Renormalization Group Equation for $f(R)$ gravity on a hyperbolic space

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International Workshop for String theory and Cosmology 2016 at Hanyang Univ., Seoul, Aug. 18, 2016

Based on

"Renormalization Group Equation for $f(R)$ gravity on a hyperbolic space,"

with Kevin Falls, arXiv:1607.08460 [hep-th]

(partly based on "A flow equation for $f(R)$ gravity and some of its exact solutions," Phys. Rev. D 92 (2015) 061501 (Rapid Communication), arXiv:1507.00968 [hep-th], "Renormalization Group Equation and scaling solutions for f(R) gravity in exponential parametrization," Eur. Phys. J. C 76 (2016) 46, arXiv:1511.09393 [hep-th] with Roberto Percacci and Gian Paolo Vacca.)

1 Introduction

A way to quantum gravity

- **Einstein theory is non-renormalizable but it is only a low-energy effective theory!**

  Higher-order terms always appear in quantum theory e.g. quantized Einstein and string theories!

  ⇒ Possible UV completion for the following reasons

- **In 4D, quadratic (higher derivative) theory is renormalizable but non-unitary! (Stelle)**
- **In 3D, there is unitary higher-derivative gravity (Bergshoeff-Hohm-Townsend)**

$$S = \frac{1}{\kappa^2}\int d^3x\sqrt{-g}\Big[\sigma R - 2\Lambda_0 + \alpha R^2 + \beta R_{\mu\nu}^2 + \frac{1}{2\mu}\mathcal{L}_{LCS}\Big],$$

Not only that, we have the possibility of renormalizability!!
For the first time, we have the possibility of having both unitary and renormalizable gravity theory, though in 3D.

- **A complete classification of all unitary and stable theories in 3D,**

  by looking at the pole residues from the action (off-shell analysis) (N.O.)

- **Unfortunately, the renormalizability fails precisely for unitary theories. They are just incompatible with each other. (Muneyuki and N.O.)**
- **In this situation, the only possible way to make sense of the quantum effects in gravity seems to be the asymptotic safety.**

Asymptotic safety

Even if finite orders in the perturbation series contain unphysical singularities, this may be avoided if we can define nonperturbative RG flow and the couplings approach a fixed point in the ultraviolet energy. (Weinberg)

- **The asymptotic safety is a wider notion than the renormalizability (includes renormalizable theories).**

Rationale: We do not stick to perturbative unitarity. It only matters if we can compute effective action without any singularity and instability.

2 Wilsonian method for renormalization group

Wilsonian RG:

Effective action describing physical phenomenon at a momentum scale k = integrate out all fluctuations of the fields with momenta larger than k.
$\Rightarrow$ effective average action Γ_k (Note: Γ_0 is the effective action.)
k: the lower limit of the functional integration (the infrared cutoff).

Most important fact

The dependence of the effective action on k gives the Wilsonian RG flow, which is free from any divergence, giving finite quantum theory.

$$k\partial_k \Gamma_k(\Phi) = \frac{1}{2}\mathbf{tr}\left[\left(\frac{\delta^2\Gamma_k}{\partial\Phi^A\Phi^B} + R_k\right)^{-1} k\partial_k R_k\right].$$

EXACT renormalization group equation!

R_k: the cutoff function.

If we find nontrivial fixed points in this formulation, this gives the UV complete theory.

We can apply this method to our theory on arbitrary background in arbitrary dimensions.

3 $f(R)$ Gravity

The problem

In order to facilitate the program, one has to truncate the theory, e.g. derivative expansion, polynomial expansion etc. and the result is background dependent.

Still there is accumulating evidence (up to 34th order in R) that there are always nontrivial fixed points.

$\Rightarrow$ Asymptotic safety program may be the right direction.

An example: The action

$$S = \int d^d x \sqrt{-g} \Big[\frac{1}{\kappa^2}(\sigma R - 2\Lambda) + \frac{1}{2\lambda} C^2 - \frac{1}{\rho} E + \frac{1}{\xi} R^2 + \tau \Box R \Big],$$

and derive the beta functions.

We find the beta functions of the dimensionless couplings in 4 dims. and nontrivial fixed points in dimensions including 3 and 4 dims. $\Rightarrow$ Asymptotic safety!!

This is certainly encouraging, however it is not enough. (truncation)

Purpose (less truncation)

Consider actions of the general form

$$S = \int d^dx\sqrt{-g}f(R),$$

and derive FRGE for the function $f(R)$ which is then determined!

Two different parametrizations of the metric fluctuation:

linear split: $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \cdots$ **most often used**

exponential split: $g_{\mu\nu} = \bar{g}_{\mu\rho}(e^h)^\rho{}_\nu, \cdots$ **new parametrization**

The latter has the advantage that there is no unphysical singularity and the result is gauge-independent.

Another problem

On the compact space, the spectrum is discrete $\frac{\ell(\ell+3)}{12}R \geqq \frac{5}{6}R(\ell \geqq 2)$ and there is no sense of coarse graining for large curvature; k^2 cannot be less than this.

There is no problem of this kind on noncompact space because the spectrum is continuous, but has not been studied.

$\Rightarrow$ Extend to noncompact space.

The procedure

1. **We first derive the quadratic terms (hessians)**
2. **then introduce gauge fixing and the corresponding FP ghost,**
3. **and derive the FRGE on the sphere and hyperbolic spaces.**

We get in both cases

$$\dot{\Gamma}_k = \frac{1}{2}\mathbf{Tr}_{(2)}\left[\frac{\dot{f}'(\bar{R})R_k(\Box)+f'(\bar{R})\dot{R}_k(\Box)}{f'(\bar{R})\left(P_k(\Box)+\beta\bar{R}+\frac{2}{d(d-1)}\bar{R}\right)}\right]+\frac{1}{2}\mathbf{Tr}_{(0)}\left[\frac{\dot{f}''(\bar{R})R_k(\Box)+f''(\bar{R})\dot{R}_k(\Box)}{f''(\bar{R})\left(P_k(\Box)+\alpha\bar{R}-\frac{1}{d-1}\bar{R}\right)+\frac{d-2}{2(d-1)}f'(\bar{R})}\right]$$
$$-\frac{1}{2}\mathbf{Tr}_{(1)}\left[\frac{\dot{R}_k(\Box)}{P_k(\Box)+\alpha\bar{R}-\frac{1}{d}\bar{R}}\right],$$

Dot: logarithmic derivative of the scale k. $P_k(\Box)=\Box+R_k(\Box), (\Box=-\bar{\nabla}^2-\alpha_s\bar{R})$, cutoff function. α_s $(s=0,1,2)$: endomorphism parameters.
The subscripts to the traces: contributions from different spin sectors.

Note that there is no gauge fixing paramaters which already cancell out.

Using heat kernel expansion or spectrum sum and the optimized cutoff $R_k(z)=(k^2-z)\theta(k^2-z)$, $r\equiv\bar{R}k^{-2}$ (dimensionless curvature), $\varphi(r)=k^{-d}f(\bar{R})$, we get

Our main result in 4 dims. for both spaces

$$32\pi^2(\dot{\varphi}-2r\varphi'+4\varphi)=\frac{c_1(\dot{\varphi}'-2r\varphi'')+c_2\varphi'}{\varphi'[6+(6\alpha+1)r]}+\frac{c_3(\dot{\varphi}''-2r\varphi''')+c_4\varphi''}{2\{\varphi''[3+(3\beta-1)r]+\varphi'\}}-\frac{c_5}{4+(4\gamma-1)r}$$

where the coefficients $c_1 - c_5$ are polynomial in r up to 3rd order on the sphere and involve polylogarithms on the hyperboloid.

$$
\begin{aligned}
c_1 &= 5 + 5\Big(3\alpha - \frac{1}{2}\Big)r + \Big(15\alpha^2 - 5\alpha - \frac{1}{72}\Big)r^2 + \Big(5\alpha^3 - \frac{5}{2}\alpha^2 - \frac{\alpha}{72} + \frac{311}{9072}\Big)r^3, \\
c_2 &= 40 + 15(6\alpha - 1)r + \Big(60\alpha^2 - 20\alpha - \frac{1}{18}\Big)r^2 + \Big(10\alpha^3 - 5\alpha^2 - \frac{\alpha}{36} + \frac{311}{4536}\Big)r^3, \\
c_3 &= \frac{1}{2}\Big[1 + \Big(3\beta + \frac{1}{2}\Big)r + \Big(3\beta^2 + \beta - \frac{511}{360}\Big)r^2 + \Big(\beta^3 + \frac{1}{2}\beta^2 - \frac{511}{360}\beta + \frac{3817}{9072}\Big)r^3\Big], \\
c_4 &= 3 + (6\beta + 1)r + \Big(3\beta^2 + \beta - \frac{511}{360}\Big)r^2, \\
c_5 &= 12 + 2(12\gamma + 1)r + \Big(12\gamma^2 + 2\gamma - \frac{607}{180}\Big)r^2.
\end{aligned}
$$

for sphere. The FRGE itself is the same for the hyperbolic space $(r < 0)$ but the heat kernel or spectrum is different. For small curvature

$$
\begin{aligned}
c_1 &\approx 5 + 5\Big(3\alpha - \frac{1}{2}\Big)r + \Big(15\alpha^2 - 5\alpha - \frac{271}{72}\Big)r^2 \left(5\alpha^3 - \frac{5\alpha^2}{2} - \frac{271\alpha}{72} - \frac{7249}{9072}\right) r^3, \\
c_2 &\approx 40 + 15(6\alpha - 1)r + \Big(60\alpha^2 - 20\alpha - \frac{271}{18}\Big)r^2 + \left(10\alpha^3 - 5\alpha^2 - \frac{271\alpha}{36} - \frac{7249}{4536}\right) r^3, \\
c_3 &\approx \frac{1}{2}\Big[1 + \Big(3\beta + \frac{1}{2}\Big)r + \Big(3\beta^2 + \beta + \frac{29}{360}\Big)r^2 + \left(\beta^3 + \frac{1}{2}\beta^2 + \frac{29\beta}{360} + \frac{37}{9072}\right) r^3\Big], \\
c_4 &\approx 3 + (6\beta + 1)r + \left(3\beta^2 + \beta + \frac{29}{360}\right) r^2, \\
c_5 &\approx 12 + 2(12\gamma + 1)r + \left(12\gamma^2 + 2\gamma - \frac{67}{180}\right) r^2.
\end{aligned}
$$

- **The FRGE itself has the same structure for both spaces!**
- **The coefficients are only slightly different for small $|r|$ due to the difference in the symmetry (Killing) vectors (which should be removed).**
- **The heat kernel or spectrum is continuous for noncompact case so that the result is exact. (The result was approximated for the compact case.)**
- **There is no problem of coarse-graining on hyperbolic space; the RHS vanishes for large $|r|$, and we can integrate down to $k = 0$ (at finite $|r|$) on noncompact space. $\Rightarrow$ the effective action**

Step in the direction of background-independence.

4 Scaling solutions in 4D

Amazing result

Fixed point theory $f(r)$ is determined by a differential equation!!

Third prder ODE: $rc_3\,\varphi'''(r) = \cdots .$

Properties of differential equations obtained from $\dot{\varphi} = 0 \Rightarrow$ fixed points.

$$\varphi(r) = \sum_{m=0}^{N} g_m r^m, \qquad \dot{\varphi}(r) = \sum_{m=0}^{N} \beta_{g_m} r^m,$$

g_m**: the k-dependent running couplings**
$\beta_{g_m} = \partial_t g_m$**: their beta functions.**

The FRGE tells us that the large-r behavior of φ is

$$\varphi \sim a_2 r^2 + a_1 r + a_0 + a_{-1}/r + \cdots \text{ at most quadratic!}$$

Though it is an ordinary differential equation, it is still difficult to solve analytically for fixed general endomorphism α, β, γ.

So we try numerical solutions.

$10^3\alpha$	$10^3\beta$	$10^3\gamma$	$10^3\tilde{g}_{0*}$	$10^3\tilde{g}_{1*}$	$10^3\tilde{g}_{2*}$	θ
−593	−73.5	−177	7.28	−8.42	1.71	3.78
−616	−70.7	−154	7.42	−8.64	1.74	3.75
−564	−80.3	−168	6.82	−8.77	1.83	3.70
−543	−87.4	−126	6.31	−9.47	2.06	3.43
−420	−100.5	−3.19	4.90	−10.2	2.83	2.93
−173	−2.98	244	4.53	−8.34	2.70	2.18
−146	−64973	250	2.90	−10.7	0.0006	2.58
−109	−22267	307	2.90	−10.4	0.0045	2.45
109	−3564	526	2.84	−7.83	0.094	C
377	−1305	794	2.57	−4.37	0.214	> 4

$10^3\alpha$	$10^3\beta$	$10^3\gamma$	10^3g_0	10^3g_1	10^3g_2	θ
−441	−46.0	−129	9.42	−3.80	0.721	433, 0.776
−463	−46.8	−46.8	9.33	−4.62	0.877	153, 0.783
767	250	1180	5.86	−2.59	0.589	0.359
1850	3090	2270	3.42	8.97	2.84	7.54
805	308	−238	5.40	5.23	−1.28	7.08
−497	−4220	278	2.96	−16.6	−0.235	2.94, 0.984
−266	−17800	252	2.91	−12.7	−0.0119	2.76, 1.74
−683	−102	−165	6.92	−9.63	2.00	8.92
−1130	−432	−354	4.67	−17.8	5.38	9.68, 4.12
2210	3240	1170	3.48	18.7	8.56	4.00

Table 1: **Left:** Exact quadratic solutions **for sphere**. **Right:** Those for **hyperbolic space**. In the last column, we report the results for the positive (real part of) critical exponents, evaluated up to 9th order polynomial expansion. The critical exponent 4 is present in all solutions and is related to the cosmological term. Those solutions with critical exponents larger than 4 are not reliable.

Common features:

1. **Both have exact solutions: First treat these parameters as unknowns to solve for. The simplest possible solutions are of the form**

$$\varphi(r) = g_0 + g_1 r + g_2 r^2 \ \cdots \quad \textbf{Similar to Starobinsky model!}$$

We obtains a system of six equations for the six unknowns g_0, g_1, g_2, α, β and γ. This system has a number of solutions $\Rightarrow$ Table 1

2. **We tried to get polynomial solutions for small $|r|$. We have a good convergence for sphere as we increase the number of terms, but the convergence is poorer for the hyperbolic space.**

3. **The coefficients of terms beyond quadratic terms in r are in general quite small. Quadratic approximation seems good enough.**

4. **We also get the effective action at $k = 0$ for the endmorphism $\alpha = -\frac{1}{6}, \beta = 0, \gamma = \frac{1}{4}$:**

$$\int d^4x\sqrt{g}f(R) = \int d^4x\sqrt{g}\left(aR^2 - \frac{371R^2\log(R/\mu^2)}{23040\pi^2}\right).$$

5. **We can also make numerical analysis for fixed α, β, γ:**

   When we solve for the differential equations of the fixed point solutions, which are third order in general, the zero's of the third order coefficients $rc_3 = 0$ give the singularities.

   Note: $$c_3 = \frac{1}{2}\left[1 + \left(3\beta + \frac{1}{2}\right)r + \left(3\beta^2 + \beta - \frac{511}{360}\right)r^2 + \left(\beta^3 + \frac{1}{2}\beta^2 - \frac{511}{360}\beta + \frac{3817}{9072}\right)r^3\right].$$

   A singularity at $r = 0$ and further fixed singularities depending on β.

   A discrete number of solutions are expected to occur when the number of fixed singularities matches the order of the equation. We managed to do this for compact manifold, but noncompact case is resistive.

Example for compact sphere:

For the choice $\beta = \frac{1}{3}$, $\alpha = -\frac{1}{6}$, $\gamma = \frac{1}{2}$, we have singularities at $r = 0, 6/5$ and 2. The solution would have three free parameters for third order diff. eq., and these are used to bypass the possible singularites, leading to a unique solution.

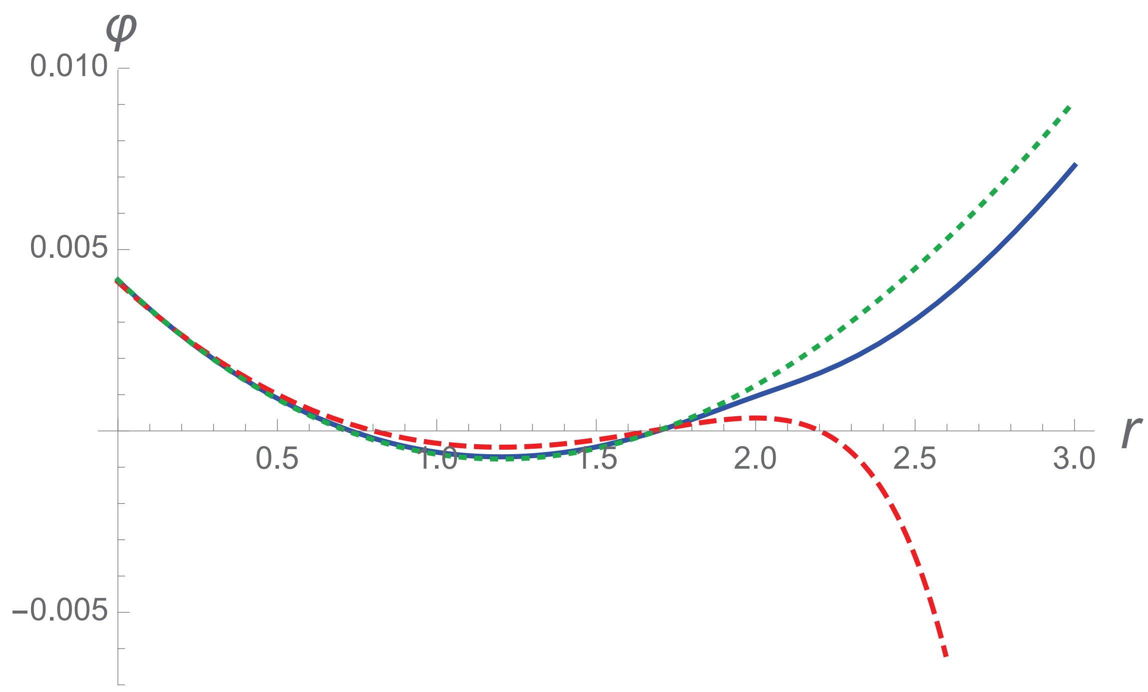

Figure 1: We compare the numerical solution (continuous blue curve) with the 16th order polynomial expansions around the origin (dashed red) and around the minimum (dotted green).

We can also study beta functions in other dimensions. Our results confirm that there are always nontrivial UV fixed point functions.

5 Conclusions

- **We have constructed a novel functional renormalization group equation for gravity which encodes the gravitational degrees of freedom in terms of general function $f(R)$ of the scalar curvature.**
- **The advantage of the new parametrization is that it gives flow equations free from unphysical singularities and to some extent gauge-independent result.**
- **The flow equations take very similar forms for compact as well as noncompact space. ⇒ Step towards background-independence.**
- **There are ultraviolet fixed points essential for Asymptotic Safety for the function $f(R)$.**
- **We have studied if this approach may be used to determine possible UV completion of gravitational theory and the result contains exact solutions similar to Starobinsky model $(R + R^2)$ (on the sphere), consistent with the current observation on inflation.**

We believe that this is a good step toward the realization of asymptotic safety.

Possible future directions:

- **Extending the analysis to more general theory (extend the theory space)**
- **Real background-independence etc.**