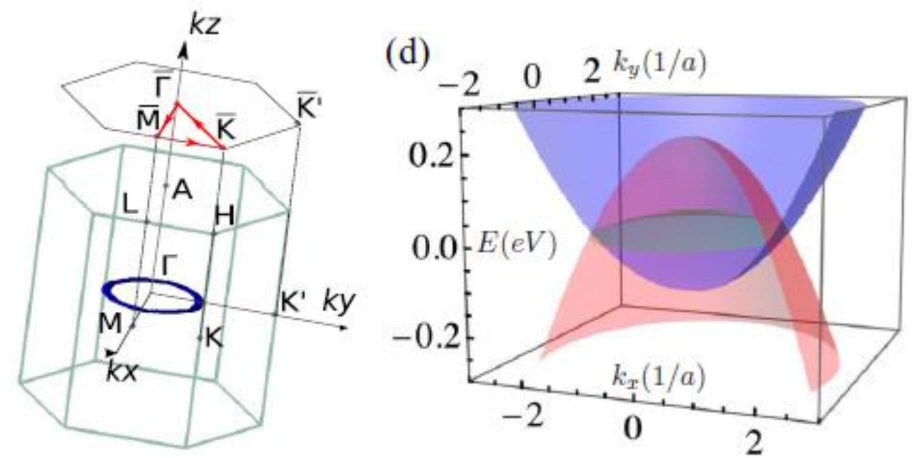
Line node semimetals

- $BaTaS_3$

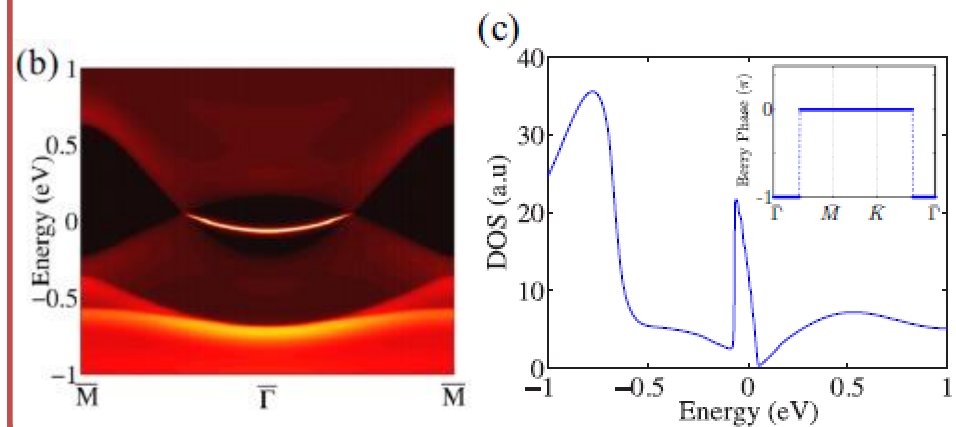


Liang, Weng, PRB (2016)

- Ca_3P_2



Surface flat band

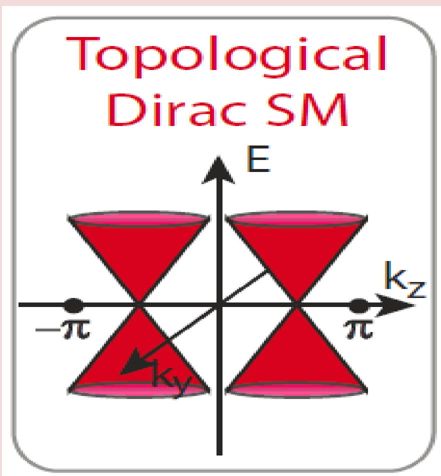


Chan, Schnyder, PRB (2016)

Classification based on crystalline symmetry

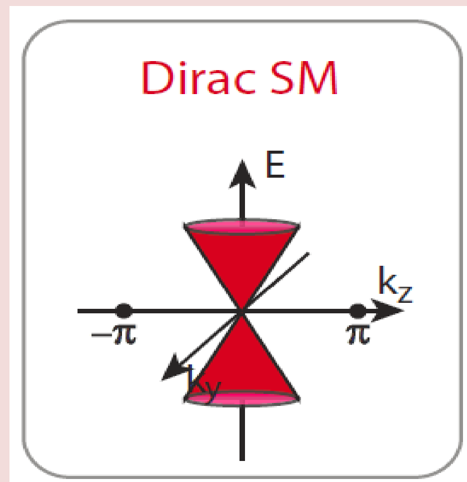
- Variety of Dirac semimetals protected by symmetries!

Class I



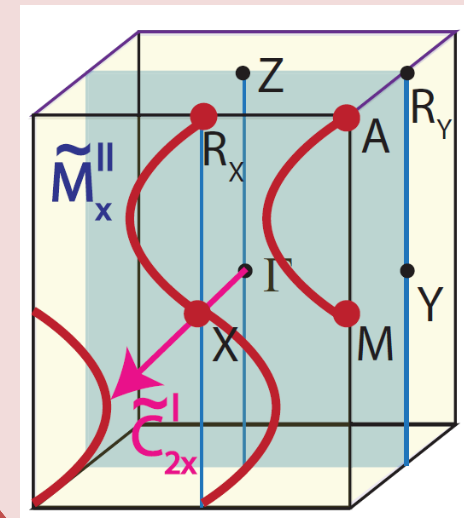
Symmorphic rotation symmetry

Class II



Type-I nonsymmorphic symmetry

Class III



Type-II nonsymmorphic symmetry

Summary

- Variety of nodal semimetals protected by symmetries
- New emerging fermionic particles
- New topological responses
- New correlated phenomena

Band crossing and phase diagram

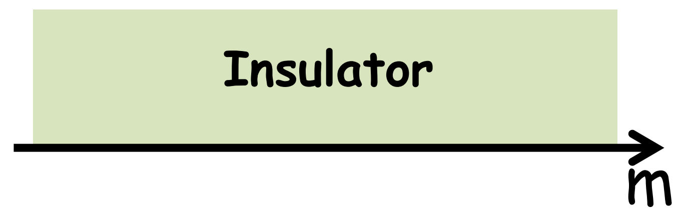
- Condition for accidental band crossing

$$\left.\begin{aligned} f_1(x_1, x_2, \ldots, x_D, m) &= 0 \\ f_2(x_1, x_2, \ldots, x_D, m) &= 0 \\ &\vdots \\ f_N(x_1, x_2, \ldots, x_D, m) &= 0 \end{aligned}\right\}$$

Solve "N" coupled equations with "D+1" variables

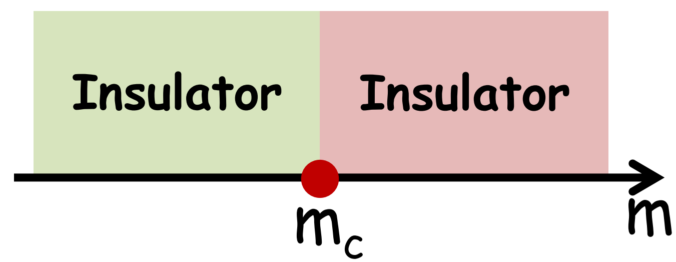
(1) $N > D+1$

- No solution
- Gapped always



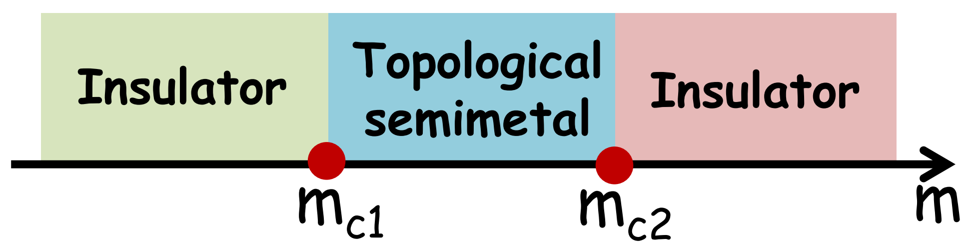
(2) $N = D+1$

- Unique solution
- Critical point



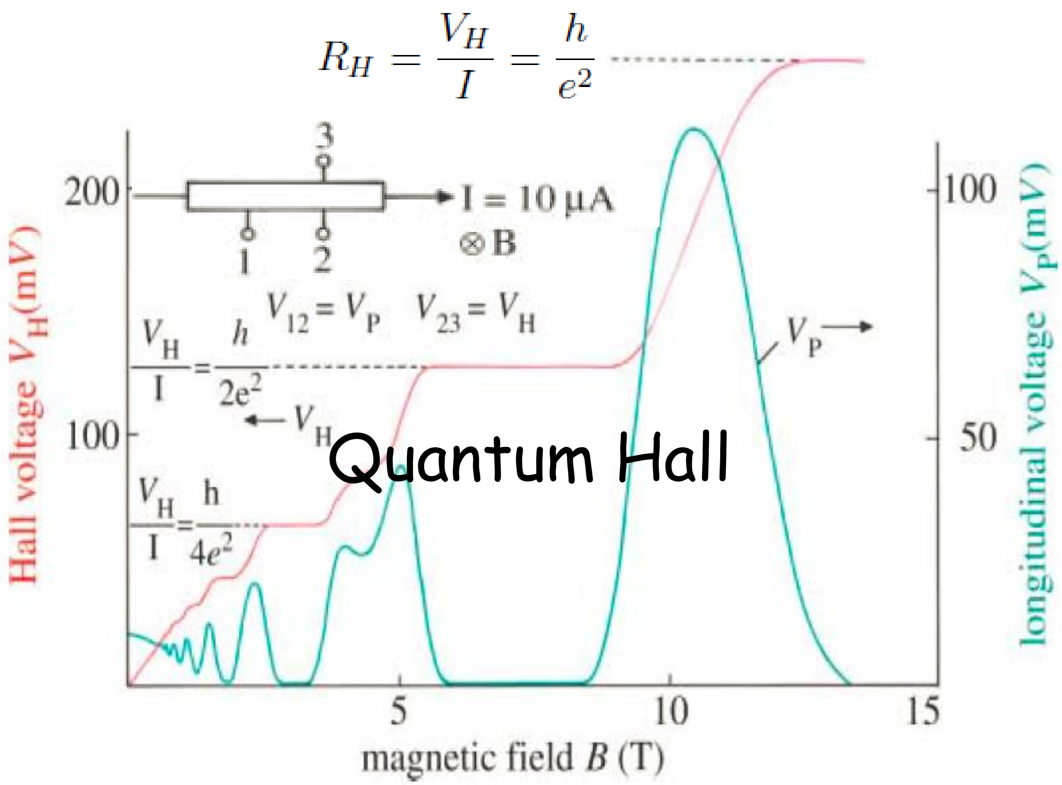
(3) $N < D+1$

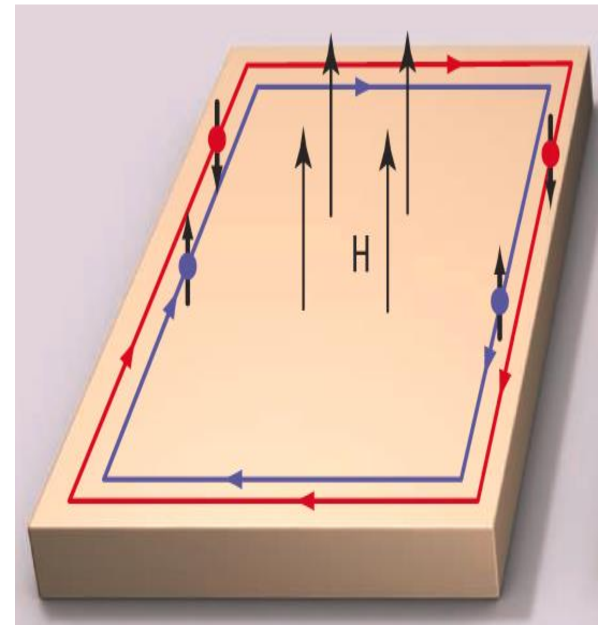
- Many solution
- Critical phase



Applications

- Dirac fermions are a basis for non-dissipative electronics!



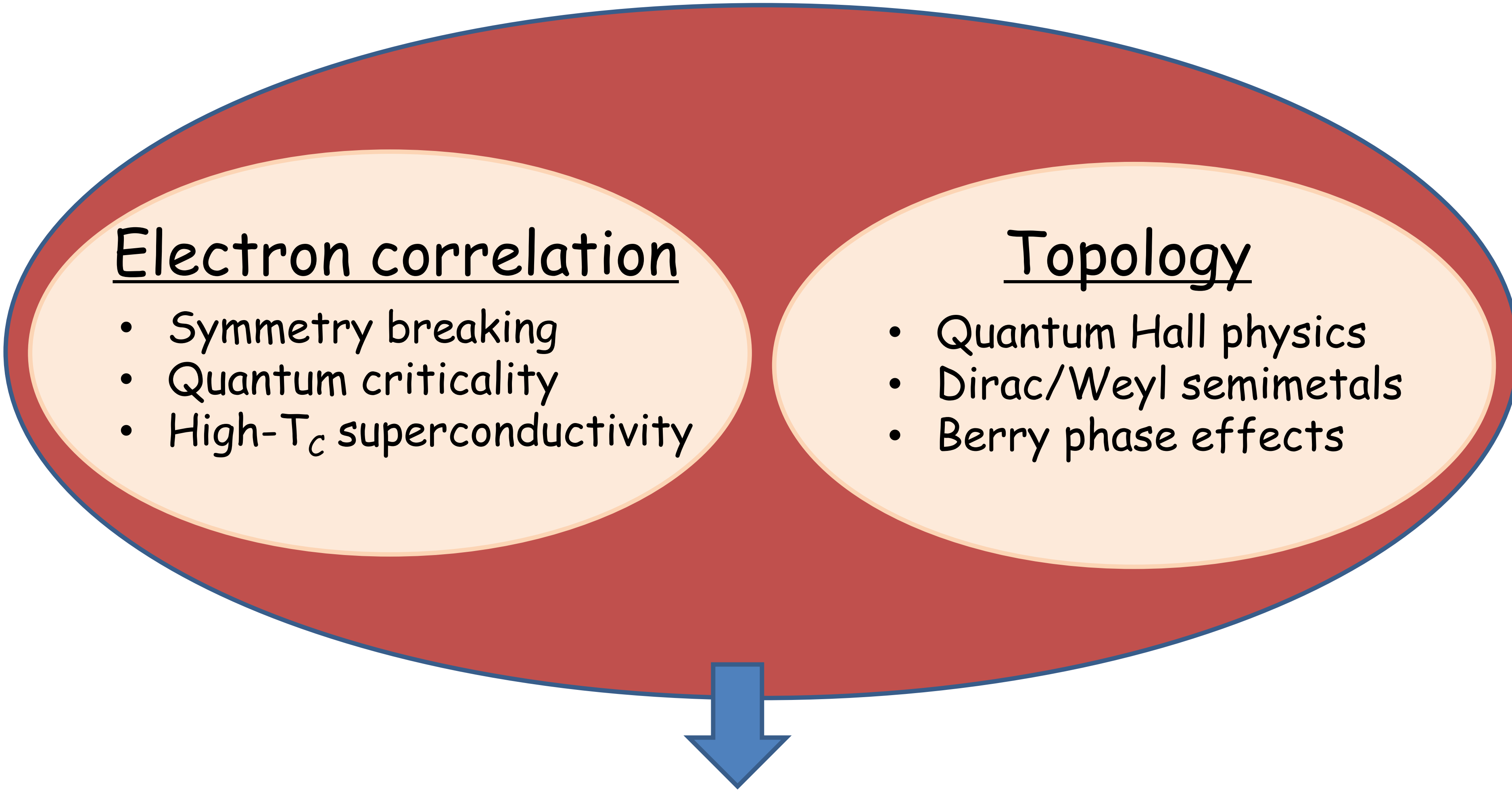


1D Chiral Dirac fermions on the sample boundary!

K. v. Klitzing, G. Dorda, and M. Pepper (1980); K. v. Klitzing (2005)

1. **Quantized Hall Resistance ($R_H = \frac{h}{e^2}\frac{1}{N_c}$, N_c=integer)**
2. **Zero longitudinal resistance (No back scattering, dissipationless)**

Emergent physics in the future

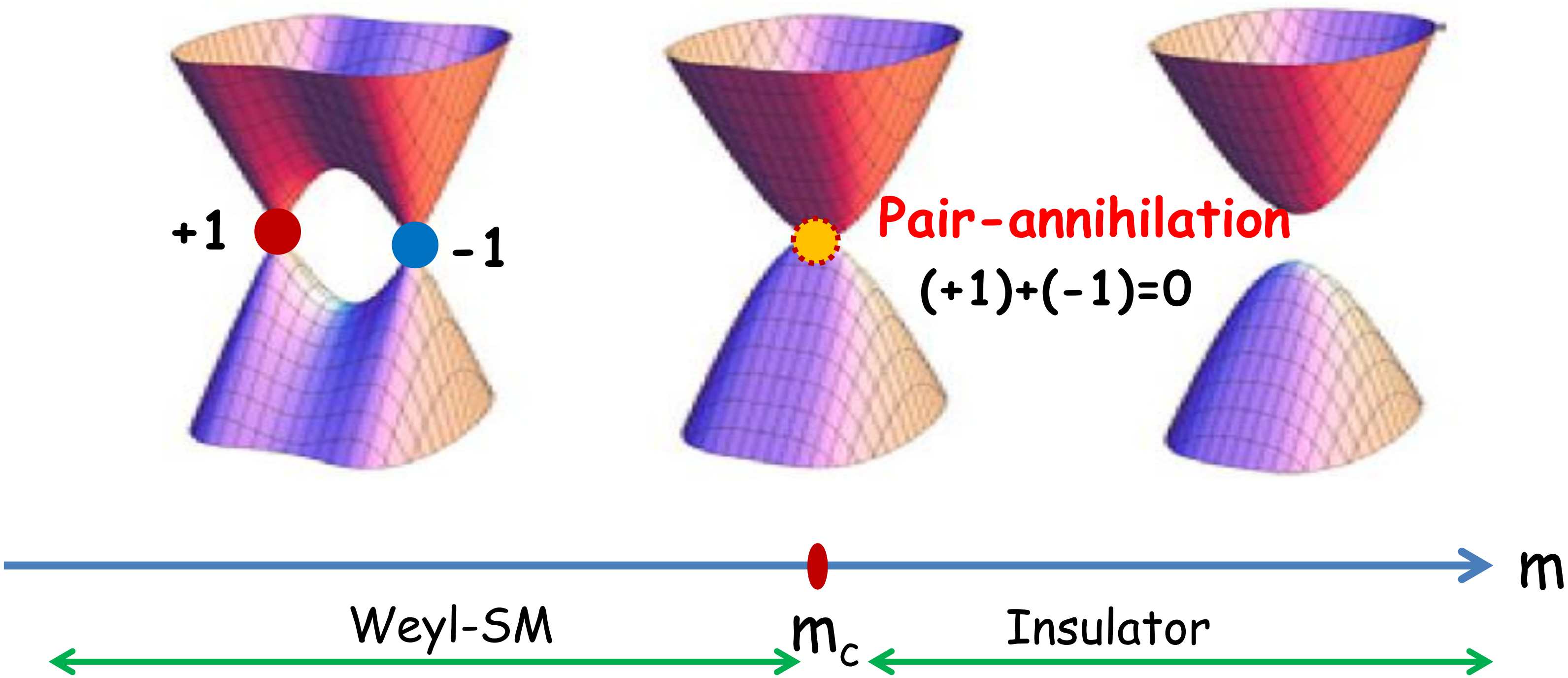


"Emergent physics
in strongly correlated topological systems"

Transition from a Weyl SM to an insulator

The chiral charge of a Weyl point guarantees its stability

→ A pair-annihilation is required



$$H = k_1 \sigma_1 + k_2 \sigma_2 + [(m-m_c) + k^2_3]\sigma_3$$