Fubini instantons in Dilatonic Einstein–Gauss–Bonnet theory of gravitation

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Outline

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- We are interested in the tunneling process on the *hilltop* type of potential with the quartic scalar field term only, which corresponds to the decay of vacuum.
- S. Fubini was studied it first in the absence of gravity and the tunneling solution is so-called Fubini instanton. We have been extended the study of Fubini instanton into the presence of Einstein and Diliatonic Einstein–Gauss–Bonnet (DEGB) gravity.
- We try to understand the effects of gravitation into the tunneling process and want to know further the modifications of the solutions under the DEGB gravity. We hope that the solutions in DEGB gravity will prescribe the problems of the solutions in Einstein gravity.
- This presentation is based on arXiv:1409.3935 and arXiv:1607.01125.

Outline

- Let us consider the following Euclidean action in the absence of gravity as follows:

$$S_E = \int_{\mathcal{M}} \sqrt{\det \eta_{\mu\nu}} d^4x \, \left[\frac{1}{2}\partial^\mu \phi \partial_\mu \phi + U(\phi) \right],$$

where the potential is given by

$$U(\phi) = -\frac{\lambda}{4}\phi^4 + U_0.$$

- In order to find the equation of motion, we assume the $\mathcal{O}(4)$ symmetry to get the maximized tunneling probability,

$$ds^2 = d\eta^2 + \eta^2 \left[d\chi^2 + \sin^2\chi \left(d\theta^2 + \sin^2\theta d\varphi^2 \right) \right].$$

- The equation of motion is

$$0 \;=\; \ddot{\phi} + \frac{3}{\eta}\dot{\phi} - U'(\phi),$$

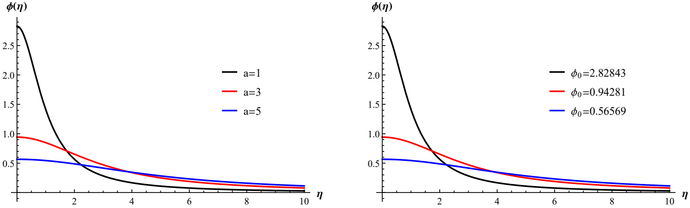
where the boundary conditions are,

$$\left.\frac{d\phi}{d\eta}\right|_{\eta=0} = 0 \quad \text{and} \quad \phi\Big|_{\eta=\infty} = 0.$$

- For the given potential, there is an analytic solution which is called *Fubini instanton* which has the following form

$$\phi(\eta) \;=\; \sqrt{\frac{8}{\lambda}}\frac{a}{\eta^2+a^2}.$$

- Here, η is Euclidean time parameter and a is an arbitrary length scale that characterizes the size of the instanton.



(a) Analytic solutions (b) Numerical solutions

Figure: Fubini instantons with the values of a and corresponding initial values of scalar field ϕ_0.

- Let us consider the Euclidean action that the scalar field is interacting with the Gauss–Bonnet (GB) term as follows:

$$S_E = -\int_{\mathcal{M}} d^4x \sqrt{\det g_{\mu\nu}} \left[\frac{R}{2\kappa} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi) + f(\phi)R^2_{GB} \right],$$

where GB term is given by $R^2_{GB} = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2$.

- The potential is same as before,

$$U(\phi) = -\frac{\lambda}{4}\phi^4 + U_0,$$

but U_0 now gives an effect to the cosmological constant.

- The coupling function is setted as follows:

$$f(\phi) = \alpha e^{-\gamma\phi},$$

where α is the GB coefficient and γ is the coupling constant between the scalar field and GB term.

- The equation of motion for scalar field and the Einstein's equations are

$$
\begin{aligned}
0 &= \nabla^2\phi - U'(\phi) + f'(\phi)R^2_{GB}, \\
0 &= \frac{1}{2\kappa}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) - \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{4}g_{\mu\nu}\partial_\rho\phi\partial^\rho\phi + \frac{1}{2}g_{\mu\nu}U(\phi) + (GB)_{\mu\nu}.
\end{aligned}
$$

- The last term of Einstein's equation is obtained by the GB term variation and that in four-dimensional space is

$$
\begin{aligned}
(GB)_{\mu\nu} &= -2(\nabla_\mu\nabla_\nu f(\phi))R + 2g_{\mu\nu}(\nabla^2 f(\phi))R + 4(\nabla_\rho\nabla_\mu f(\phi))R_\nu{}^\rho \\
&\quad +4(\nabla_\rho\nabla_\nu f(\phi))R_\mu{}^\rho - 4(\nabla^2 f(\phi))R_{\mu\nu} - 4g_{\mu\nu}(\nabla_\rho\nabla_\sigma f(\phi))R^{\rho\sigma} \\
&\quad +4(\nabla^\rho\nabla^\sigma f(\phi))R_{\mu\rho\nu\sigma}.
\end{aligned}
$$

- There were terms linear in $f(\phi)$ but those terms are cancelled each other in four-dimensional space.
- Thus, $(GB)_{\mu\nu}$ in four-dimensional space appears only when the scalar field and GB term are non-minimally coupled.

Fubini instantons in DEGB gravity

- We also consider the $\mathcal{O}(4)$-symmetric metric

$$ds^2 = d\eta^2 + \rho(\eta)^2\left(d\theta^2 + \sin^2\theta(d\chi^2 + \sin^2\chi d\varphi^2)\right).$$

which minimizes the Euclidean action.

- It give the scalar curvature and GB term as follows:

$$R = -\frac{\dot{\rho}^2 - 1 + \rho\ddot{\rho}}{\rho^2}, \qquad \text{and} \qquad R^2_{GB} = 24\frac{\ddot{\rho}(\dot{\rho}^2 - 1)}{\rho^3}.$$

- The equation of motion for ϕ and (η, η), (χ, χ) components of Einstein's equation are

$$\begin{aligned} 0 &= \ddot{\phi} + 3\frac{\dot{\rho}}{\rho}\dot{\phi} - U'(\phi) + 24f'(\phi)\frac{\ddot{\rho}(\dot{\rho}^2 - 1)}{\rho^3}, \\ 0 &= \frac{3}{2\kappa}\frac{(\dot{\rho}^2 - 1)}{\rho^2} - \frac{1}{4}\dot{\phi}^2 + \frac{1}{2}U(\phi) - 12\dot{f}(\phi)\frac{\dot{\rho}(\dot{\rho}^2 - 1)}{\rho^3}, \\ 0 &= \frac{\dot{\rho}^2 - 1 + 2\rho\ddot{\rho}}{2\kappa} + \frac{\rho^2}{4}\dot{\phi}^2 + \frac{\rho^2}{2}U(\phi) - 8\dot{f}(\phi)\dot{\rho}\ddot{\rho} - 4\ddot{f}(\phi)(\dot{\rho}^2 - 1). \end{aligned}$$

Setup

- We have to impose appropriate boundary conditions which are depending on the background spaces, i.e. AdS, flat, and dS space.
- For the flat and AdS background, we can impose the boundary conditions as follows:

$$\rho(0) \;=\; 0, \quad \dot\rho(0) \;=\; 1, \quad \dot\phi(0) \;=\; 0, \quad \text{and} \quad \phi(\infty) \;=\; \phi_v.$$

where ϕ_v is the value of scalar field at vacuum.

- For dS background, we can impose the boundary conditions as follows:

$$\rho(0) \;=\; 0, \quad \rho(\eta_{\max}) \;=\; 0, \quad \dot\phi(0) \;=\; 0, \quad \text{and} \quad \dot\phi(\eta_{\max}) \;=\; 0.$$

- In order to use a numerical method, make the variables dimensionless such as

$$\lambda U_0 \to U_0, \quad \sqrt{\lambda}\phi \to \phi, \quad \frac{\kappa}{\lambda} \to \kappa, \quad \lambda\alpha \to \alpha, \quad \text{and} \quad \frac{\gamma}{\sqrt{\lambda}} \to \gamma.$$

- We set the initial value of η to be ϵ where $\epsilon \ll 1$ to avoid the initial divergence:

$$\phi(\epsilon) \;\approx\; \phi_0 - \frac{\epsilon^2}{8}\lambda\phi_0^3 + \cdots, \qquad \phi'(\epsilon) \;\approx\; -\frac{\epsilon}{4}\lambda\phi_0^3 + \cdots,$$

$$\rho(\epsilon) \;\approx\; \epsilon + \cdots, \qquad \rho'(\epsilon) \;\approx\; 1 - \frac{\epsilon^2}{6}\kappa U(\phi_0) + \cdots.$$

- We have to cut the Euclidean time for the numerical calculation because the evolution parameter is infinite in AdS background.
- In order to find the initial value of scalar field ϕ_0 which satisfy the boundary condition in the equations of motion, we use the *shooting method*.

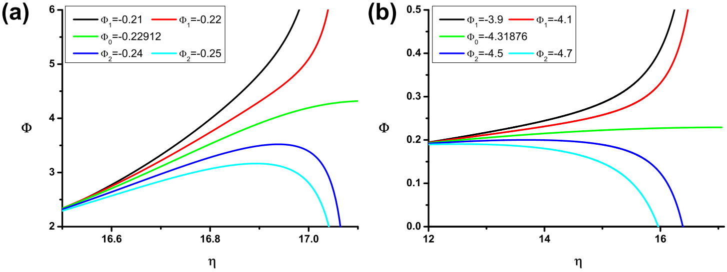


Figure: Shooting method examples in dS background

Tunneling probability

- To get the tunneling probability, we need to calculate Euclidean action

$$S_E = \int_{\mathcal{M}} \sqrt{g_E}\, d^4x \left[-\frac{R_E}{2\kappa} + \frac{1}{2}\phi'^2 + U \right] = 2\pi^2 \int \rho^3 d\eta [-U].$$

- In the semiclassical approximation, the decay probability is represented as Ae^{-B} where the exponent B is the difference between Euclidean action of a bounce solution and background action, $B = S_E^{bs} - S_E^{bg}$.

- From the technical reason in AdS background, we change the expression of the exponent B by using constraint equation

$$B = 2\pi^2 \int \rho^2 d\rho \left[\frac{-U}{\sqrt{\frac{1}{\rho^2} + \frac{\kappa}{3}\left(\frac{1}{2}\phi'^2 - U\right)}} - \frac{-U_0}{\sqrt{\frac{1}{\rho^2} + \frac{\kappa}{3}(-U_0)}} \right].$$

Outline

- We first determine the value of ϕ_v with the given potential $U(\phi)$ and GB coupling function $f(\phi)$.
- Since ϕ_v represent the vacuum, it is a static solution of equations of motion.
- We first obtain the analytic form of $\rho(\eta)$ by solving static equations of motion. For AdS and dS background,

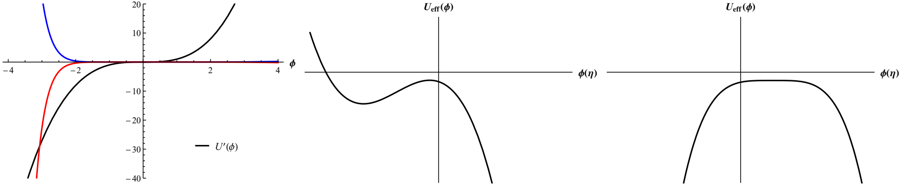
$$\rho(\eta) = \sqrt{\frac{3}{\Lambda}} \sinh \sqrt{\frac{\Lambda}{3}}\eta, \quad \text{and} \quad \rho(\eta) = \sqrt{\frac{3}{\Lambda}} \sin \sqrt{\frac{\Lambda}{3}}\eta.$$

- By substituting those scale functions into the scalar field equation of motion, we can simplify that such as

$$0 = U'(\phi_v) + \frac{8}{3}\alpha\gamma e^{-\gamma\phi_v}\Lambda^2,$$

where the cosmological constant is defined by

$$\Lambda = \kappa U(\phi_v).$$



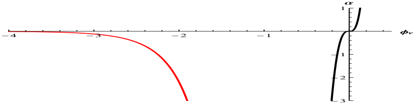
(a) Finding vacuum states

(b) Sketch of effective potential for $\alpha < 0$

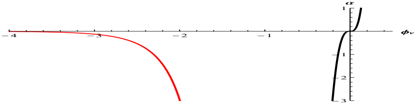
(c) Sketch of effective potential for $\alpha > 0$

Figure: (a) The red and blue line represent the first term in Eq. (15) for the case of $\alpha < 0$ and $\alpha > 0$, and the black line represents the second term in Eq. (15). Since they cross each other twice and once for red and blue, respectively, there exist two vacuums for $\alpha < 0$ and one vacuum for $\alpha > 0$. Those are the static solutions in Eq. (15). The parameters are fixed as $\kappa = 0.1$, $\alpha = \pm 0.1$, $\gamma = 1.0$ and $U_0 = -0.3$. (b) Through the previous plot, we sketch the rough figure of the expected effective potential for $\alpha < 0$ and (c) for $\alpha > 0$.

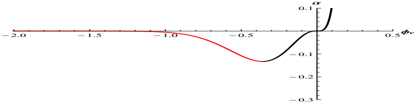
Vacuum



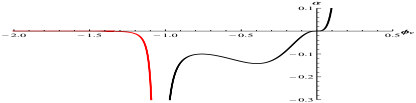
(a) α vs. ϕ_v with $\gamma = 1.0$ and $U_0 = -0.3$



(b) α vs. ϕ_v with $\gamma = 1.0$ and $U_0 = 0.3$



(c) α vs. ϕ_v with $\gamma = 8.0$ and $U_0 = -0.3$



(d) α vs. ϕ_v with $\gamma = 8.0$ and $U_0 = 0.3$

Figure: (a) The black and red line represent the vacuum states which are obtained from the simplified scalar field equation. The red line is appeared only for $\alpha < 0$. This plot shows the values of vacuum state with respect to α with fixed parameter $\gamma = 1.0$ for AdS background and (b) for dS background. (c) When $\gamma = 8.0$, it is shown that the number of vacuums become one at specific negative α and disappear when α further decreases for AdS background and (d) the number of vacuums increases at specific range of negative α and decreases again when α further decreases for dS background. $\kappa = 1.0$ for all figures.

Solutions for Einstein gravity

- We numerically solve the coupled equations of the scalar field ϕ and scale factor ρ which comes from gravity.
- As we said before, there are three parameters κ, U_0 and ϕ_0. Each parameters correspond to the gravitational constant, cosmological constant, and initial value of scalar field, respectively.
- In AdS background, *any* set of parameter values always give the solutions with finite number of oscillation. So, we are focused to classify the number of solutions. Especially, we are interested in the oscillating solutions which correspond to the boundary of oscillating numbers.
- In dS background, *specific* set of parameter values give the solutions with different number of oscillation. Thus, we are focused to find the solutions. Interestingly, there are two types of solutions which are symmetric and asymmetric.
- For each solutions, we get the action difference B. It might be infinite or finite.

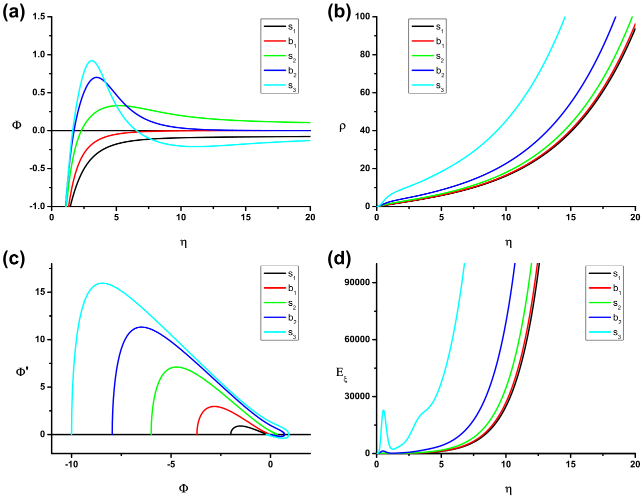


Figure: (a) Numerical solutions for ϕ, (b) for ρ, (c) phase diagram of ϕ' vs. ϕ, and (d) Euclidean energy E_ξ in AdS background. We take $\kappa = 0.30$ and $U_0 = -0.30$.

- Late time behavior of ϕ is linear in log-log scale for all solutions. This means that the solutions are approaching to zero when the time goes to infinity.
- From the analysis of action difference B, we notice that the marginal solutions only have the finite action difference and the others looks have infinity.

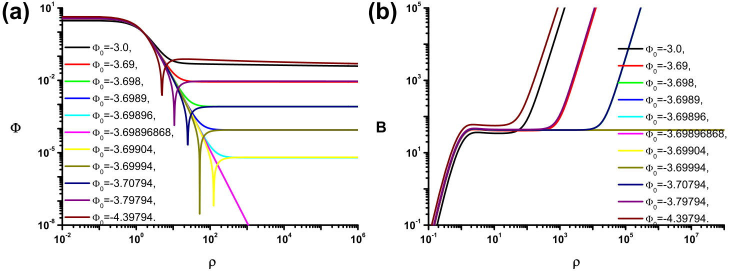


Figure: Log-Log graph of (a) ϕ and (b) B versus ρ for AdS solutions.

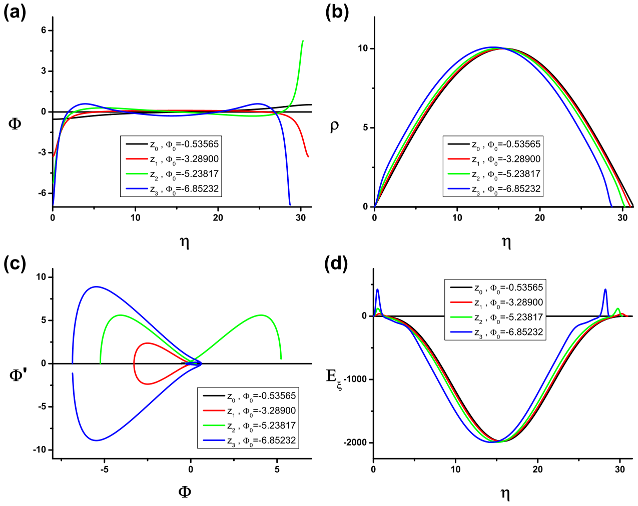


Figure: (a) Numerical solutions for ϕ, (b) for ρ, (c) phase diagram of ϕ' vs. ϕ, and (d) Euclidean energy E_ξ of Z_2 symmetric cases in dS background.

Solutions in dS background

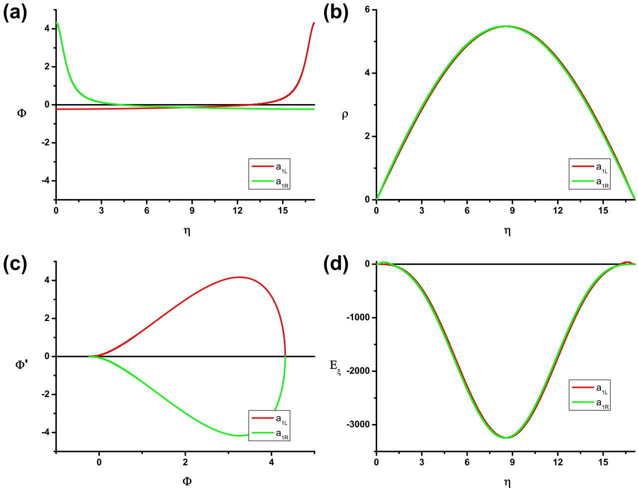


Figure: (a) Numerical solutions for ϕ, (b) for ρ, (c) phase diagram of ϕ' vs. ϕ, and (d) Euclidean energy E_ξ of Z_2 asymmetric cases in dS background.

Solutions in dS background

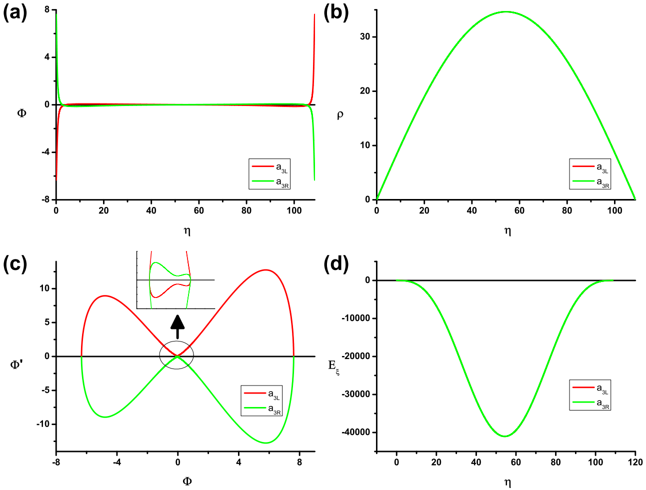


Figure: (a) Numerical solutions for ϕ, (b) for ρ, (c) phase diagram of ϕ' vs. ϕ, and (d) Euclidean energy E_ξ of Z_2 asymmetric cases in dS background.

- We are tested the behavior of solutions with different parameter values. By using this result, we draw a map which visualize the information about solutions.

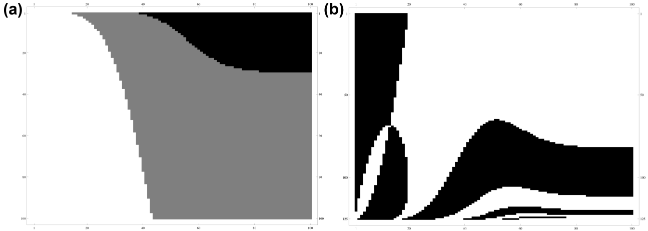


Figure: Matrix plots in (a) AdS background with white, gray and black colors which are correspond to the different number of oscillation such as 1, 2, and 3, respectively, and in (b) dS background with white and black colors which are correspond to the direction of divergence such as negative and positive, respectively

- Through the matrix plot, we modify our code to find the phase diagram easily.

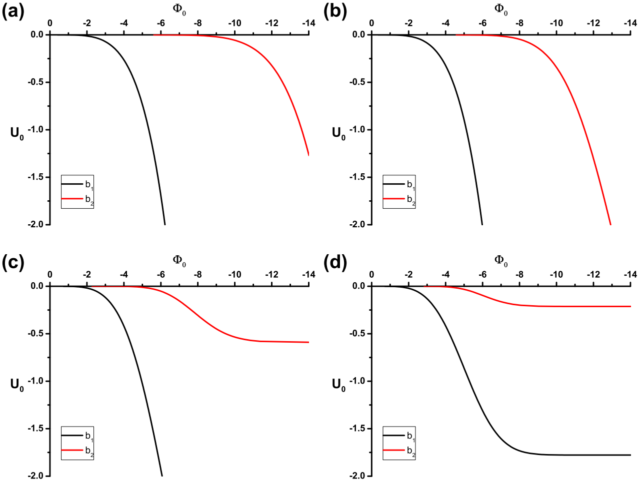


Figure: Parametric phase diagram in AdS background with (a) $\kappa = 0.05$, (b) $\kappa = 0.10$, (c) $\kappa = 0.30$, and (d) $\kappa = 0.50$, respectively

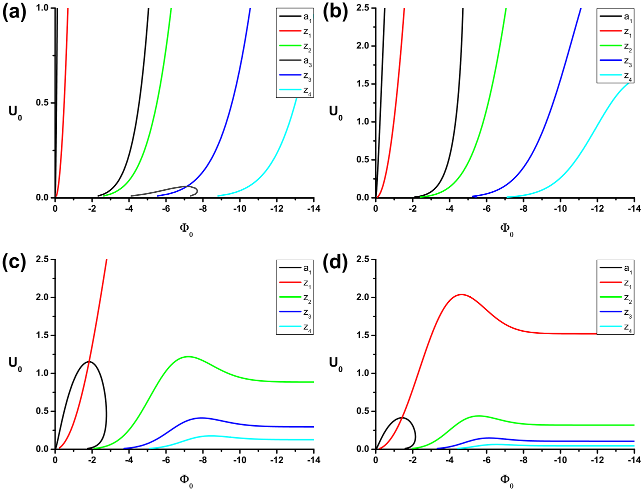
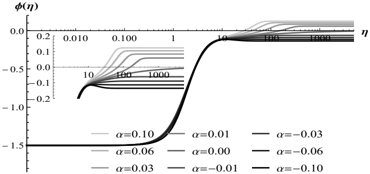
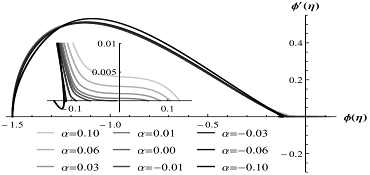


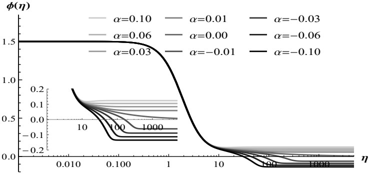
Figure: Parametric phase diagrams in dS background with (a) $\kappa = 0.05$, (b) $\kappa = 0.10$, (c) $\kappa = 0.30$, and (d) $\kappa = 0.50$, respectively



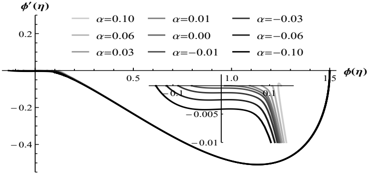
(a) ϕ vs. η for $\phi_0 < 0$



(b) ϕ' vs. ϕ for $\phi_0 < 0$



(c) ϕ vs. η for $\phi_0 > 0$



(d) ϕ' vs. ϕ for $\phi_0 > 0$

Figure: Solutions are plotted with respect to α. The other parameters are fixed as $\kappa = 0.1$, $\gamma = 1.0$ and $U_0 = -0.3$.

- We calculate the exponent B which is an Euclidean action difference between the solution and background where the form is given by

$$B = 2\pi^2 \int_0^{\rho_{\rm m}} d\rho \frac{\rho^3}{\dot{\rho}} \left(S_E|_{\eta\to\rho^{-1}} - S_E|_{\eta\to\sqrt{\frac{3}{|\Lambda|}}\sinh^{-1}\sqrt{\frac{|\Lambda|}{3}}\rho,\phi\to\phi_v} \right)$$

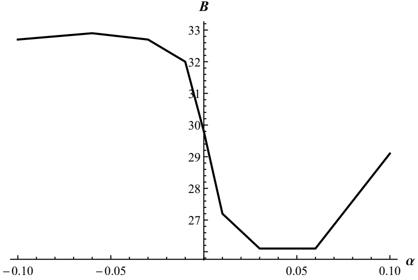
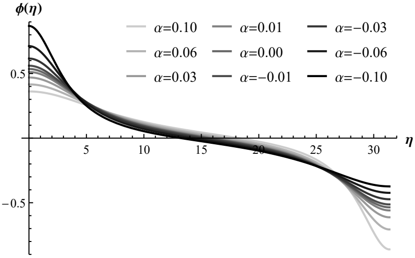
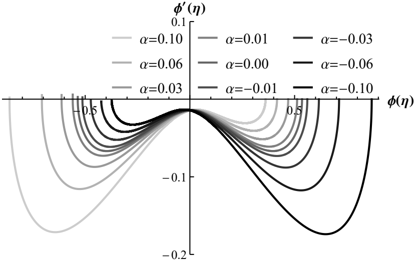


Figure: Exponent B of decay rate for the marginal solutions in AdS background with respect to α.



(a) Plot of $\phi(\eta)$ vs. η



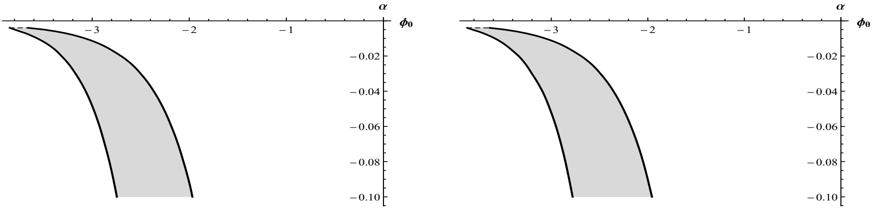
(b) Plot of $\phi'(\eta)$ vs. $\phi(\eta)$

Figure: Selected type of solutions in dS background with respect to α. The parameters are fixed as $\kappa = 0.1$, $\gamma = 1.0$ and $U_0 = 0.3$.

Forbidden solutions

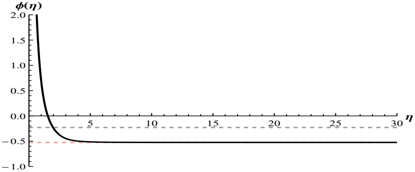
- Recall the equations of motion,

$$
\begin{aligned}
0 &= \ddot{\phi} + 3\frac{\dot{\rho}}{\rho}\dot{\phi} - U'(\phi) + 24 f'(\phi)\frac{\ddot{\rho}(\dot{\rho}^2 - 1)}{\rho^3}, \\
0 &= \frac{3}{2\kappa}\frac{(\dot{\rho}^2 - 1)}{\rho^2} - \frac{1}{4}\dot{\phi}^2 + \frac{1}{2}U(\phi) - 12\dot{f}(\phi)\frac{\dot{\rho}(\dot{\rho}^2 - 1)}{\rho^3}, \\
0 &= \frac{\dot{\rho}^2 - 1 + 2\rho\ddot{\rho}}{2\kappa} + \frac{\rho^2}{4}\dot{\phi}^2 + \frac{\rho^2}{2}U(\phi) - 8\dot{f}(\phi)\dot{\rho}\ddot{\rho} - 4\ddot{f}(\phi)(\dot{\rho}^2 - 1).
\end{aligned}
$$

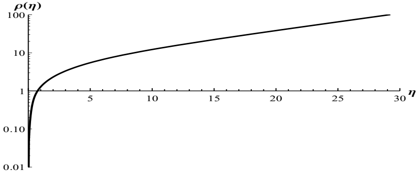


(a) For AdS background (b) For dS background

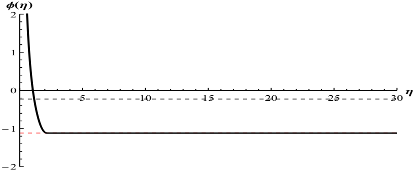
Figure: Solutions are forbidden in gray regions. The parameters are fixed as $\kappa = 0.1$, $\gamma = 1.0$ and $U_0 = \mp 0.3$.



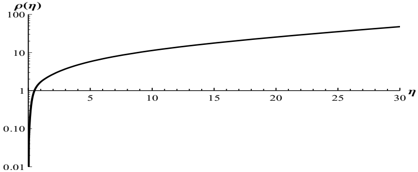
(a) $\phi(\eta)$ vs. η for AdS background



(b) $\rho(\eta)$ vs. η for AdS background



(c) $\phi(\eta)$ vs. η for dS background



(d) $\rho(\eta)$ vs. η for dS background

Figure: New type of solutions in AdS and dS background are appeared. The parameters are fixed as $\kappa = 0.1$, $\alpha = -0.1$, $\gamma = 8.0$ and $U_0 = \pm 0.3$.

Outline

Summary and Discussion

- In the absence of gravity, a hilltop type of quartic potential makes infinitely many solutions which is called Fubini instanton.
- However, inclusion of the Einstein gravity changes the situation abruptly. Depending on the sign of Λ, solution space changed or get reduced.
- In AdS background, there always exist the oscillating solutions but the oscillating numbers are fixed by the choice of parameters. This means that the solution space is changed.
- Interestingly, the action difference B is finite when the solution is a marginal solution. The other solutions looks have infinite action difference.
- In dS background, there are Z_2 symmetric and asymmetric solutions with specific set of parameter values. This means that the solution space is reduced.

- We have also studied about Fubini instantons in DEGB gravity which makes several changes of the solutions in Einstein gravity.
- In AdS background, the solution converges to moved vacuum where the moved direction depends on the value of α. Still the oscillating behavior appears.
- In dS background, the solution shows an oscillating behavior but now Z_2-symmetry is broken.
- From the coupling function, there appears new vacuum and it allows to have a new type of solution.
- We hope to control the decay probability by adjusting the GB parameters α and γ. Indeed, the exponent B is decreased or increased by changing the value of α.
- Further studies are needed to investigate more detail characteristics of AdS, dS and new type of solutions.