Correspondences between Gravity and Quantum Entanglement

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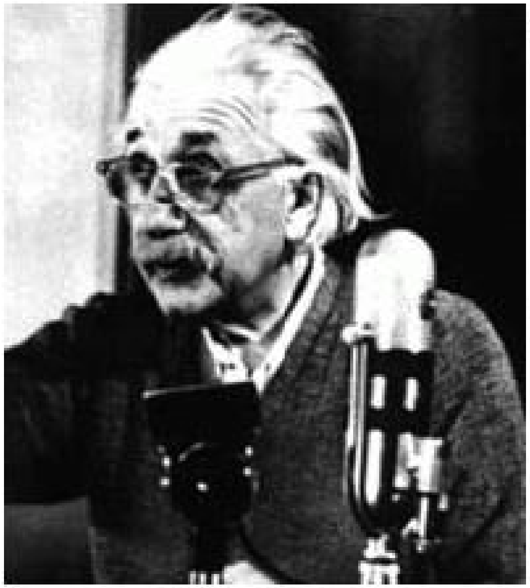
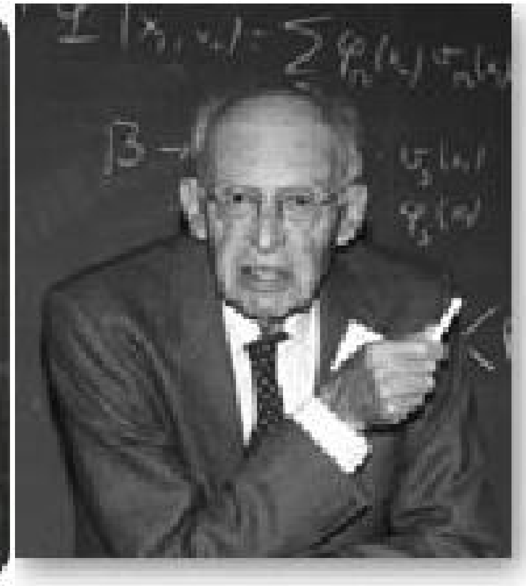
ABSTRACT

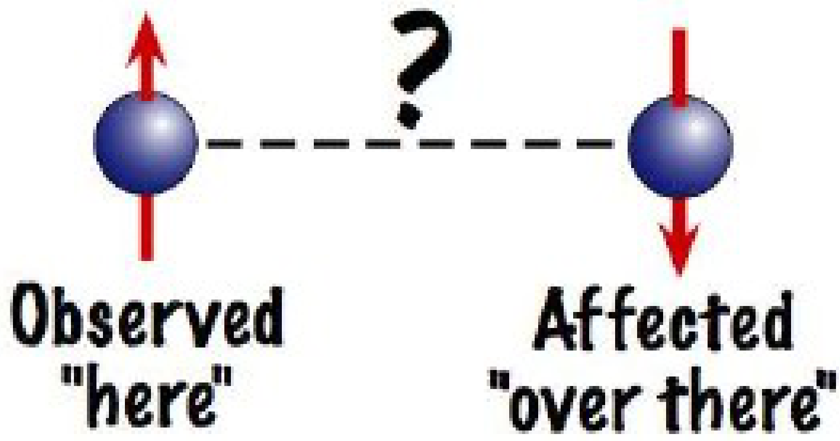
A picture of gravity as an analog of quantum entanglement has been a subject of great interest. In this talk we provide a simple model of a quantum-entangled system, built by means of a new method, "Information Geometry": a kind of differential geometry specially devised to construct virtual manifolds that represent the physical states of our quantum system. We compare our model with the gravity-analogs based on AdS/CFT, presented by Ryu and Takayanagi, Van Raamsdonk, etc., and find remarkable correspondences between them. Among other things, (i) the correlation of degrees of freedom and (ii) the entanglement entropy show excellent agreement between the two different physical phenomena: (i) the exponentially decaying pattern suggests a quantitative connection between entanglement measures and the structure of the dual spacetime, (ii) the information content of a region depends on its surface area rather than on its volume - holographic principle.

1. Einstein-Podolsky-Rosen Paradox and Quantum Entanglement





- EPR argument - Einstein's critique of the orthodox Copenhagen interpretation of quantum mechanics: violation of classical causality.
- EPR paradox draws on a phenomenon known as quantum entanglement, to show that measurements performed on spatially separated parts of a quantum system can apparently have an instantaneous influence on one another.
- This effect is known as non-local behavior (or quantum weirdness or spooky action at a distance).



- Entangled pair - any change to one particle will be instantly reflected in the other, no matter how far apart they might be:
  e.g. anti-alignment of spins of an electron-positron pair from pion decay.

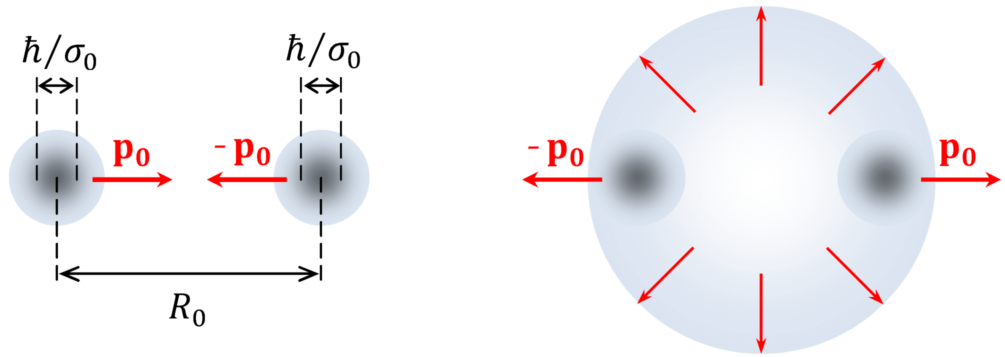
- This seems to run counter to a central tenet of Einstein's theory of relativity: nothing, not even information, can travel faster than the speed of light.

- The notion of entanglement leads to correlation

$$\langle\psi\,|AB|\,\psi\rangle - \langle\psi\,|A|\,\psi\rangle\,\langle\psi\,|B|\,\psi\rangle \neq 0,$$

  given observables A and B [***Kaplan, arXiv:quant-ph/0508078v1***].

2. Quantum Entangled Wave-Packets and Probability Distribution Functions



Before collision

After collision

Before collision:

$$\psi_{\text{before}}\left(\mathbf{p}_1,\mathbf{p}_2\right)=\left(\frac{1}{2\pi\sigma_\text{o}^2}\right)^{3/2}\exp\left[-\frac{\left(\mathbf{p}_1-\mathbf{p}_\text{o}\right)^2+\left(\mathbf{p}_2+\mathbf{p}_\text{o}\right)^2}{4\sigma_\text{o}^2}\right]e^{i\left[-\frac{(p_1-p_\text{o})R_\text{o}}{2\hbar}+\frac{(p_2+p_\text{o})R_\text{o}}{2\hbar}\right]}$$

After collision [*Wang et al., Phys. Rev. A73, 034302 (2006)*]:

$$\psi_{\text{after}}\left(\mathbf{p}_1,\mathbf{p}_2,t\right)=\left(N\right)^{-1/2}\left[\psi_{\text{before}}\left(\mathbf{p}_1,\mathbf{p}_2\right)e^{-i\frac{p_1^2+p_2^2}{2\hbar m}t}+\varepsilon\psi_{\text{scat}}\left(\mathbf{p}_1,\mathbf{p}_2,t\right)\right];$$

$$\varepsilon\psi_{\mathrm{scat}}(\mathbf{p}_1,\mathbf{p}_2,t) \approx \left(\frac{1}{2\pi\sigma_{\mathrm{o}}^2}\right)^{3/2}\exp\left[-\frac{\mathbf{P}^2+4\left(\mathbf{p}-p_{\mathrm{o}}\hat{\mathbf{p}}\right)^2}{8\sigma_{\mathrm{o}}^2}\right] \times\frac{4i\left(\hbar p_{\mathrm{o}}-i\sigma_{\mathrm{o}}^2R_{\mathrm{o}}\right)p^2f(p)}{\hbar^2\sigma_{\mathrm{o}}^2}e^{-i\left[\frac{(p-p_{\mathrm{o}})R_{\mathrm{o}}}{\hbar}+\frac{K^2}{2\hbar M}t+\frac{k^2}{2\hbar\mu}t\right]}.$$

$\mathbf{P}\equiv\mathbf{p}_1+\mathbf{p}_2$, $\mathbf{p}\equiv\frac{1}{2}(\mathbf{p}_1-\mathbf{p}_2)$, $M=2m$, $\mu=m/2$,

$f(p)\equiv\frac{\hbar\left(e^{i2\theta(p)}-1\right)}{2ip}$: **$s$-wave scattering amplitude,** $\theta(p)$: **s-wave scattering phase shift.**

Effectively reducing to 1-D,

$$P_{\mathrm{QM}}^{\mathrm{before}} = \left|\psi_{\mathrm{before}}(p_1,\,p_2)\right|^2=\frac{1}{2\pi\sigma_{\mathrm{o}}^2}\exp\left[-\frac{(p_1-p_{\mathrm{o}})^2+(p_2+p_{\mathrm{o}})^2}{2\sigma_{\mathrm{o}}^2}\right],$$

$$P_{\mathrm{QM}}^{\mathrm{after}} = \left|\psi_{\mathrm{after}}(p_1,\,p_2,t)\right|^2$$

$$\simeq \frac{1}{2\pi\sigma_{\mathrm{o}}^2\sqrt{1-r_{\mathbf{QM}}^2}}\exp\left[-\frac{(p_1-p_{\mathrm{o}})^2-2r_{\mathbf{QM}}(p_1-p_{\mathrm{o}})(p_2+p_{\mathrm{o}})+(p_2+p_{\mathrm{o}})^2}{2\left(1-r_{\mathbf{QM}}^2\right)\sigma_{\mathrm{o}}^2}\right]$$

with $r_{\mathbf{QM}}\equiv\sqrt{8\left(2p_{\mathrm{o}}^2+\sigma_{\mathrm{o}}^2\right)R_{\mathrm{o}}\left|f(p)\right|/\hbar^2}\ll 1$.

3. Information Geometry of Quantum Systems

If microvariables (observables) ξ_1 and ξ_2 are uncorrelated to each other,

$$P_0\left(\xi_1,\ \xi_2 \middle| \left\langle \xi_1 \right\rangle,\ \sigma_1,\ \left\langle \xi_2 \right\rangle,\ \sigma_2\right) = \frac{1}{2\pi\sigma_1\sigma_2}\exp\left[-\frac{\left(\xi_1-\left\langle \xi_1 \right\rangle\right)^2}{2\sigma_1^2}-\frac{\left(\xi_2-\left\langle \xi_2 \right\rangle\right)^2}{2\sigma_2^2}\right]$$

$$\Leftrightarrow \quad P_{\mathrm{QM}}^{\mathrm{before}} = \left|\psi_{\mathrm{before}}\left(p_1,\ p_2\right)\right|^2 = \frac{1}{2\pi\sigma_{\mathrm{o}}^2}\exp\left[-\frac{\left(p_1-p_{\mathrm{o}}\right)^2}{2\sigma_{\mathrm{o}}^2}-\frac{\left(p_2+p_{\mathrm{o}}\right)^2}{2\sigma_{\mathrm{o}}^2}\right].$$

However, if ξ_1 and ξ_2 are correlated to each other,

$$P_r\left(\xi_1,\ \xi_2 \middle| \left\langle \xi_1 \right\rangle,\ \sigma_1,\ \left\langle \xi_2 \right\rangle,\ \sigma_2\right) = \frac{\exp\left\{-\frac{1}{2\left(1-r^2\right)}\left[\frac{\left(\xi_1-\left\langle \xi_1 \right\rangle\right)^2}{\sigma_1^2}-\frac{2r\left(\xi_1-\left\langle \xi_1 \right\rangle\right)\left(\xi_2-\left\langle \xi_2 \right\rangle\right)}{\sigma_1\sigma_2}+\frac{\left(\xi_2-\left\langle \xi_2 \right\rangle\right)^2}{\sigma_2^2}\right]\right\}}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}}$$

$$\Leftrightarrow \quad P_{\mathrm{QM}}^{\mathrm{after}} = \left|\psi_{\mathrm{after}}\left(p_1,\ p_2, t\right)\right|^2 = \frac{\exp\left\{-\frac{1}{2\left(1-r_{\mathrm{QM}}^2\right)}\left[\frac{\left(p_1-p_{\mathrm{o}}\right)^2}{\sigma_{\mathrm{o}}^2}-\frac{2r_{\mathrm{QM}}\left(p_1-p_{\mathrm{o}}\right)\left(p_2+p_{\mathrm{o}}\right)}{\sigma_{\mathrm{o}}^2}+\frac{\left(p_2+p_{\mathrm{o}}\right)^2}{\sigma_{\mathrm{o}}^2}\right]\right\}}{2\pi\sigma_{\mathrm{o}}^2\sqrt{1-r_{\mathrm{QM}}^2}}$$

with $r = r\left(\xi_1,\ \xi_2\right) \equiv \frac{\left\langle \xi_1\xi_2 \right\rangle - \left\langle \xi_1 \right\rangle\left\langle \xi_2 \right\rangle}{\sigma_1\sigma_2}$, $\sigma_i = \sqrt{\left\langle \left(\xi_i-\left\langle \xi_i \right\rangle\right)^2 \right\rangle}$ $(i=1,2)$ **and** $r \in (-1,\ 1)$

$\Leftrightarrow \quad r_{\mathrm{QM}} = \sqrt{8\left(2p_{\mathrm{o}}^2+\sigma_{\mathrm{o}}^2\right)R_{\mathrm{o}}\left|f\left(p\right)\right|/\hbar^2} \ll 1$ (**weak correlation** $\Leftrightarrow$ **weak scattering**).

We can model our QM systems by Gaussian statistical systems via $P_{\mathrm{QM}}^{\mathrm{before}} = P_{(0)}$ and $P_{\mathrm{QM}}^{\mathrm{after}} = P_{(r)}$ with $r_{\mathbf{QM}} = r \ll 1$ (**weak scattering** $\Leftrightarrow$ **weak correlation**) and $\sigma_1 = \sigma_2 = \sigma$.

Then out of

$$P_{(0)}\left(p_1,\, p_2|\mu_1,\, \mu_2,\, \sigma\right) = \frac{1}{2\pi\sigma^2}\exp\left[-\frac{\left(p_1-\mu_1\right)^2}{2\sigma^2}-\frac{\left(p_2-\mu_2\right)^2}{2\sigma^2}\right],$$

$$P_{(r)}\left(p_1,\, p_2|\mu_1,\, \mu_2,\, \sigma\right) = \frac{\exp\left\{-\frac{1}{2\left(1-r^2\right)}\left[\frac{\left(p_1-\mu_1\right)^2}{\sigma^2}-\frac{2r\left(p_1-\mu_1\right)\left(p_2-\mu_2\right)}{\sigma^2}+\frac{\left(p_2-\mu_2\right)^2}{\sigma^2}\right]\right\}}{2\pi\sigma^2\sqrt{1-r^2}}$$

we can construct

$$g_{\mu\nu}\left(\Theta\right) = \int dX\, P\left(X|\Theta\right)\partial_\mu \ln P\left(X|\Theta\right)\partial_\nu \ln P\left(X|\Theta\right);\ \partial_\mu = \frac{\partial}{\partial\Theta^\mu},$$

the ***Fisher-Rao metric*** **associated with** $P_{(0)}$ **and** $P_{(r)}$**:**

$$g_{\mu\nu}\left(\mu_1,\, \mu_2,\, \sigma;0\right) = \frac{1}{\sigma^2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix},\quad g_{\mu\nu}\left(\mu_1,\, \mu_2,\, \sigma;\, r\right) = \frac{1}{\sigma^2}\begin{pmatrix} \frac{1}{1-r^2} & -\frac{r}{1-r^2} & 0 \\ -\frac{r}{1-r^2} & \frac{1}{1-r^2} & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

The geodesic equations for $\Theta^{\mu}=(\mu_1,\,\mu_2,\,\sigma)$ on $\mathcal{M}^{\text{3D}}_{\textbf{corr.}}$ with $g_{\mu\nu}(\mu_1,\,\mu_2,\,\sigma;\,r)$ read

$$\frac{d^2\Theta^{\mu}}{d\tau^2}+\Gamma^{\mu}_{\nu\rho}\frac{d\Theta^{\nu}}{d\tau}\frac{d\Theta^{\rho}}{d\tau}=0$$

$$\begin{aligned}
\Leftrightarrow\quad 0 &= \frac{d^2\mu_1(\tau)}{d\tau^2}-\frac{2}{\sigma(\tau)}\frac{d\mu_1(\tau)}{d\tau}\frac{d\sigma(\tau)}{d\tau},\\
0 &= \frac{d^2\mu_2(\tau)}{d\tau^2}-\frac{2}{\sigma(\tau)}\frac{d\mu_2(\tau)}{d\tau}\frac{d\sigma(\tau)}{d\tau},\\
0 &= \frac{d^2\sigma(\tau)}{d\tau^2}-\frac{1}{\sigma(\tau)}\left(\frac{d\sigma(\tau)}{d\tau}\right)^2+\frac{1}{4\sigma(\tau)(1-r^2)}\left[\left(\frac{d\mu_1(\tau)}{d\tau}\right)^2+\left(\frac{d\mu_2(\tau)}{d\tau}\right)^2\right]+\\
&\quad -\frac{r}{2\sigma(\tau)(1-r^2)}\frac{d\mu_1(\tau)}{d\tau}\frac{d\mu_2(\tau)}{d\tau}.
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow\quad \mu_1' &= C_1\sigma^2,\\
0 &= \mu_1'+\frac{C_1}{4(r^2-1)}\left[\frac{C_2}{C_1}\left(2r-\frac{C_2}{C_1}\right)-1\right]\mu_1^2+2D_1\mu_1+E_1\,;\ (1\leftrightarrow 2).
\end{aligned}$$

($\leftarrow$ **Riccati equations**)

Two sets of solutions are joined at the junction, $\tau = 0$ (at the instant of collision):

(i) Uncorrelated Gaussian system; $\tau < 0$ (before collision)

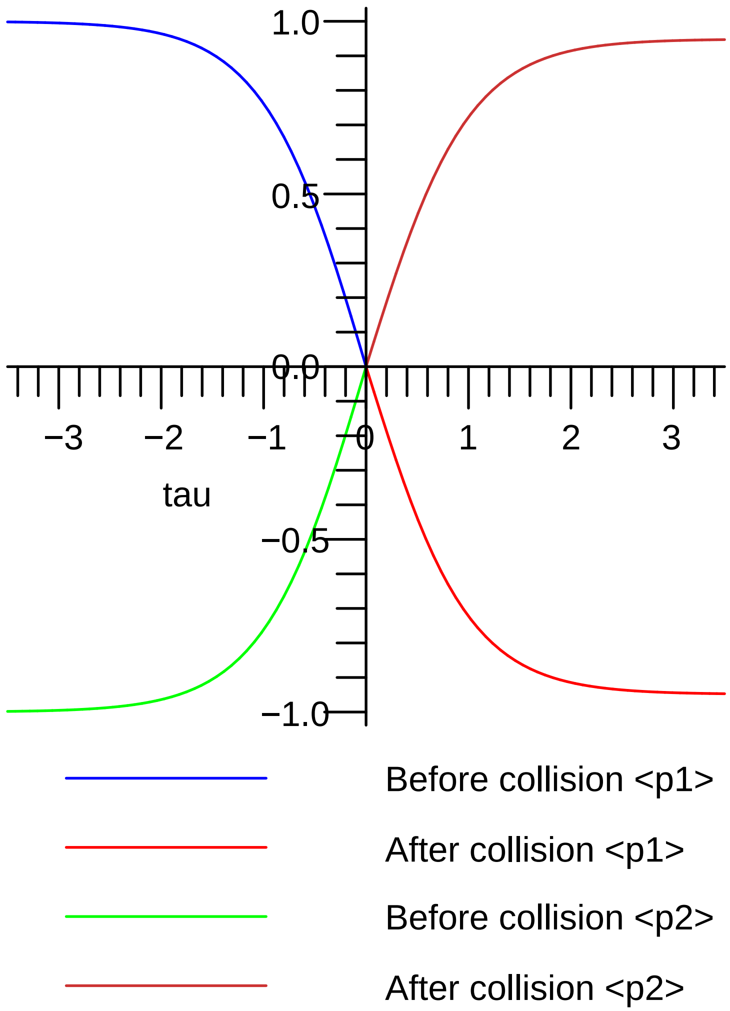
$$\begin{aligned}
\langle p_{1\,\mathrm{before}}(\tau)\rangle &= \mu_1(\tau;0) = -\sqrt{p_{\mathrm{o}}^2 + 2\sigma_{\mathrm{o}}^2}\,\tanh(A_{\mathrm{o}}\tau)\,,\\
\langle p_{2\,\mathrm{before}}(\tau)\rangle &= \mu_2(\tau;0) = \sqrt{p_{\mathrm{o}}^2 + 2\sigma_{\mathrm{o}}^2}\,\tanh(A_{\mathrm{o}}\tau)\,,\\
\langle \sigma_{\mathrm{before}}(\tau)\rangle &= \sigma(\tau;0) = \frac{1}{\sqrt{2}}\sqrt{p_{\mathrm{o}}^2 + 2\sigma_{\mathrm{o}}^2}\,\mathrm{sech}(A_{\mathrm{o}}\tau)\,,
\end{aligned}$$

(ii) Correlated Gaussian system; $\tau \geq 0$ (after collision)

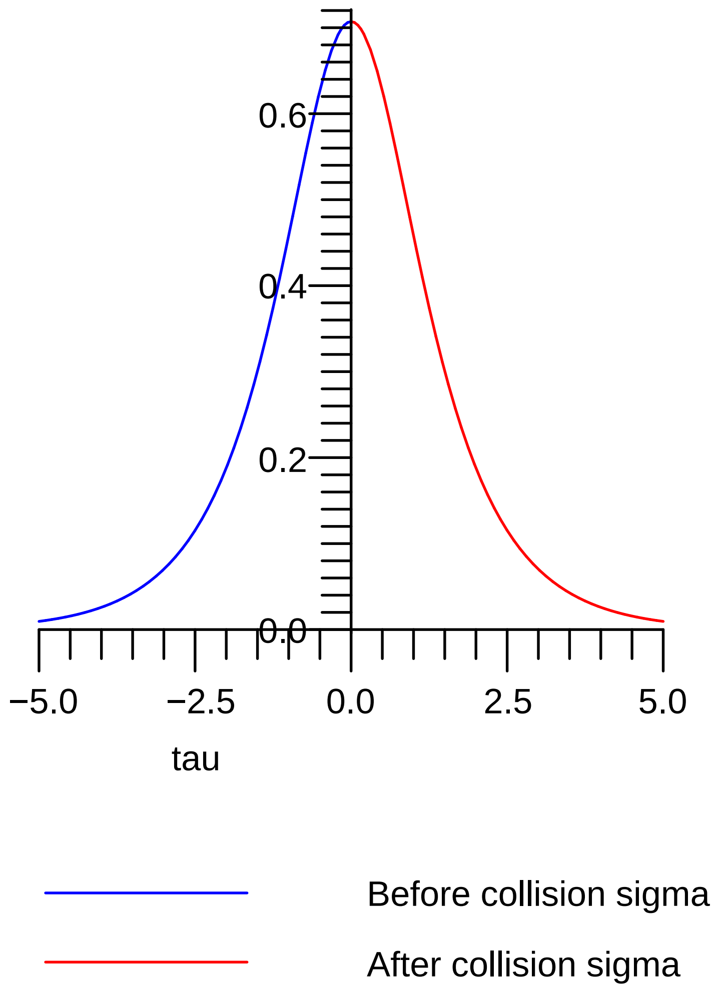
$$\begin{aligned}
\langle p_{1\,\mathrm{after}}(\tau)\rangle &= \mu_1(\tau;r) = -\sqrt{(1-r)\left(p_{\mathrm{o}}^2 + 2\sigma_{\mathrm{o}}^2\right)}\,\tanh(A_{\mathrm{o}}\tau)\,,\\
\langle p_{2\,\mathrm{after}}(\tau)\rangle &= \mu_2(\tau;r) = \sqrt{(1-r)\left(p_{\mathrm{o}}^2 + 2\sigma_{\mathrm{o}}^2\right)}\,\tanh(A_{\mathrm{o}}\tau)\,,\\
\langle \sigma_{\mathrm{after}}(\tau)\rangle &= \sigma(\tau;r) = \frac{1}{\sqrt{2}}\sqrt{p_{\mathrm{o}}^2 + 2\sigma_{\mathrm{o}}^2}\,\mathrm{sech}(A_{\mathrm{o}}\tau)\,,
\end{aligned}$$

where $p_{\mathrm{o}} \equiv \langle p_{1\,\mathrm{before}}(-\tau_{\mathrm{o}})\rangle$, $\sigma_{\mathrm{o}} \equiv \langle \sigma_{\mathrm{before}}(-\tau_{\mathrm{o}})\rangle$ **and**

$$\begin{aligned}
A_{\mathrm{o}} &= \frac{1}{\tau_{\mathrm{o}}}\sinh^{-1}\left(\frac{p_{\mathrm{o}}}{\sqrt{2}\sigma_{\mathrm{o}}}\right)\\
&\overset{\frac{\sigma_{\mathrm{o}}}{p_{\mathrm{o}}}\ll 1}{=} \frac{1}{\tau_{\mathrm{o}}}\left\{\ln\left(\frac{\sqrt{2}p_{\mathrm{o}}}{\sigma_{\mathrm{o}}}\right) + \frac{1}{2}\left(\frac{\sigma_{\mathrm{o}}}{p_{\mathrm{o}}}\right)^2 - \frac{3}{8}\left(\frac{\sigma_{\mathrm{o}}}{p_{\mathrm{o}}}\right)^4 + \mathcal{O}\left[\left(\frac{\sigma_{\mathrm{o}}}{p_{\mathrm{o}}}\right)^6\right]\right\}.
\end{aligned}$$



Plots of $\langle p_1(\tau) \rangle$ and $\langle p_2(\tau) \rangle$



Plot of $\sigma(\tau)$

4. Application of Information Geometry to Quantum Entanglement

Momentum curves attenuate after collision due to the correlation:

$$\sqrt{p_{\mathrm{o}}^2 + 2\sigma_{\mathrm{o}}^2}\tanh\left(A_{\mathrm{o}}\tau\right) \quad (\tau < 0) \quad \rightarrow \quad \sqrt{(1-r)\left(p_{\mathrm{o}}^2 + 2\sigma_{\mathrm{o}}^2\right)}\tanh\left(A_{\mathrm{o}}\tau\right) \quad (\tau \geq 0).$$

That is, the correlation renders $p_{\mathrm{o}} \rightarrow \sqrt{1-r}p_{\mathrm{o}}$.

Draw a connection between the correlation and s-wave scattering potential such that

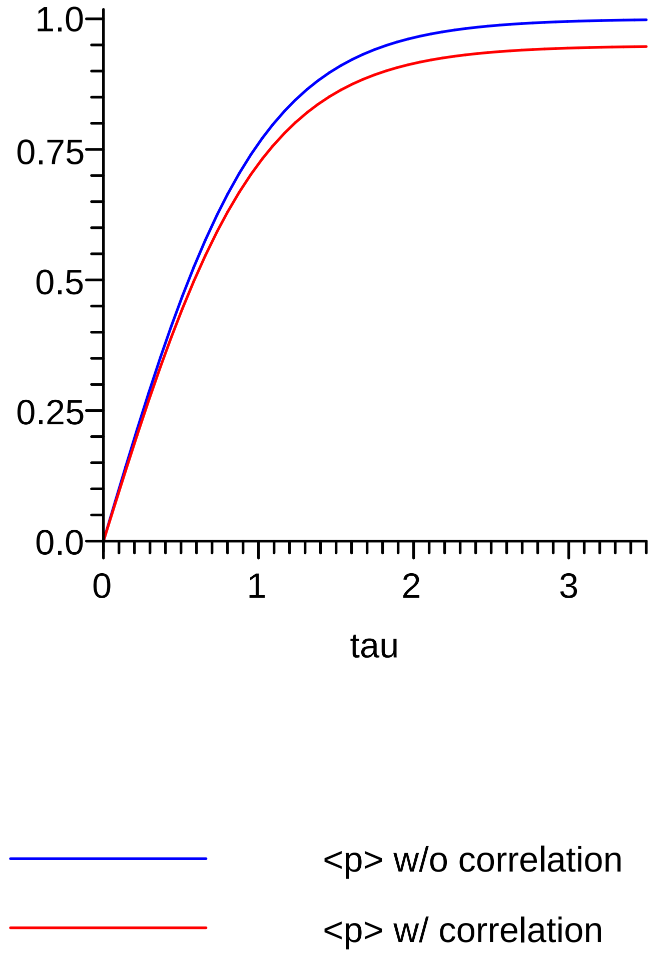
$$k_r \cot\left(k_r L\right) = k_{\mathrm{o}} \cot\left(k_{\mathrm{o}} L + \theta\right),$$

where

$$k_r \equiv \frac{\sqrt{1-r}p_{\mathrm{o}}}{\hbar} = \frac{\sqrt{2\mu\left(\mathcal{E}-\mathcal{V}\right)}}{\hbar}, \ 0 < x < L,$$

$$k_{\mathrm{o}} \equiv \frac{p_{\mathrm{o}}}{\hbar} = \frac{\sqrt{2\mu\mathcal{E}}}{\hbar}, \ x > L,$$

$\mathcal{E} = p_{\mathrm{o}}^2/\left(2\mu\right)$, $\mathcal{V}$: potential height, L: potential range, θ: scattering phase shift.



Plot of $\langle p(\tau)\rangle$ with and without correlation

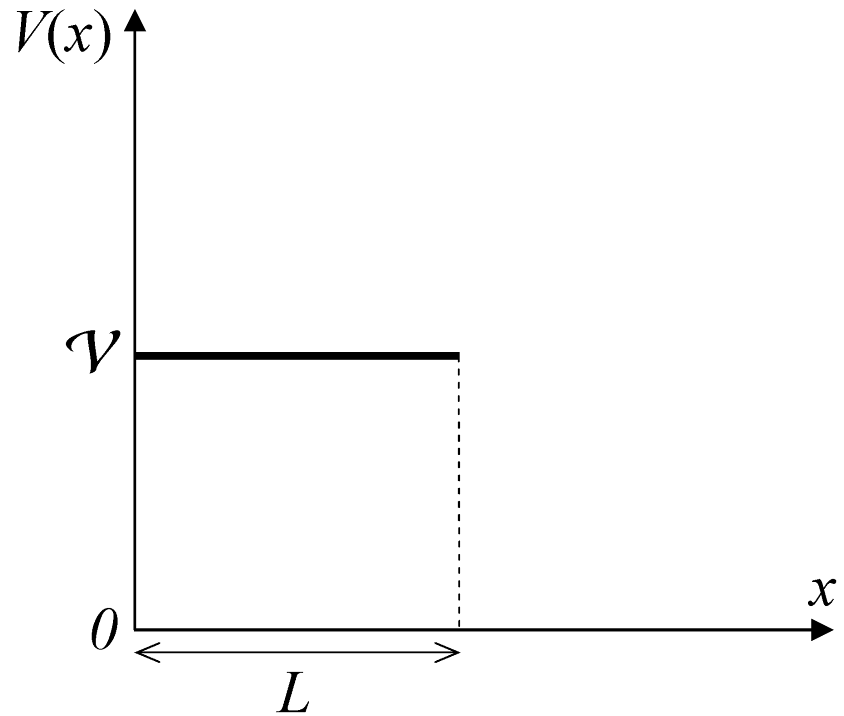


Illustration of scattering potential

Then we obtain

- **scattering potential height:** $\mathcal{V} = r\frac{p_\text{o}^2}{2\mu}$,
- **scattering phase shift:** $\theta \approx r\frac{p_\text{o}^3 L^3}{3\hbar^3}$,
- **scattering cross-section:** $\Sigma = 4\pi |f|^2 \approx r^2 \frac{4\pi p_\text{o}^4 L^6}{9\hbar^4}$,
- **purity (a measure of entanglement):**
$$\begin{aligned}\mathcal{P} &= \text{Tr}\left(\left[\text{Tr}_2\left(\rho_{12}\right)\right]^2\right)\\ &= \iiiint \psi\left(p_1, p_2, t\right)\psi\left(p_3, p_4, t\right)\psi^*\left(p_1, p_4, t\right)\psi^*\left(p_3, p_2, t\right) dp_1 dp_2 dp_3 dp_4\\ &= 1 - \frac{1}{2}r^2 + \mathcal{O}\left(r^4\right)\end{aligned}$$

with

$$r = r_{\mathbf{QM}} = \sqrt{8\left(2p_\text{o}^2 + \sigma_\text{o}^2\right) R_\text{o} \left|f\left(p\right)\right| / \hbar^2} \ll 1.$$

7. Conclusions and Discussion

- Information about low energy quantum scattering and entanglement is encoded in the statistical correlation. Information geometry provides a useful tool to analyze the correlation.

- Quantum entanglement can be interpreted as a perturbation in statistical momentum space geometry, which is analogous to linearized gravity. Information geometry utilizes this analogy to provide an interpretation of our quantum-entangled system, which shows good agreement with a well-known QM analysis.

- Our entanglement model shows remarkable correspondences with the gravity-analogs based on AdS/CFT by Ryu and Takayanagi, Van Raamsdonk, etc.:
  (i) correlation of degrees of freedom: the exponentially decaying pattern suggests a quantitative connection between entanglement measures and the structure of the dual spacetime.
  (ii) entanglement entropy: the information content of a region depends on its surface area rather than on its volume - holographic principle.
  Some other issues have yet to be investigated: modular Hamiltonian, geometrical structures, etc.