Healthy degenerate theories with higher derivatives

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$$c = \hbar = M_G^2 = 1/(8\pi G) = 1$$

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Introduction

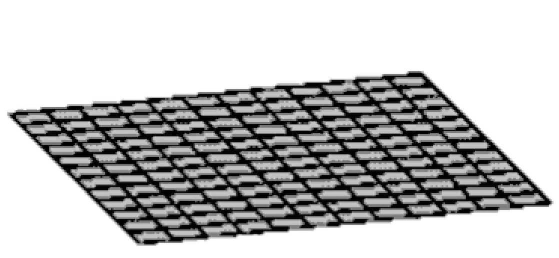
Inflation

Inflation, characterized as quasi De Sitter expansion, can naturally solve the problems of the standard big bang cosmology.

- **The horizon problem**
- **The flatness problem**
- **The origin of density fluctuations**
- **The monopole problem**
- …

Generic predictions of inflation

- Spatially flat universe



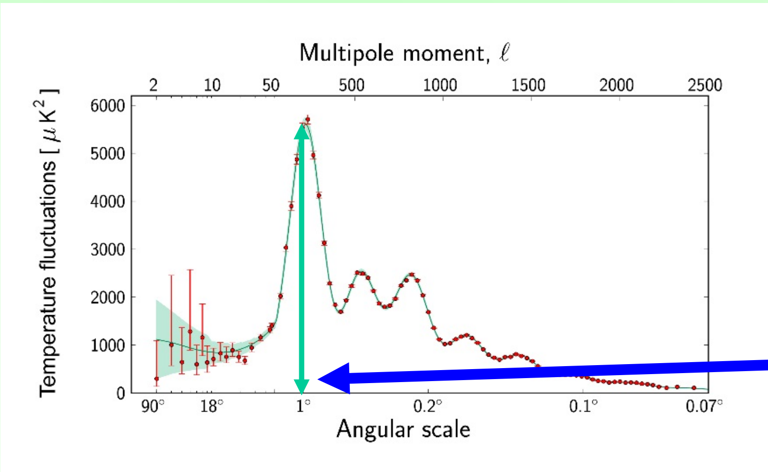
- Almost scale invariant, adiabatic, and Gaussian primordial density fluctuations

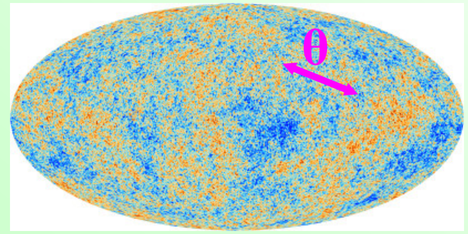
- Almost scale invariant and Gaussian primordial tensor fluctuations

Generates anisotropy of CMBR.

Inflation is strongly supported by CMB observations

Planck TT correlation :





Green line : prediction by inflation

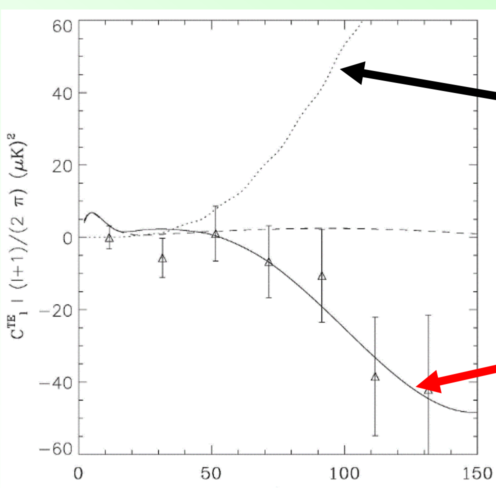
Red points : observation by PLANCK

Angle $\theta \sim 180^\circ / l$

Total energy density ⇔ Geometry of our Universe

Our Universe is spatially flat as predicted by inflation !!

WMAP TE correlation :



Causal seed models

Superhorizon models

(adiabatic perturbations)

Unfortunately, primordial tensor perturbations have not yet been observed.

Next task is to identify the inflaton, a scalar field which caused inflation.

How to identify the inflaton

- **Top down approach**

To construct the unique inflation model from the ultimate theory.

- **Bottom up approach**

To consider the most general model of inflation. Then, we can constrain inflation models (or to single out the true model finally) from the observational results.

We would like to find the most general model of inflation based on a scalar-tensor theory, which is also useful for dark energy models.

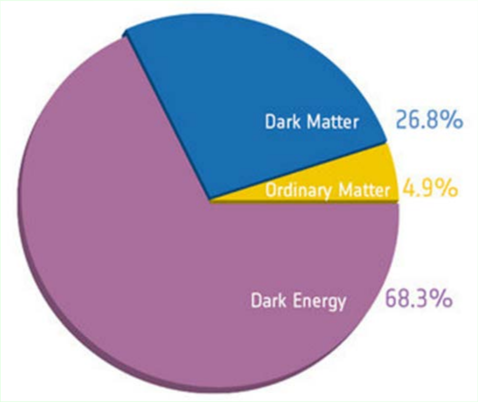
What is dark energy ?

The Universe is now accelerating !!

- **Dark Energy is introduced**

or

- **GR may be modified in the IR limit**



PLANCK

One possibility : Scalar-tensor theory

If GR is modified around the present Hubble scale, the present Universe may look accelerating apparently.

In order to survey all of the possibilities, it is interesting to pursue the most general scalar-tensor theory.

How widely can we extend scalar tensor theory ?

- **A kinetic term of an inflaton is not necessarily canonical.**

$$\mathcal{L} = X - V(\phi), \quad X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \quad \Longrightarrow \quad \mathcal{L} = K(\phi, X)$$

(k-inflation)

(Armendariz-Picon et.al. 1999)

- **An inflaton is not necessarily minimally coupled to gravity.**

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}M_G^2 R + \mathcal{L}_\phi\right) \quad \Longrightarrow \quad \Delta S = \int d^4x\sqrt{-g}f(\phi)R$$

(Higgs inflation)

(Cervantes-Cota & Dehnen 1995, Bezrukov & M. Shaposhnikov 2008)

- **Action may include higher derivatives.**

$$\mathcal{L} = K(\phi, X) \quad \Longrightarrow \quad \Delta\mathcal{L} = G(\phi, X)\Box\phi$$

Theories with higher derivatives are quite dangerous in general.

Lagrangian

Why does Lagrangian generally depend on only
a position q and its velocity dot{q} ?

Newton recognized that an acceleration, which is given by the second time derivative of a position, is related to the Force :

$$m\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} = \boldsymbol{F}\left(\boldsymbol{x}, \dot{\boldsymbol{x}}\right).$$

The Euler-Lagrange equation gives an equation of motion up to the second time derivative if a Lagrangian is given by L = L(q,dot{q},t).

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0, \quad \Longrightarrow \quad \ddot{q} = \ddot{q}\left(\dot{q}, q\right) \quad \Longrightarrow \quad q(t) = Q\left(\dot{q}_0, q_0, t\right).$$

(if $p := \frac{\partial L}{\partial \dot{q}}$ depends on dot{q} ⇔ non-degenerate condition.)

What happens if Lagrangian depends on higher derivative terms ?

Ostrogradski's theorem

Assume that $L = L(q, \dot{q}, \ddot{q})$ **and** $\frac{\partial L}{\partial \ddot{q}}$ **depends on** $\ddot{q}$ **:**

(Non-degeneracy)

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\mathrm{d}^2}{\mathrm{d}^2 t}\left(\frac{\partial L}{\partial \ddot{q}}\right) = 0, \quad \Longrightarrow \quad q^{(4)} = q^{(4)}\left(q^{(3)}, \ddot{q}, \dot{q}, q\right).$$

Canonical variables :

$$\begin{cases} Q_1 := q, & P_1 := \dfrac{\partial L}{\partial \dot{q}} - \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial \ddot{q}}, \\ Q_2 := \dot{q}, & P_2 := \dfrac{\partial L}{\partial \ddot{q}}. \end{cases}$$

Non-degeneracy $\Leftrightarrow$ $\ddot{q} = \ddot{q}\left(q, \dot{q}, \dfrac{\partial L}{\partial \ddot{q}}\right)$ $\Leftrightarrow$ $\ddot{q} = \ddot{q}\,(Q_1, Q_2, P_2)$

Hamiltonian:

$$\begin{aligned} H(Q_1, Q_2, P_1, P_2) &:= P_1 \dot{q} + P_2 \ddot{q} - L \\ &= P_1 Q_2 + P_2 \ddot{q}(Q_1, Q_2, P_2) - L(Q_1, Q_2, \ddot{q}(Q_1, Q_2, P_2)). \end{aligned}$$

These canonical variables really satisfy the canonical EOM : $\dot{Q}_i = \dfrac{\partial H}{\partial P_i}, \quad \dot{P}_i = -\dfrac{\partial H}{\partial Q_i}.$

P1 depends linearly on H so that no system of this form can be stable !!

N.B. $\dfrac{\partial L}{\partial \phi} - \partial_\mu \left(\dfrac{\partial L}{\partial(\partial_\mu \phi)}\right) + \partial_\mu \partial_\nu \left(\dfrac{\partial L}{\partial(\partial_\mu \partial_\nu \phi)}\right) = 0.$ $\Longrightarrow$ $\dfrac{i}{(p^2+m_1^2)(p^2+m_2^2)} = \dfrac{1}{m_2^2 - m_1^2}\left(\dfrac{i}{p^2+m_1^2} - \dfrac{i}{p^2+m_2^2}\right).$

(propagators)

Loophole of Ostrogradski's theorem

We can break the non-degeneracy condition which requires that $\frac{\partial L}{\partial \ddot{q}}$ depends on ddot{q}.

(NB: another interesting possibility is infinite derivative theory)

In case Lagrangian depends on only a position q and its velocity dot{q}, degeneracy implies that EOM is first order, which represents not the dynamics but the constraint.

$$\Updownarrow$$

In case Lagrangian depends on q, dot{q}, ddot{q}, degeneracy implies that EOM can be (more than) second order, which can represent the dynamics.

What is the most general scalar-tensor theory ?

Generalized Galileon = Horndeski

Deffayet et al. 2011 **equivalence** Horndeski 1974

Kobayashi, MY, Yokoyama 2011

$$
\begin{cases}
\mathcal{L}_2 = \boxed{K(\phi, X),} \\
\mathcal{L}_3 = -\boxed{G_3(\phi, X)}\Box\phi, \\
\mathcal{L}_4 = \boxed{G_4(\phi, X)}R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \\
\mathcal{L}_5 = \boxed{G_5(\phi, X)}G_{\mu\nu}\nabla^\mu\nabla^\nu\phi \\
\qquad -\frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right].
\end{cases}
$$

$$
X = -\frac{1}{2}(\nabla\phi)^2, \qquad G_{iX} \equiv \partial G_i/\partial X.
$$

This is the most general scalar tensor theory whose EOMs are up to second order ?

NB :
- **$G_4 = M_G^2 / 2$ yields the Einstein-Hilbert action**
- **$G_4 = f(\varphi)$ yields a non-minimal coupling of the form $f(\varphi)R$**
- **The new Higgs inflation with $G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ comes from $G_5 \propto \varphi$ after integration by parts.**

Beyond Horndeski theory

Gleyzes et al. 2014
Gao 2014

Gleyzes et al. (GLPV) pointed out that there is extended theory with the number of propagating degrees of freedom unchanged, even though apparent equations of motion of the theory are higher order.

Gao pointed out another direction of extension, in which only time derivatives should be second order while spacial ones can be higher.

All of these systems are degenerate to avoid ghost.

We are interested in an issue whether further extension is possible.

Healthy degenerate theories
with higher derivatives for point particles

(as a first step with keeping in mind
future extension to scalar-tensor theory)

Degeneracy condition

Assume that $L = L(q, \dot{q}, \ddot{q})$ **and** $\frac{\partial L}{\partial \ddot{q}}$ **depends on** $\ddot{q}$ **: (Non-degeneracy)**

$$\Longrightarrow \quad \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\mathrm{d}^2}{\mathrm{d}^2 t}\left(\frac{\partial L}{\partial \ddot{q}}\right) = 0, \quad \Longrightarrow \quad q^{(4)} = q^{(4)}\left(q^{(3)}, \ddot{q}, \dot{q}, q\right).$$

On the other hand, if we require degeneracy, L should not depend on ddot{q} after integration by parts for single particle.

$$L = f(q, \dot{q})\,\ddot{q} = \left(f_0(q) + f_1(q)\dot{q} + f_2(q)\dot{q}^2 + \cdots + f_n(q)\dot{q}^n + \cdots\right)\ddot{q}$$

$$\downarrow$$

$$\frac{d}{dt}\left(\frac{f_n(q)\dot{q}^{n+1}}{n+1}\right) - \frac{f_n'(q)\dot{q}^{n+2}}{n+1}$$

Though a linear term in ddot{q} is allowed apparently, it can be recast into a term without ddot{q} after integration by parts.

This discussion applies only for a single particle case. What is a condition for healthy higher order (multiple) particle theory ?

(At least, more than two DOF. Scalar & graviton, φ(x,t) ➔ φ^{i}(t))

discrete limit

Lagrangian with two variables

Langlois & Noui 2015

$$L(\ddot{\phi}, \dot{\phi}, \phi; \dot{q}, q)$$

One regular (q) and one special variable (φ)

(gravity ←→ scalar)

φ and q generically obey fourth order and second order equations of motion, respectively → Ostrogradsky instability

Let us derive the conditions to escape such an instability by Hamiltonian analysis, starting from the equivalent Lagrangian:

$$L_{eq}^{(1)} \equiv L(\dot{Q}, Q, \phi; \dot{q}, q) + \lambda(\dot{\phi} - Q).$$

Hamiltonian analysis

$$L_{eq}^{(1)} \equiv L(\dot{Q}, Q, \phi; \dot{q}, q) + \lambda(\dot{\phi} - Q).$$

4 variables : Q, φ, q, λ

4 canonical momenta :

$$P = \frac{\partial L}{\partial \dot{Q}} \equiv L_{\dot{Q}} \ , \ p = \frac{\partial L}{\partial \dot{q}} \equiv L_{\dot{q}} \ , \ \pi = \frac{\partial L_{eq}^{(1)}}{\partial \dot{\phi}} = \lambda \ , \ \rho = \frac{\partial L_{eq}^{(1)}}{\partial \dot{\lambda}} = 0 \, .$$

(only non-vanishing) elementary Poisson brackets

$$\{Q, P\} = \{q, p\} = \{\phi, \pi\} = \{\lambda, \rho\} = 1 \, .$$

Two primary constraints :

$$\Phi = \pi - \lambda \approx 0 \ , \ \Psi = \rho \approx 0 \, , \ (\{\Phi, \Psi\} = -1)$$

(Legendre trans.)

$$H = H_0 + \pi Q \quad \text{with} \quad H_0 = P\dot{Q} + p\dot{q} - L(\dot{Q}, Q, \phi; \dot{q}, q)$$

Hamiltonian analysis II

$$P = \frac{\partial L}{\partial \dot{Q}} \equiv L_{\dot{Q}} \ , \ p = \frac{\partial L}{\partial \dot{q}} \equiv L_{\dot{q}} \ , \quad \left(\pi = \frac{\partial L_{eq}^{(1)}}{\partial \dot{\phi}} = \lambda \ , \ \rho = \frac{\partial L_{eq}^{(1)}}{\partial \dot{\lambda}} = 0 \right)$$

$$(\Phi = \pi - \lambda \approx 0 \ , \ \Psi = \rho \approx 0)$$

When we can express dot{Q} & dot{q} in terms of P & p (with fixed Q & q), there is no further constraint.

⟺ **det K ≠ 0 : non-degeneracy condition**

$$\begin{pmatrix} \delta P \\ \delta p \end{pmatrix} = K \begin{pmatrix} \delta \dot{Q} \\ \delta \dot{q} \end{pmatrix} . \qquad K \equiv \begin{pmatrix} L_{\dot{Q}\dot{Q}} & L_{\dot{q}\dot{Q}} \\ L_{\dot{q}\dot{Q}} & L_{\dot{q}\dot{q}} \end{pmatrix} \ \text{with} \ L_{xy} \equiv \frac{\partial^2 L}{\partial x \partial y} .$$

⇨

$$H_T = H(P, p, \pi, Q, q, \phi) + \mu\Phi + \nu\Psi$$

Time invariance ⇨

$$\dot{\Phi} = \{\Phi, H\} - \nu \approx 0, \quad \dot{\Psi} = \mu \approx 0.$$

(Two Lagrange multipliers are fixed ➔ no further constraint)

⇨ **Only two second class constraints ➔ (8-2)/2 = 3 DOF.**
Then, one of them is ghost !! (π linearly appears in H)

Degenerate Lagrangian

For healthy theories, the following condition is necessary :

$$\det K = L_{\dot{Q}\dot{Q}} L_{\dot{q}\dot{q}} - L_{\dot{q}\dot{Q}}^2 = 0 .$$

$\Longleftrightarrow$

$\Longleftarrow$ **: trivial**

- **For** $L_{\dot{q}\dot{q}} = \dfrac{\partial p}{\partial \dot{q}} \neq 0 \Longrightarrow \dot{q} = \varphi(p, \dot{Q}, Q, q, \phi)$

  (implicit function theorem)

  $\Longrightarrow P = \dfrac{\partial L}{\partial \dot{Q}} = P(\dot{Q}, \dot{q}, Q, q, \phi) = F(\dot{Q}, p, Q, q, \phi)$

  $\xRightarrow{\det K = 0} \partial F / \partial \dot{Q}|_p = 0$ **(Otherwise,** $\dot{Q} = \dot{Q}(P, p, Q, q, \phi)$ **)**

  $\Longrightarrow \Xi \equiv P - F(p, Q, \phi, q) \approx 0.$

- **For** $L_{\dot{Q}\dot{Q}} = \dfrac{\partial P}{\partial \dot{Q}} \neq 0 \Longrightarrow \Pi \equiv p - G(P, Q, \phi, q) \approx 0$

- **For** $L_{\dot{q}\dot{q}} = \dfrac{\partial p}{\partial \dot{q}} = 0$ & $L_{\dot{Q}\dot{Q}} = \dfrac{\partial P}{\partial \dot{Q}} = 0 \xRightarrow{\det K = 0} L_{\dot{q}\dot{Q}} = \dfrac{\partial p}{\partial \dot{Q}} = L_{\dot{Q}\dot{q}} = \dfrac{\partial P}{\partial \dot{q}} = 0$ $(K = 0)$

  $\Longrightarrow \Xi \equiv P - f(Q, \phi, q) \approx 0, \ \tilde{\Pi} \equiv p - g(Q, \phi, q) \approx 0$

Additional primary constraint

$\Xi \equiv P - F(p, Q, \phi, q) \approx 0.$ **(Other cases are almost the same)**

$$H_T = H(P, p, \pi, Q, q, \phi) + \mu\Phi + \nu\Psi + \xi\Xi$$

$$(\Phi = \pi - \lambda \approx 0, \;\; \Psi = \rho \approx 0)$$

Time invariance

$$\begin{cases} \dot{\Phi} = \{\Phi, H\} - \nu + \xi F_\phi \approx 0, \\ \dot{\Psi} = \mu \approx 0, \qquad (\{\Phi, \Xi\} = F_\phi) \\ \dot{\Xi} = \{\Xi, H\} - \overset{0}{\mu} F_\phi \approx 0. \end{cases}$$

Fix μ & ν

$$\Theta \equiv \{\Xi, H\} = -\pi + \{\Xi, H_0\} - F_\phi Q \approx 0$$

(new constraint, which determines π in terms of the other phase space variables)

$$\dot{\Theta} = \{\Theta, H\} + \xi\{\Theta, \Xi\} \approx 0$$

- **For** $\Delta \equiv \{\Theta, \Xi\} \neq 0$ **Fix ξ** **(no further constraint)**

$$D_{ij} = \{\chi_i, \chi_j\} = \begin{pmatrix} 0 & -1 & F_\phi & \{\Phi, \Theta\} \\ 1 & 0 & 0 & 0 \\ -F_\phi & 0 & 0 & -\Delta \\ \{\Theta, \Phi\} & 0 & \Delta & 0 \end{pmatrix}$$

$\chi_i \in (\Phi, \Psi, \Xi, \Theta)$

$\det D = \Delta^2 \neq 0.$

All constrains are second class ➔ (8-4)/2 = 2 DOF

Additional primary constraint II

$$\dot{\Theta} = \{\Theta, H\} + \xi\{\Theta, \Xi\} \approx 0$$

- **For** $\Delta \equiv \{\Theta, \Xi\} = 0$

① $\Gamma \equiv \{\Theta, H\}$ **does not vanish automatically.**

➡ $\Gamma \approx 0$ **: new (tertiary) constraint**

➡ **Check whether** $\dot{\Gamma} \approx 0$ **generate new constraints.**
In any case, DOF is strictly less than 2.

② $\Gamma \equiv \{\Theta, H\}$ **vanishes automatically.**

$$D_{ij} = \{\chi_i, \chi_j\} = \begin{pmatrix} 0 & -1 & F_\phi & \{\Phi, \Theta\} \\ 1 & 0 & 0 & 0 \\ -F_\phi & 0 & 0 & -\Delta = 0 \\ \{\Theta, \Phi\} & 0 & \Delta = 0 & 0 \end{pmatrix}$$

$$\chi_i \in (\Phi, \Psi, \Xi, \Theta)$$

➡ **rank D = 2**

➡ **Two second class & two first class ➔ (8-2x2-2) / 2 = 1 DOF**

No ghost DOF !!

Short summary

$$L(\ddot{\phi}, \dot{\phi}, \phi; \dot{q}, q)$$

One regular (q) and one special variable (φ)

$$\det K = L_{\dot{Q}\dot{Q}} L_{\dot{q}\dot{q}} - L^2_{\dot{q}\dot{Q}} = 0 \qquad \left(\dot{Q} = \ddot{\phi}\right)$$

$\Longleftrightarrow$ **necessary & sufficient condition for healthy theory**

In this case, we can show that the Euler-Lagrange equations for both variables can be reduced to second order system.

Multiple regular & multiple special variables

(Multiple regular & one special variable case is essential the same)

$$L(\ddot{\phi}, \dot{\phi}, \phi; \dot{q}^i, q^i) \quad (i = 1, \cdots, m)$$

Lagrangian with multiple regular & multiple special variables

$$L(\ddot{\phi}^a, \dot{\phi}^a, \phi^a; \dot{q}^i, q^i) \;\; (a = 1, \cdots, n; \;\; i = 1, \cdots, m)$$

Multiple regular (q^i) and special variables (φ^a)

φ^a and q^i generically obey fourth order and second order equations of motion, respectively ➔ Ostrogradsky instability

Let us derive the conditions to escape such an instability by Hamiltonian analysis, starting from the equivalent Lagrangian:

$$L^{(1)}_{eq}(\dot{Q}^a, Q^a; \dot{\phi}^a, \phi^a; \dot{q}^i, q^i, \lambda_a) \equiv L(\dot{Q}^a, Q^a, \phi^a; \dot{q}^i, q^i) + \lambda_a(\dot{\phi}^a - Q^a).$$

Hamiltonian analysis

$$L^{(1)}_{eq}(\dot{Q}^a, Q^a; \dot{\phi}^a, \phi^a; \dot{q}^i, q^i, \lambda_a) \equiv L(\dot{Q}^a, Q^a, \phi^a; \dot{q}^i, q^i) + \lambda_a(\dot{\phi}^a - Q^a).$$

(3n+m) variables : Q^a, φ^a, q^i, λ^a

(3n+m) canonical momenta :

$$P_a = \frac{\partial L}{\partial \dot{Q}^a} \equiv L_{\dot{Q}} \ , \ p_i = \frac{\partial L}{\partial \dot{q}^i} \equiv L_{\dot{q}} \ , \ \pi_a = \frac{\partial L^{(1)}_{eq}}{\partial \dot{\phi}^a} = \lambda_a \ , \ \rho_a = \frac{\partial L^{(1)}_{eq}}{\partial \dot{\lambda}^a} = 0 \, .$$

(only non-vanishing) elementary Poisson brackets

$$\{Q^a, P_b\} = \{\phi^a, \pi_b\} = \{\lambda^a, \rho_b\} = \delta^a_b \ , \ \{q^i, p_j\} = \delta^i_j.$$

Two sets of n primary constraints :

$$\Phi_a = \pi_a - \lambda_a \approx 0 \ , \ \Psi_a = \rho_a \approx 0 \, , \ (\{\Phi_a, \Psi_b\} = -\delta_{ab})$$

(Legendre trans.)

$$H = H_0 + \pi_a Q^a \quad \text{with} \quad H_0 = P_a \dot{Q}^a + p_a \dot{q}^a - L(\dot{Q}^a, Q^a, \phi^a; \dot{q}^i, q^i)$$

Hamiltonian analysis II

$$P_a = \frac{\partial L}{\partial \dot{Q}^a} \equiv L_{\dot{Q}^a} \ , \ \ p_i = \frac{\partial L}{\partial \dot{q}^i} \equiv L_{\dot{q}^i} \ , \ \ \left(\pi_a = \frac{\partial L_{eq}^{(1)}}{\partial \dot{\phi}^a} = \lambda \ , \ \ \rho_a = \frac{\partial L_{eq}^{(1)}}{\partial \dot{\lambda}^a} = 0 \right)$$

$$(\Phi_a = \pi_a - \lambda_a \approx 0 \ , \ \ \Psi_a = \rho_a \approx 0)$$

When we can express dot{Q}^{a} & dot{q}^{i} in terms of P_a & p_i (with fixed Q^a & q^i), there is no further constraint.

⟺ **det K ≠ 0**

$$\begin{pmatrix} \delta P_a \\ \delta p_i \end{pmatrix} = K \begin{pmatrix} \delta \dot{Q}^b \\ \delta \dot{q}^j \end{pmatrix} . \qquad K \equiv \begin{pmatrix} L_{\dot{Q}^a \dot{Q}^b} & L_{\dot{Q}^a \dot{q}^j} \\ L_{\dot{q}^i \dot{Q}^b} & L_{\dot{q}^i \dot{q}^j} \end{pmatrix} = \begin{pmatrix} L_{ab} & L_{aj} \\ L_{ib} & L_{ij} \end{pmatrix} .$$

⇒ $$H_T = H(P_a, p_i, \pi_a, Q^a, q^i, \phi^a) + \mu^a \Phi_a + \nu^a \Psi_a$$

$\dot{\Phi}_a \approx 0, \dot{\Psi}_a \approx 0$

⇒ $$\mu^a = -\{\Psi_a, H\} = 0, \quad \nu^a = \{\Phi_a, H\}$$

(Two sets of n Lagrange multipliers are fixed ➔ no further constraint)

⇒ **Only two sets of n second class constraints**

➔ (6n+2m-2n)/2 = (2n+m)DOF. n ghosts appear !!

Degenerate Lagrangian

For healthy theories, we have to eliminate n DOF from constraints

One way : $\dim(\mathrm{Ker}\,K) = n$

$$\left[\begin{pmatrix}\delta P_a\\ \delta p_i\end{pmatrix} = K\begin{pmatrix}\delta \dot{Q}^b\\ \delta \dot{q}^j\end{pmatrix},\quad K \equiv \begin{pmatrix} L_{\dot{Q}^a\dot{Q}^b} & L_{\dot{Q}^a\dot{q}^j}\\ L_{\dot{q}^i\dot{Q}^b} & L_{\dot{q}^i\dot{q}^j}\end{pmatrix} = \begin{pmatrix} L_{ab} & L_{aj}\\ L_{ib} & L_{ij}\end{pmatrix}\right]$$

Assume det Lij ≠ 0 (q^i is a regular variable)

$$\Longleftrightarrow \quad L_{ab} - L_{ai}L^{ij}L_{jb} = 0$$

$$\because \quad \left(\det K = \det(L_{ab} - L_{ai}L^{ij}L_{jb})\,\det L_{ij}\right)$$

(all n eigenvalues are zero) **(m non-zero eigenvalues)**

Assume det Lij ≠ 0

$$\Longleftrightarrow \quad \Xi_a \equiv P_a - F_a(p_i, Q^b, \phi^b, q^i) \approx 0.$$

Degenerate Lagrangian II

- $L_{ab} - L_{ai}L^{ij}L_{jb} = 0 \xrightarrow{\text{Assume det Lij} \neq 0} \Xi_a \equiv P_a - F_a(p_i, Q^b, \phi^b, q^i) \approx 0.$

  $\because$ **det Lij ≠ 0** $\Longrightarrow$ $\dot{q}^i = \varphi^i(p_j, \dot{Q}^a, Q^a, q^j, \phi^a)$

  (implicit function theorem)

  $$\Longrightarrow P_a = \frac{\partial L}{\partial \dot{Q}^a} = P_a(\dot{Q}^b, \dot{q}^i, Q^b, q^i, \phi^b) = F_a(\dot{Q}^b, p_i, Q^b, q^i, \phi^b)$$

  $$\xRightarrow{\det K = 0} \partial F_a / \partial \dot{Q}^b|_{p_i} = 0$$ **(Otherwise,** $\dot{Q}^a = \dot{Q}^a(P_b, p_i, Q^b, q_i, \phi^b)$**)**

  $$\Longrightarrow \Xi_a \equiv P_a - F_a(p_i, Q^b, \phi^b, q^i) \approx 0.$$

- $L_{ab} - L_{ai}L^{ij}L_{jb} = 0 \xleftarrow{\text{Assume det Lij} \neq 0} \Xi_a \equiv P_a - F_a(p_i, Q^b, \phi^b, q^i) \approx 0.$

  $$\because \quad L_{ab} = \frac{\partial P_a}{\partial \dot{Q}^b} = L_{ib}\frac{\partial F_a}{\partial p_i}, \quad \text{and} \quad L_{ia} = \frac{\partial P_a}{\partial \dot{q}^i} = L_{ij}\frac{\partial F_a}{\partial p_j}.$$

  $$\Longrightarrow L_{ai}L^{ij}L_{jb} = \frac{\partial F_a}{\partial p_k}L_{ik}L^{ij}L_{jb} = \frac{\partial F_a}{\partial p_k}L_{kb} = L_{ab}.$$

Additional primary constraints

(Motohashi & Suyama 2014)

$$\Xi_a \equiv P_a - F_a(p_i, Q^b, \phi^b, q^i) \approx 0.$$

$$\Rightarrow \quad H_T = H(P_a, p_i, \pi_a, Q^a, q^i, \phi^a) + \mu^a \Phi_a + \nu^a \Psi_a + \xi^a \Xi_a$$

$$(\Phi_a = \pi_a - \lambda_a \approx 0, \ \ \Psi_a = \rho_a \approx 0)$$

$$\Rightarrow \quad \begin{cases} \dot{\Phi}_a = \{\Phi_a, H\} - \nu^a + \xi^b \partial F_b / \partial \phi^a \approx 0, \\ \dot{\Psi}_a = \mu^a \approx 0, \qquad (\{\Phi_a, \Xi_b\} = \partial F_b/\partial \phi^a) \\ \dot{\Xi}_a = \{\Xi_a, H\} + \xi^b \underbrace{\{\Xi_a, \Xi_b\}}_{\equiv M_{ab}} \approx 0. \end{cases} \quad \Rightarrow \text{Fix } \mu^a \ \& \ \nu^a$$

If det $M_{ab} \neq 0$, all ξ^a are fixed and no secondary constraints.

⇒ **Not sufficient number of constraints to eliminate all ghosts.**

⇒ **Simplest case for healthy theory : $M_{ab} = 0$.**

⇒ $$\Theta_a \equiv \{\Xi_a, H\} = -\pi_a + \{\Xi_a, H_0\} - \partial F_a / \partial \phi_b \, Q^b \approx 0$$

(new constraint, which fixes all π_a in terms of the other phase space variables)

Additional primary constraint II

$$\dot{\Theta}_a = \{\Theta_a, H\} + \xi^b \underbrace{\{\Theta_a, \Xi_b\}}_{\Delta_{ab}} \approx 0$$

- **For** $\det \Delta_{ab} \neq 0$ ➡ **Fix ξ^b** **(no further constraint)**

➡ $$D_{ij} = \{\chi_i, \chi_j\} = \begin{pmatrix} 0 & -\delta_{ab} & \partial F_b/\partial\phi^a & \{\Phi_a, \Theta_b\} \\ \delta_{ab} & 0 & 0 & 0 \\ -F_\phi & 0 & 0 & -\Delta_{ab} \\ \{\Theta_a, \Phi_b\} & 0 & \Delta_{ab} & 0 \end{pmatrix}$$ ➡ $\det D = (\det \Delta)^2 \neq 0.$

$\chi_i \in (\Phi_a, \Psi_a, \Xi_a, \Theta_a)$

➡ **All constrains are second class**

➔ (2(3n+m) - 4n) / 2 = (n+m) DOF

- **For** $\det \Delta_{ab} = 0$

➡ **new (tertiary) constraints or first class constrains**

➡ **Less than (n+m) DOF**

No ghost DOF !!

Short summary II

$$L(\ddot{\phi}^a, \dot{\phi}^a, \phi^a; \dot{q}^i, q^i)$$

multiple regular (q^i) and multiple special variable (φ^a)

- $\det K = \det(L_{ab} - L_{ai}L^{ij}L_{jb}) \det L_{ij} = 0$ **is not clearly enough.**

$$\left[\begin{pmatrix} \delta P_a \\ \delta p_i \end{pmatrix} = K \begin{pmatrix} \delta \dot{Q}^b \\ \delta \dot{q}^j \end{pmatrix}, \quad K \equiv \begin{pmatrix} L_{\dot{Q}^a \dot{Q}^b} & L_{\dot{Q}^a \dot{q}^j} \\ L_{\dot{q}^i \dot{Q}^b} & L_{\dot{q}^i \dot{q}^j} \end{pmatrix} = \begin{pmatrix} L_{ab} & L_{aj} \\ L_{ib} & L_{ij} \end{pmatrix}\right]$$

- $\dim(\mathrm{Ker}\, K) = n \Longleftrightarrow L_{ab} - L_{ai}L^{ij}L_{jb} = 0$

  Assume det Lij ≠ 0

  is not sufficient condition for healthy theory

- $M_{ab} = \{\Xi_a, \Xi_b\} = -\frac{\partial F_a}{\partial Q^b} + \frac{\partial F_b}{\partial Q^a} + \frac{\partial F_a}{\partial q_i}\frac{\partial F_b}{\partial p_i} - \frac{\partial F_a}{\partial p_i}\frac{\partial F_b}{\partial q_i} = 0.$

sufficient

In this case, we can show that the Euler-Lagrange equations for all variables can be reduced to second order system.

Summary

- **We have investigated how to obtain healthy degenerate theory with higher derivative.**
- **For single regular and single special variables, the degeneracy condition is necessary and sufficient for healthy theory.**
- **For multiple regular and multiple special variables, we have given sufficient condition for healthy theory. In this case, not only degeneracy condition but also other secondary conditions are necessary**