Black Holes by Perfect Fluid

Inyong Cho
(SeoulTech)

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on-going work with **Hyeong-Chan Kim** (KNUT)

Outline

1. Introduction to S3/H3

2. Black Hole Solutions in S3/H3
   with Static Perfect Fluid

3. Geodesics

4. Stability

5. Conclusions

Introduction (S3 and H3)

Metric

$$x_1^2 + x_2^2 + x_3^2 \pm x_4^2 = \pm R_0^2$$

Spatial Embedding, +: S3, -: H3

(t, r, θ, ϕ) coordinate system,

$$ds^2 = \mp dt^2 + \frac{dr^2}{1 - kr^2/R_0^2} + r^2 d\Omega_2^2$$

k=+1: S3, k=-1: H3

(t, χ, θ, ϕ) coordinate system

S3-I

$$r = R_0 \sin\chi, \qquad (r \leq R_0),$$
$$ds^2 = -dt^2 + R_0^2 d\chi^2 + R_0^2 \sin^2\chi d\Omega_2^2.$$

S3-II

$$r = R_0 \cosh\chi, \qquad (R_0 \leq r < \infty),$$
$$ds^2 = +dt^2 - R_0^2 d\chi^2 + R_0^2 \cosh^2\chi d\Omega_2^2.$$

H3

$$r = R_0 \sinh\chi, \qquad (0 \leq r < \infty),$$
$$ds^2 = -dt^2 + R_0^2 d\chi^2 + R_0^2 \sinh^2\chi d\Omega_2^2.$$

Effective Energy-Momentum Tensor

$$\bar{G}^{\mu}_{\nu} = \mp\frac{1}{R_0^2}\mathrm{diag}(3,1,1,1) \equiv 8\pi\bar{T}^{\mu}_{\nu}$$

: constant, -: S3 (ρ>0), +: H3 (ρ<0)

$$p = -\frac{1}{3}\rho = const.$$

: Eq. of State

R3: Schwarzschild BH

if M=0, Minkowski Space: spatially flat

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega_2^2$$

Can we have such a black hole parameterized by MASS in S3/H3?

$$\bar{T}^{\mu}_{\nu} = \mp\frac{1}{8\pi R_0^2}\mathrm{diag}(3,1,1,1)$$

with effective EM tensor

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + r^2 d\Omega_2^2.$$

search BH solution

NO such a solution exists

Black Holes in S3/H3 with Static Perfect Fluid

Introduce "Static Perfect Fluid" satisfying S3/H3 Equation of State

(t, r, θ, ϕ) coordinate system,

metric ansatz

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + r^2 d\Omega_2^2.$$

EM tensor

$$T^\mu_\nu = \mathrm{diag}\big[-\rho(r), p(r), p(r), p(r)\big],$$

$$p(r) = -\frac{1}{3}\rho(r)$$

Require S3/H3 Eq. of State

Einstein Eqs.

$$G^0_0 = -\frac{1}{r^2} + \frac{1}{r^2 g} - \frac{g'}{rg^2} = -8\pi\rho(r),$$

$$G^1_1 = -\frac{1}{r^2} + \frac{1}{r^2 g} + \frac{f'}{rfg} = 8\pi p(r),$$

$$G^2_2 = \frac{f'}{2rfg} - \frac{f'^2}{4f^2 g} - \frac{g'}{2rg^2} - \frac{f'g'}{4fg^2} + \frac{f''}{2fg} = 8\pi p(r),$$

⇒

Solutions

$$\rho(r) = -\frac{3}{8\pi\alpha}\left(1 \mp \frac{2\alpha|\beta|}{r}\left[\beta(r^2+\alpha)\right]^{1/2}\right),$$

$$f(r) = \frac{\rho(r)}{\rho_c},$$

$$g^{-1}(r) = -\frac{8\pi}{3}(r^2+\alpha)\rho(r).$$

☺ MAPLE users
☹ Mathematica users

Define new parameters : $R_0^2 \equiv |\alpha|, \quad K \equiv 2R_0^2|\beta|^{3/2}.$

Mass Parameter

3 types of solutions

S3-I $r = R_0 \sin\chi \quad (0 \le \chi \le \pi).$ $\alpha < 0,\ \beta < 0,\ \text{and}\ 1 - 4\alpha^2\beta^3 > 0$

$$\rho(\chi) = \frac{3}{8\pi R_0^2}(1 - K\cot\chi), \quad \equiv F$$

$$f(\chi) = \frac{\rho(\chi)}{\rho_c}, \qquad (\rho_c > 0),$$

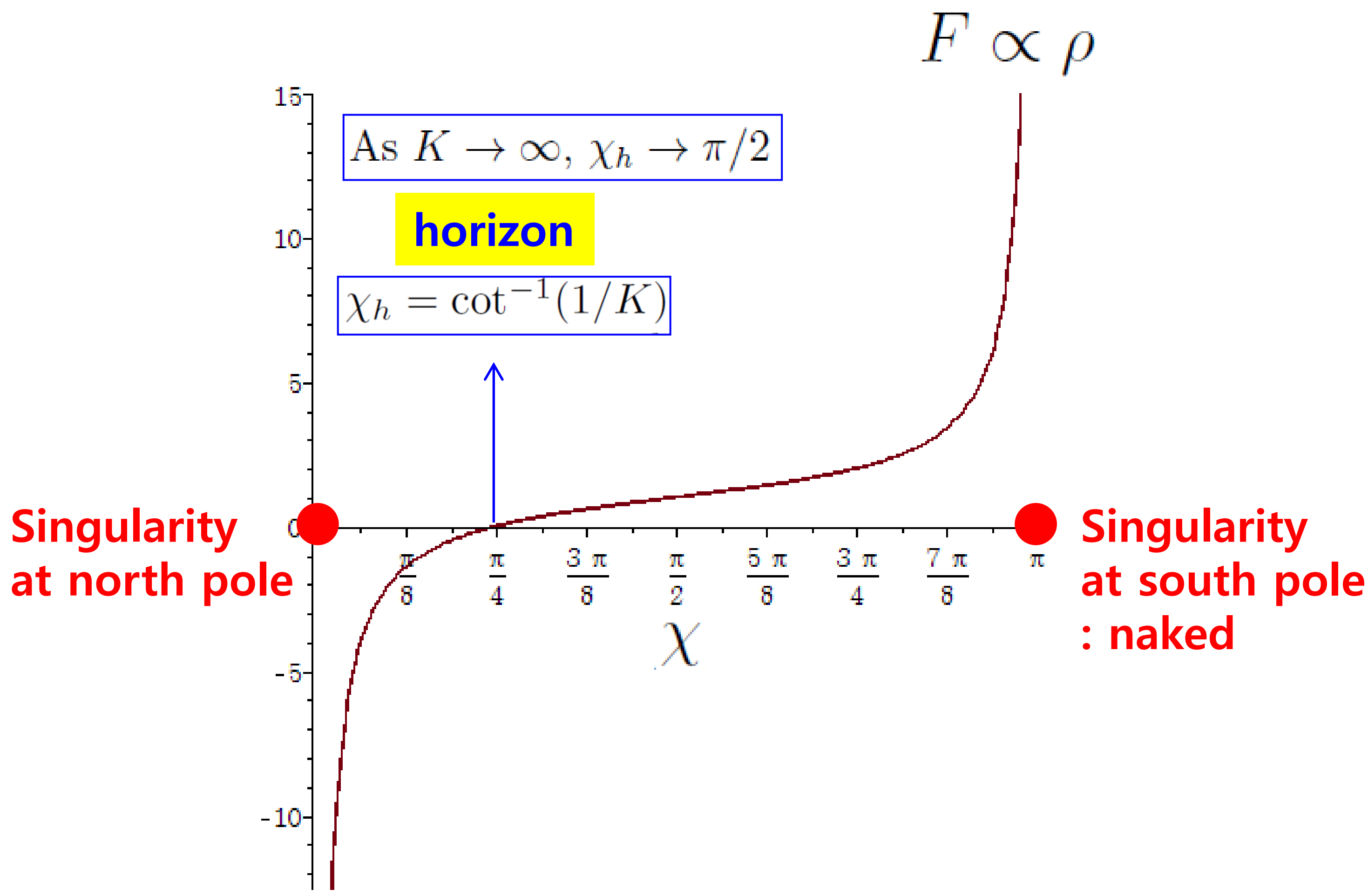
$$g(\chi) = \frac{3}{8\pi\rho(\chi)},$$

$$ds^2 = -\frac{3}{8\pi R_0^2\rho_c}(1 - K\cot\chi)dt^2 + \frac{R_0^2}{1 - K\cot\chi}d\chi^2 + R_0^2\sin^2\chi d\Omega_2^2.$$

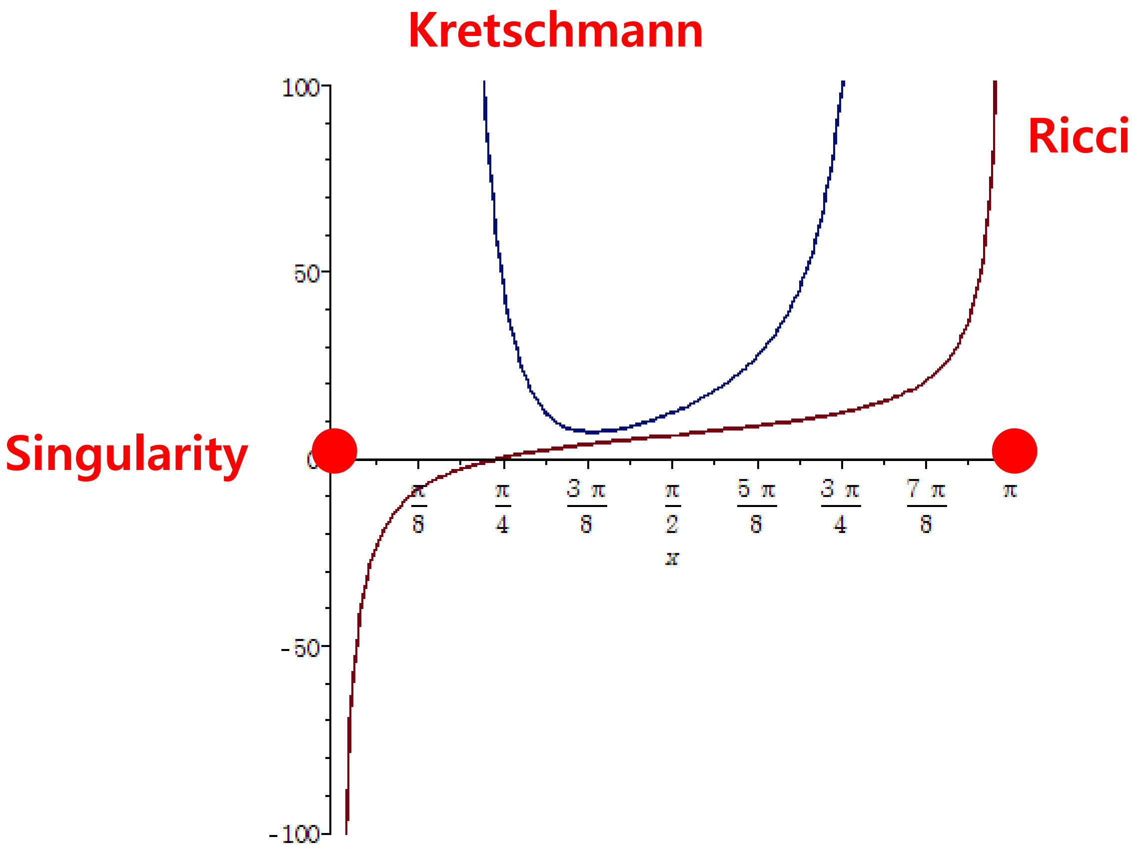
$\equiv F$

Black Hole Solution

If K=0, it is S3


$F \propto \rho$
15
10
5
0
-5
-10
As $K \to \infty$, $\chi_h \to \pi/2$
horizon
$\chi_h = \cot^{-1}(1/K)$
$\frac{\pi}{8}$ $\frac{\pi}{4}$ $\frac{3\pi}{8}$ $\frac{\pi}{2}$ $\frac{5\pi}{8}$ $\frac{3\pi}{4}$ $\frac{7\pi}{8}$ π
χ
Singularity at north pole
Singularity at south pole : naked

Curvature



Kretschmann
Ricci
Singularity
100
50
0
-50
-100
π/8
π/4
3π/8
π/2
5π/8
3π/4
7π/8
π
x

S3-II

$r = R_0 \cosh\chi \quad (\chi \geq 0).$

$\alpha < 0,\ \beta > 0,\ \text{and}\ 1 - 4\alpha^2\beta^3 < 0.$

$$\rho(\chi) = \frac{3}{8\pi R_0^2}\left(1 \mp K \tanh\chi\right),$$

$$f(\chi) = \frac{\rho(\chi)}{\rho_c}, \quad (\rho_c < 0),$$

$$g(\chi) = -\frac{3}{8\pi\rho(\chi)},$$

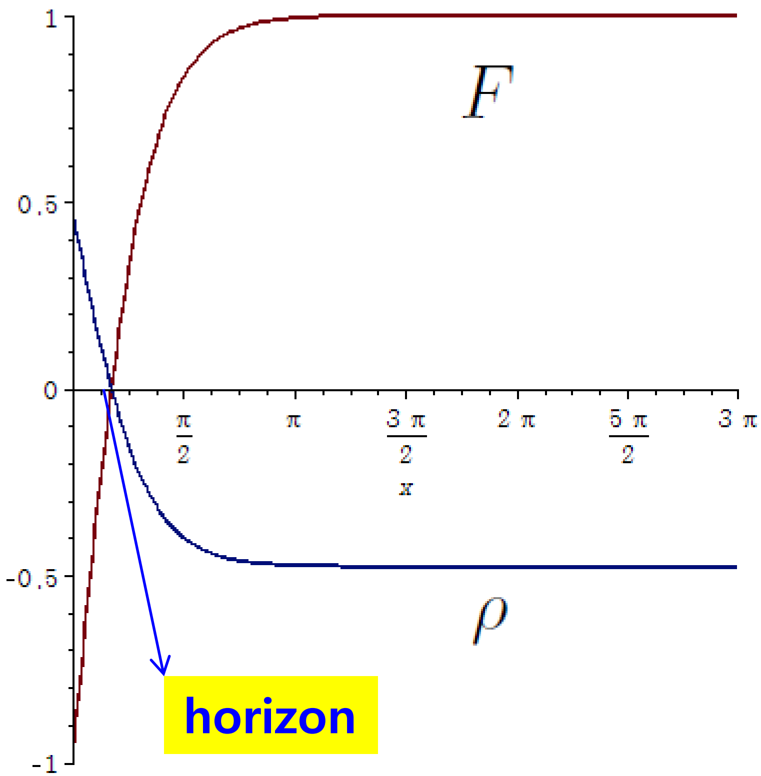
$$ds^2 = -\frac{3}{8\pi R_0^2 \rho_c}\left(1 \mp K \tanh\chi\right) dt^2 + \frac{R_0^2}{-\left(1 \mp K \tanh\chi\right)} d\chi^2 + R_0^2 \cosh^2\chi\, d\Omega_2^2.$$

Black Hole Solution: -, K>1

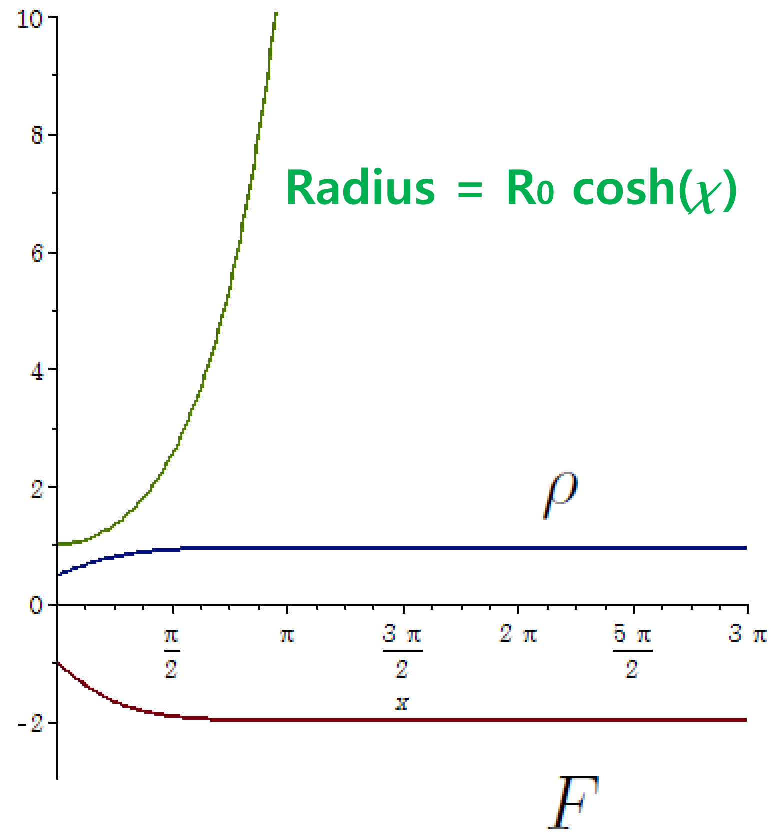
Cosmological Solution: -, K<1, or +

χ: time coordinate
Role of t and χ is changed!!!
Finite size at χ=0
Expanding Universe from a finite size!

Black Hole Solution: -, K>1

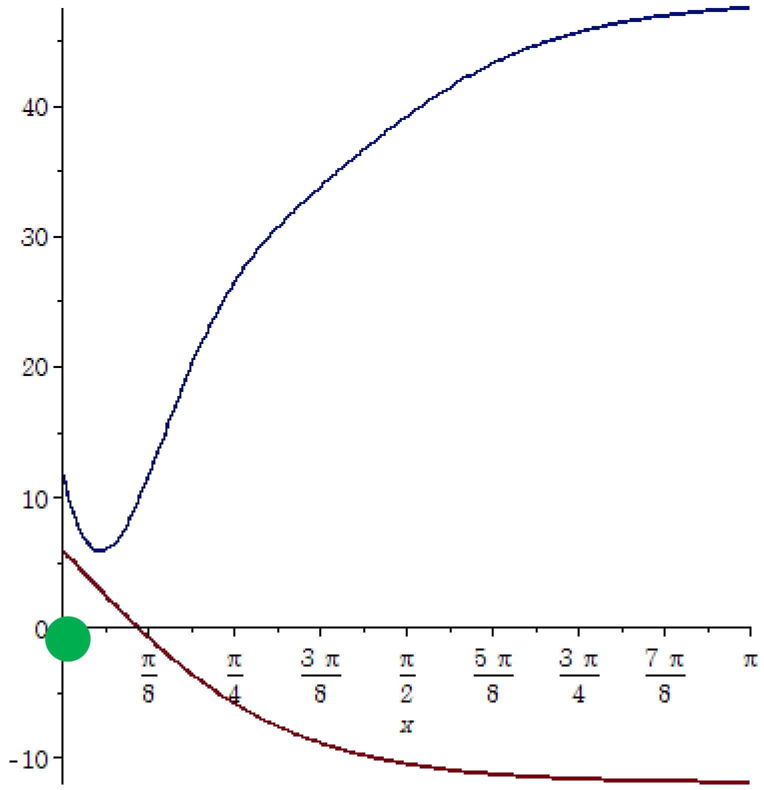


Cosmological Solution: +

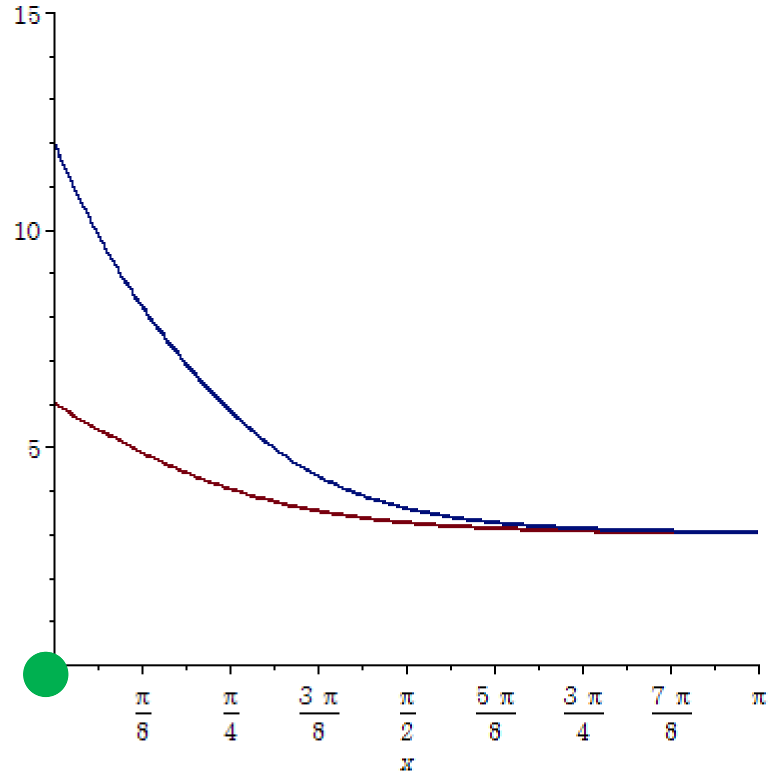


χ is time coordinate

Curvature



regular



H3

$r = R_0 \sinh\chi \quad (\chi \geq 0).$

$\alpha > 0,\ \beta > 0,\ \text{and}\ 1 - 4\alpha^2\beta^3 > 0.$

$$\rho(\chi) = -\frac{3}{8\pi R_0^2}\left(1 \mp K\coth\chi\right),$$

$$f(\chi) = \frac{\rho(\chi)}{\rho_c}, \quad (\rho_c < 0),$$

$$g(\chi) = -\frac{3}{8\pi\rho(\chi)},$$

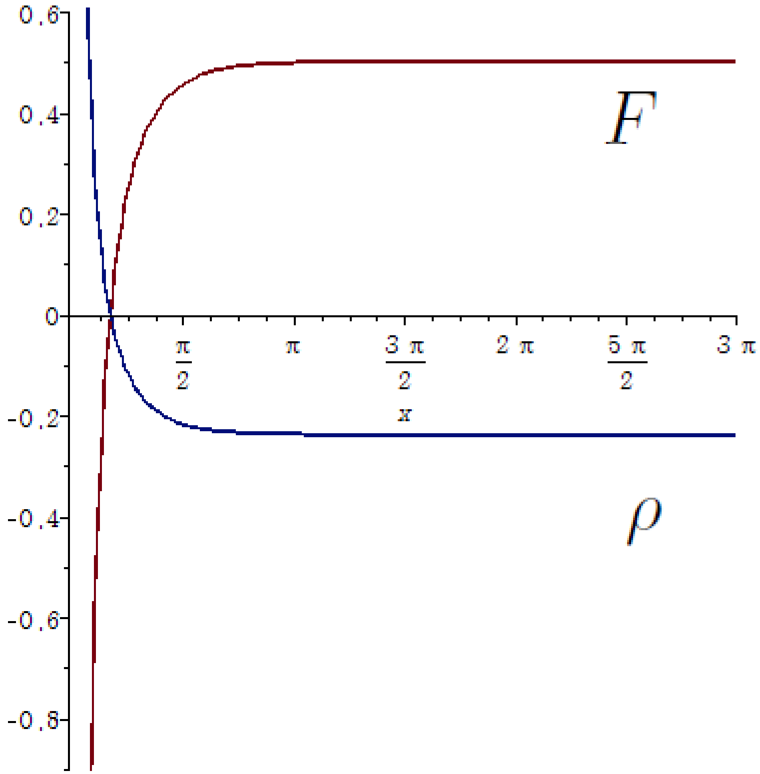
$$ds^2 = -\frac{3}{8\pi R_0^2(-\rho_c)}\left(1 \mp K\coth\chi\right)dt^2 + \frac{R_0^2}{1 \mp K\coth\chi}d\chi^2 + R_0^2\sinh^2\chi\, d\Omega_2^2.$$

Black Hole Solution: -, K<1

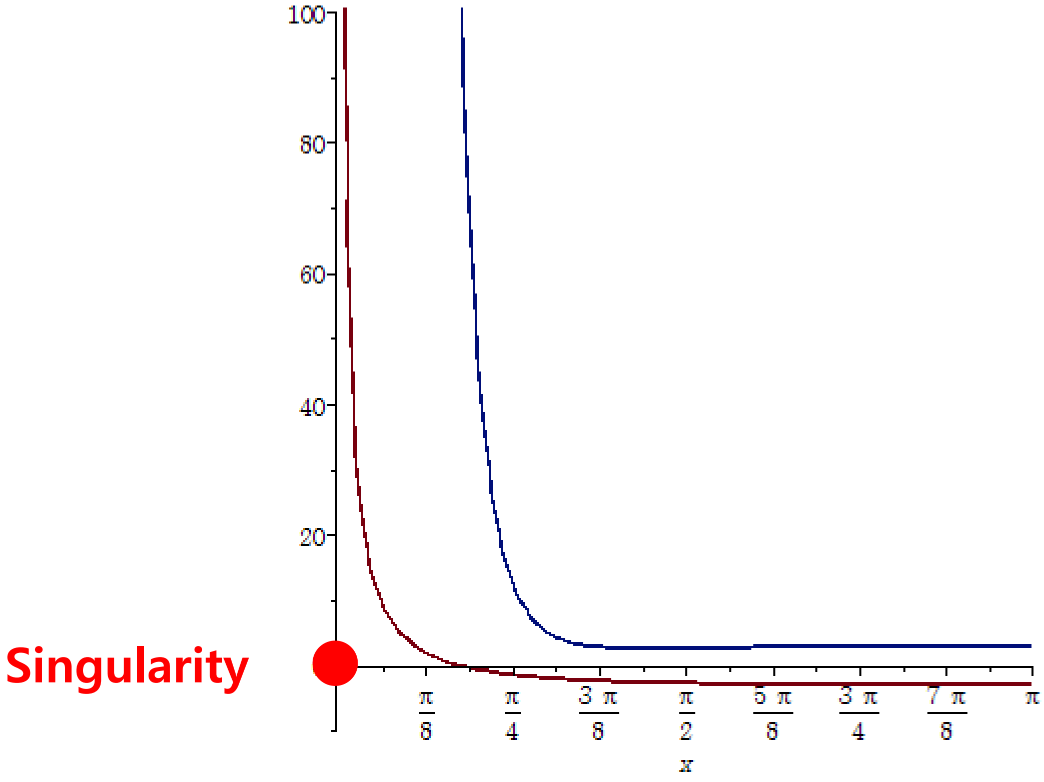
Cosmological Solution: -, K>1 **ρ>0 & ρ(0)=∞ : INITIAL SINGULARITY**

Regular Solution: + **ρ<0 & ρ(0)=∞ : NOT Interesting**

Black Hole Solution: -. K<0



Curvature



Horizon Structure and Mass

Schwarzschild Black Hole

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2$$

Near-horizon behavior

$$g^{-1}(r) = 1 - \frac{2M}{r} = 1 - \frac{r_{sh}}{r} = \frac{1}{r_{sh}}(r - r_{sh}) - \frac{1}{r_{sh}^2}(r - r_{sh})^2 + \cdots .$$

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + r^2 d\Omega_2^2.$$

$$\rho(r) = -\frac{3}{8\pi\alpha}\left(1 \mp \frac{2\alpha|\beta|}{r}\left[\beta(r^2 + \alpha)\right]^{1/2}\right),$$

$$f(r) = \frac{\rho(r)}{\rho_c},$$

$$g^{-1}(r) = -\frac{8\pi}{3}(r^2 + \alpha)\rho(r).$$

$$g^{-1}(r) = \frac{1}{r_h}(r - r_h) + \cdots ,$$

$$r_h = \pm 2\alpha|\beta|\sqrt{\frac{\alpha\beta}{1 - 4\alpha^2\beta^3}},$$

: same structure with Sch. BH

Identifying r_h=r_sh, we can relate K with M:

$$K = \left(\frac{R_0^2}{4M^2} - 1\right)^{-1/2}, \quad \left(-\frac{R_0^2}{4M^2} + 1\right)^{-1/2}, \quad \left(\frac{R_0^2}{4M^2} + 1\right)^{-1/2}.$$

S3-I S3-II H3

$$K=\left(\frac{R_0^2}{4M^2}-1\right)^{-1/2},\quad \left(-\frac{R_0^2}{4M^2}+1\right)^{-1/2},\quad \left(\frac{R_0^2}{4M^2}+1\right)^{-1/2}.$$

S3-I **S3-II** **H3**

From this relation between K & M,

S3-I: M_max = $R_0/2$

→ K = ∞

→ horizon at equator: maximum size of S3-I BH

If M > M_max, S3-II black holes

Geodesics

$$ds^2 = -\frac{3}{8\pi R_0^2 \rho_c} F(\chi) dt^2 + g(\chi) d\chi^2 + R_0^2 b^2(\chi) d\Omega_2^2$$

$$t\text{-eq. :} \quad \frac{1}{F(\chi)} \frac{d}{d\lambda} \left[F(\chi) \frac{dt}{d\lambda} \right] = 0,$$

$$\phi\text{-eq. :} \quad \frac{1}{b^2(\chi)} \frac{d}{d\lambda} \left[b^2(\chi) \frac{d\phi}{d\lambda} \right] = 0.$$

$$F(\chi) \frac{dt}{d\lambda} = \text{const.} \equiv E, \qquad b^2(\chi) \frac{d\phi}{d\lambda} = \text{const.} \equiv L.$$

On the $\theta = \pi/2$ plane, the χ-equation becomes

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = -\varepsilon,$$

$\varepsilon = 1, 0$ for timelike and null geodesics,

$$\frac{1}{2} \left(\frac{d\chi}{d\lambda} \right)^2 + V(\chi) = \frac{3E^2}{16\pi R_0^4 |\rho_c|} \equiv \tilde{E}^2,$$

$$V(\phi) = \frac{1}{2} F(\chi) \left[\frac{L^2}{b^2(\chi)} + \frac{\varepsilon}{R_0^2} \right].$$

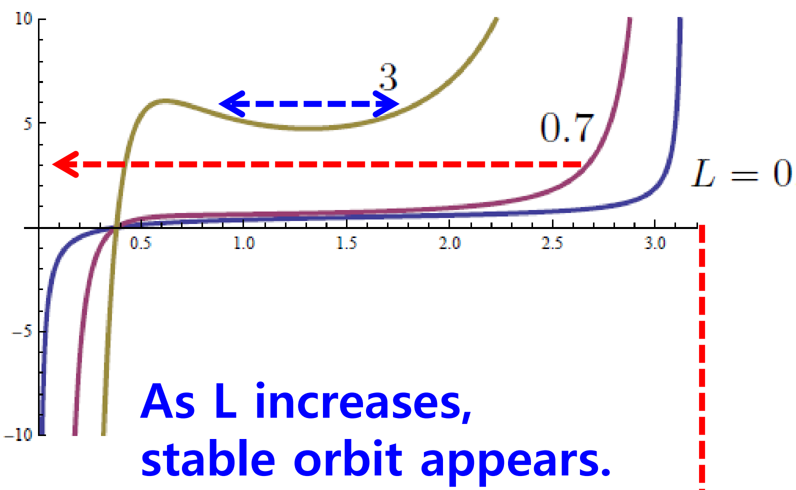
Effective Potential

S3-I

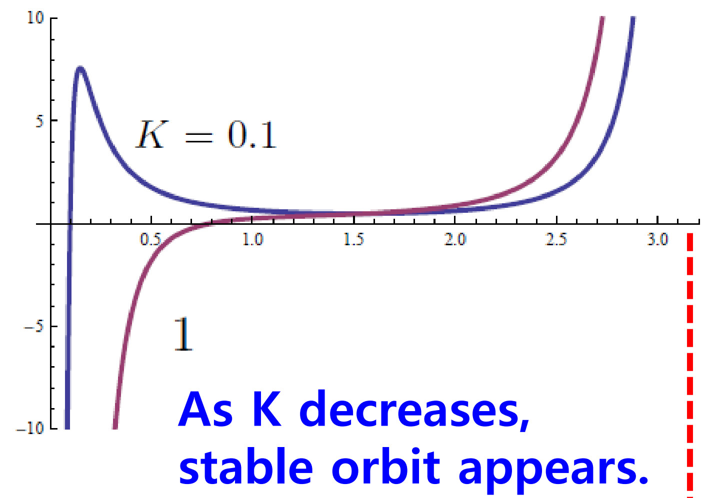
$$F(\chi) = 1 - K\cot\chi, \qquad b(\chi) = \sin\chi,$$

$$V(\phi) = \frac{1}{2}(1 - K\cot\chi)\left(\frac{L^2}{\sin^2\chi} + \frac{\varepsilon}{R_0^2}\right).$$

Timelike **Null**



(a) S_3 timelike geodesics for $R_0 = 1$ and $K = 0.4$:



(b) S_3 null geodesics for $R_0 = 1$ and $L = 1$:

NO geodesic can reach the singularity at south pole!!!

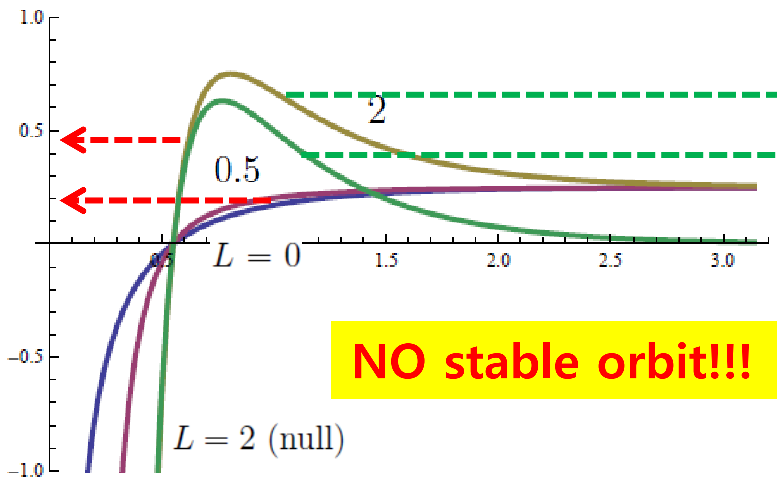
H3

$$F(\chi) = 1 - K \coth \chi, \qquad b(\chi) = \sinh \chi,$$

$$V(\phi) = \frac{1}{2}(1 - K \coth \chi)\left(\frac{L^2}{\sinh^2 \chi} + \frac{\varepsilon}{R_0^2}\right).$$

Similar with Sch. BH

As L increases,
bouncing orbit appears.
(Null: always)



can escape

NO stable orbit!!!

(c) H_3 timelike and null geodesics for $R_0 = 1$ and $K = 0.5$:

Stability

Spherical Scalar Perturbations

$$ds^2 = -f(t,\chi)dt^2 + g(t,\chi)d\chi^2 + R_0^2 b^2(\chi)d\Omega_2^2,$$

$$f(t,\chi) = f_0(\chi) + \epsilon f_1(t,\chi),$$
$$g(t,\chi) = R_0^2\left[g_0(\chi) + \epsilon g_1(t,\chi)\right],$$

$$f_0(\chi) = \frac{\rho_0(\chi)}{\rho_c} = \frac{3}{8\pi R_0^2 \rho_c} F(\chi),$$
$$g_0(\chi) = \frac{s}{F(\chi)},$$

background

s=+1: S3-I
s=-1: S3-II, H3

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p g^{\mu\nu},$$

$$\rho(t,\chi) = \rho_0(\chi) + \epsilon\rho_1(t,\chi),$$

source of perturbation

$$u^\mu = [u^0(t,\chi), u^1(t,\chi), 0, 0].$$

normalization $u^\mu u_\mu = -1$

comoving fluid

$1/\sqrt{f_0(\chi)}$ $\quad -f_1/(2f_0^{3/2})$ from normalization

$$u^0(t,\chi) = u_0^0(\chi) + \epsilon u_1^0(t,\chi),$$
$$u^1(t,\chi) = u_0^1(\chi) + \epsilon u_1^1(t,\chi).$$

$$-\sqrt{\frac{2\pi R_0^2\rho_c}{3}}\frac{\dot{g}_1 b' F}{s^2 b\sqrt{F}}.$$

from Einstein's eq.: (0,1)

After manipulating the Einstein equations, one finally gets the single master perturbation equation.
If we introduce the perturbation in the form,

$$g_1(t,\chi) = e^{i\omega t}\varphi(\chi),$$

the perturbation equation becomes

$$-F^2\varphi'' - \left(3FF' + 2F^2\frac{b''}{b'}\right)\varphi' + \left[\frac{\omega^2}{S} - 2FF'' - FF'\left(5\frac{b''}{b'} - \frac{b'}{b}\right) - 2F^2\left(\frac{b'''}{b'} - \frac{b''^2}{b'^2} + \frac{b''}{b} - \frac{b'^2}{b^2}\right)\right]\varphi = 0,$$

For all cases $S = 1/(8\pi R_0^4\rho_c s) > 0$

We rescale the radial coordinate and the amplitude function by

$$z = \int_0^\chi \frac{d\chi}{2F(\chi)},$$
$$\Phi(z) = N\frac{F(\chi)b'(\chi)}{z}\varphi(\chi),$$

The perturbation equation is cast into the Schrodinger-type equation,

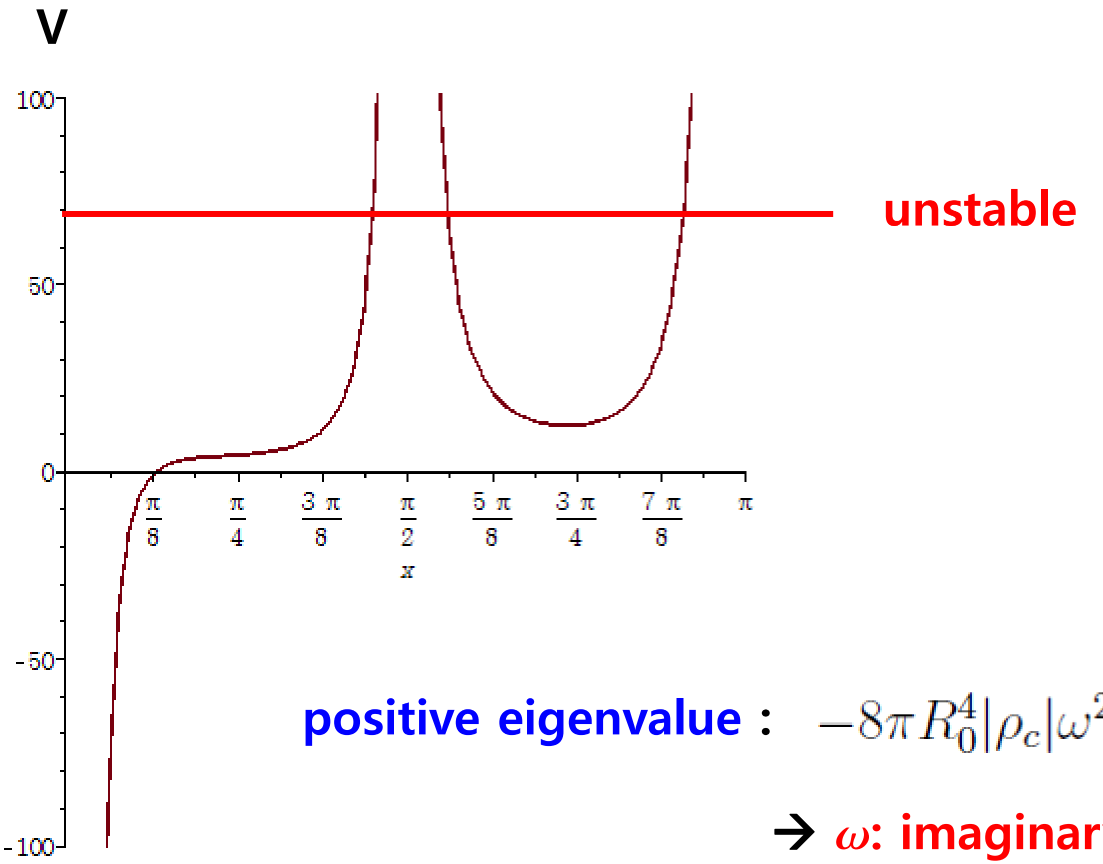
$$\left[-\frac{1}{2}\frac{d^2}{dz^2} - \frac{1}{z}\frac{d}{dz} + V(z)\right]\Phi(z) = -\frac{\omega^2}{S}\Phi(z) = -8\pi R_0^4|\rho_c|\omega^2\Phi(z),$$

eigenvalue

$$V[z(\chi)] = F^2\left[-\frac{F''}{F} + \left(\frac{F'}{F}\right)^2 - \frac{F'}{F}\left(2\frac{b''}{b'} - \frac{b'}{b}\right) - \frac{b'''}{b'} + 2\left(\frac{b''}{b'}\right)^2 - 2\frac{b''}{b} + 2\left(\frac{b'}{b}\right)^2\right]$$

: effective potential

S3-I

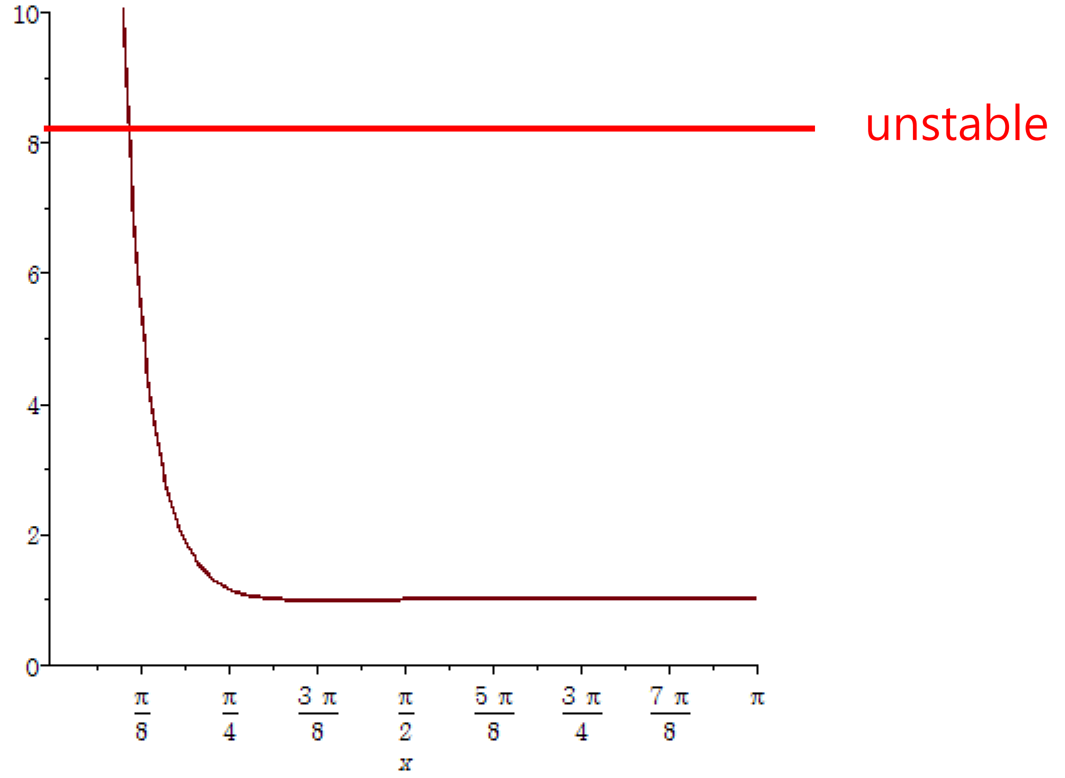


positive eigenvalue : $-8\pi R_0^4|\rho_c|\omega^2 > 0$

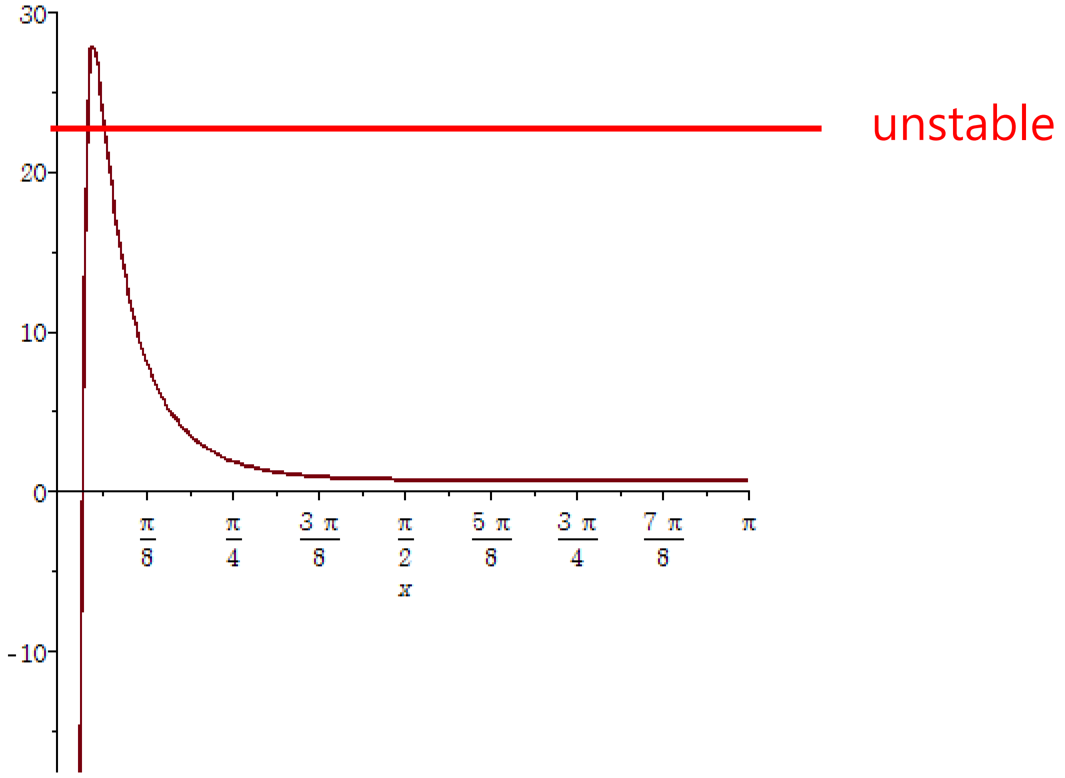
→ ω: imaginary

→ UNSTABLE!!

S3-II



H3



$$\left[-\frac{1}{2}\frac{d^2}{dz^2} - \frac{1}{z}\frac{d}{dz} + V(z)\right]\Phi(z) = -\frac{\omega^2}{S}\Phi(z) = -8\pi R_0^4|\rho_c|\omega^2\Phi(z),$$

eigenvalue

since this is negative, there exists always unstable modes for any type of V

Fate of Black Holes

1) BH may collapse: may leave some cosmological remnants

2) BH may remain while BACKGROUND expands
   :- since the b.g. matter is perfect fluid,
      the instability may imply the Friedmann expansion
   :- BH may sustain its nature in expanding b.g. Universe
   :- good and interesting
   :- requires NUMERICAL SIMULATIONS

Conclusions

1) There are black hole solution with static perfect fluid
   with spatial topology S3 & H3

2) S3-I
   - :- only black hole solution
     - :- singularities at both poles
       - :- the one is hidden by horizon, the other is naked
     - :- the naked singularity is visible to the observer
     - :- however, observer never falls into the naked singularity
     - :- observer may fall into BH, or can have a stable orbit

3) S3-II
   - :- black hole solution
     - : not very interesting b/c ρ<0 in the regular region
   - :- cosmological solution
     - : expanding from a finite size

4) H3
   - :- black hole solution : single singularity behind horizon, no stable orbit
   - :- cosmological solution: initial singularity
   - :- regular solution: central singularity