Conductivity in an anisotropic medium

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Workshop, Hanyang University, August 17^{th}, 2016

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Outline

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J. i. Koga, K. Maeda and K. Tomoda, Phys. Rev. D **89**, no. 10, 104024 (2014) doi:10.1103/PhysRevD.89.104024 [arXiv:1401.6501 [hep-th]].

S. A. Hartnoll, C. P. Herzog and G. T. Horowitz, JHEP **0812**, 015 (2008) doi:10.1088/1126-6708/2008/12/015 [arXiv:0810.1563 [hep-th]].

K. Y. Kim, K. K. Kim, Y. Seo and S. J. Sin, JHEP **1507**, 027 (2015) doi:10.1007/JHEP07(2015)027 [arXiv:1502.05386 [hep-th]].

D. T. Son and A. O. Starinets, JHEP **0209**, 042 (2002) doi:10.1088/1126-6708/2002/09/042 [hep-th/0205051].

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Translation Invariant

- AdS/CFT
  → Transport Coefficients
  → Translation Invariant (Infinite DC conductivity)
- Break Translation Invariant
  → IBC (Inhomogeneous at boundary field)→ (PDE)
  → HBC (Ex. Massive gravity)→ (ODE)
- EMDA Model
  → Axion Fields are linear in spatial direction that turn on Anisotropic background solution and give HBC
  → Effect of anisotropic background on conductivities.
  → $SL(2,R)$ transformation $\Longrightarrow$ momentum dissipation produced by massive gravity

We would like to consider the following Action

Action

$$S = \int d^4x\sqrt{-g}\left(R + \frac{6}{L^2} - 2(\nabla\phi)^2 - \frac{1}{2}e^{4\phi}\sum_{i=1}^{2}(\nabla\tilde{a}_i)^2 - e^{-2\phi}F^2\right)$$

Metric

$$ds^2 = \frac{L^2}{z^2}\left(-g(z)dt^2 + g(z)^{-1}dz^2 + e^{A(z)+B(z)}dx^2 + e^{A(z)-B(z)}dy^2\right)$$

Einstein and Field Equations

$$R_{\mu\nu} = -\frac{3}{L^2} g_{\mu\nu} + 2\nabla_\mu \phi \nabla_\nu \phi + \frac{1}{2} e^{4\phi} \nabla_\mu \tilde{a} \nabla_\nu \tilde{a} + 2 e^{-2\phi} F_{\mu\rho} F_\nu{}^\rho - \frac{1}{2} g_{\mu\nu} e^{-2\phi} F^2 ,$$

$$\Box \phi - \frac{1}{2} e^{4\phi} \sum_{i=1}^{2} (\nabla \tilde{a}_i)^2 + \frac{1}{2} e^{-2\phi} F^2 = 0 ,$$

$$\Box \tilde{a}_1 + 4 \nabla_\mu \phi \nabla^\mu \tilde{a}_1 = 0 ,$$

$$\Box \tilde{a}_2 + 4 \nabla_\mu \phi \nabla^\mu \tilde{a}_2 = 0 ,$$

$$\nabla_\mu (e^{-2\phi} F^{\mu\nu}) = 0 ,$$

where we choose ansatz

$$\phi = \phi(z), \quad \tilde{a}_1 = \alpha_1 x, \quad \tilde{a}_2 = \alpha_2 y, \quad A_\mu dx^\mu = A_t dt ,$$

and with a solution

$$F_{zt} = A_t' = \rho_z L e^{-A+2\phi} .$$

Background EOM

$$0 = 2A'' + (A')^2 + (B')^2 + 4(\phi')^2 ,$$

$$0 = gzB'' + (g(zA' - 2) + zg')B' + \frac{1}{2}ze^{-A-B+4\phi}\left(\alpha_1^2 - \alpha_2^2 e^{2B}\right) ,$$

$$0 = (4z - 2z^2A')g' + \left(-z^2(A')^2 + 8zA' + z^2(B')^2 + 4z^2(\phi')^2 - 12\right)g$$
$$- \alpha_1^2 z^2 e^{-A-B+4\phi} - \alpha_2^2 z^2 e^{-A+B+4\phi} - 4\rho_z^2 z^4 e^{2\phi-2A} + 12 ,$$

$$0 = e^A g\phi'' + \left(e^A gA' + e^A g' - \frac{2e^A g}{z}\right)\phi' - \rho_z^2 z^2 e^{2\phi-A}$$
$$- \frac{1}{2}\alpha_1^2 e^{4\phi-B} - \frac{1}{2}\alpha_2^2 e^{B+4\phi} .$$

Scaling

$$\phi \to \phi - \phi_0 \,, \qquad \alpha_1 \to e^{2\phi_0}\alpha_1 \,, \qquad \alpha_2 \to e^{2\phi_0}\alpha_2 \,, \qquad \rho_z \to e^{\phi_0}\rho_z \,,$$

where ϕ_0 is arbitrary value, and the scaling at AdS boundary is

$$x \to e^{-(A(0)+B(0))/2}x \,, \qquad y \to e^{-(A(0)-B(0))/2}y \,,$$
$$\alpha_1 \to \alpha_1 e^{-(A(0)+B(0))/2} \,, \qquad \alpha_2 \to \alpha_2 e^{-(A(0)-B(0))/2} \,.$$

Horizon Regularity

Before scaling, we first set

$$A(1) = B(1) = \phi(1) = 0\,.$$

and the fact that

$$g(1) = 0\,.$$

Regularity of Black Brane Solution becomes

$$A'(1) = -\frac{12 - 4\rho_z^2 - \alpha_1^2 - \alpha_2^2 - 4\kappa}{2\kappa}\,, \quad B'(1) = \frac{\alpha_1^2 - \alpha_2^2}{2\kappa}\,,$$
$$\phi'(1) = \frac{-2\rho_z^2 - \alpha_1^2 - \alpha_2^2}{2\kappa}$$

where we denote $\kappa = -g'(1)$. For a given κ, ρ_z, α_1, and α_2, we obtain $A'(0) = 0$; then $\phi(0) = A(0) = B(0) = 0$ after scaling.

Initial Data as a function of κ, ($\alpha_1 = 2$, $\alpha_2 = 0$)

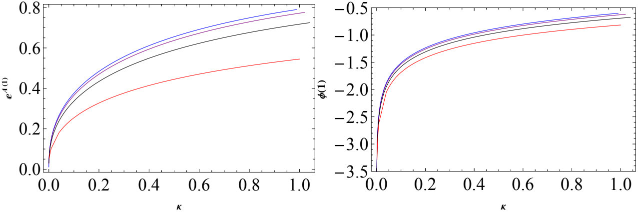


Figure: Fix $\alpha_1 = 2$, $\alpha_2 = 0$ (Blue), $\alpha_2 = 1$ (Purple), $\alpha_2 = 2$ (Black), $\alpha_2 = 4$ (Red)

Initial Data as a function of κ, ($\alpha_1 = 2$, $\alpha_2 = 0$)

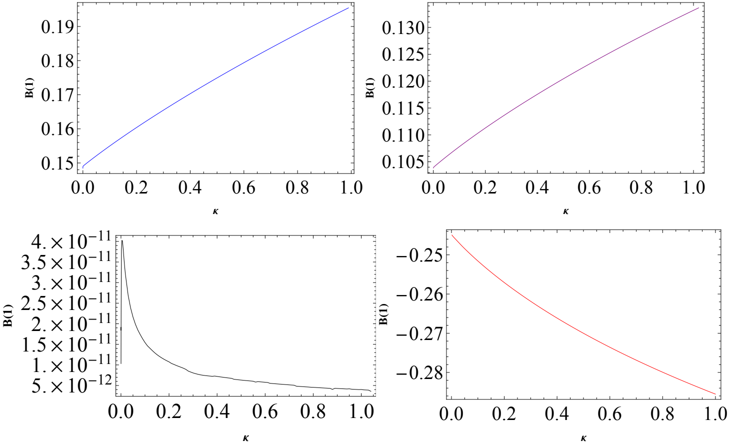


Figure: Fix $\alpha_1 = 2$, $\alpha_2 = 0$ (Blue), $\alpha_2 = 1$ (Purple), $\alpha_2 = 2$ (Black), $\alpha_2 = 4$ (Red)

Initial Data as a function of α_2, ($\alpha_1 = 2$, $\kappa = 0.5$)

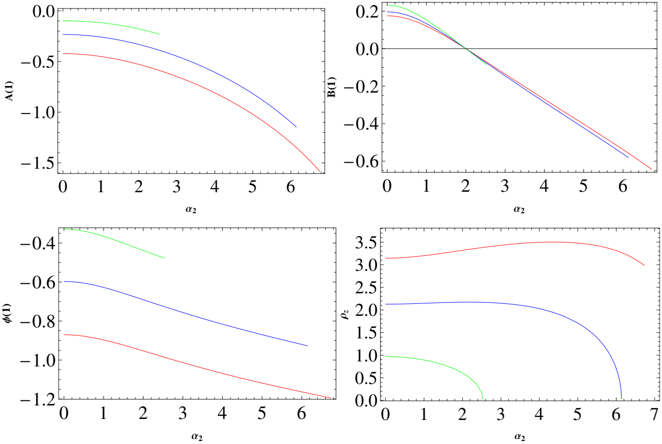


Figure: Fix $\alpha_1 = 2$, $\kappa = 0.5$ (Red), $\kappa = 1$ (Blue), and $\kappa = 2$ (Green)

Evolution of Background Fields

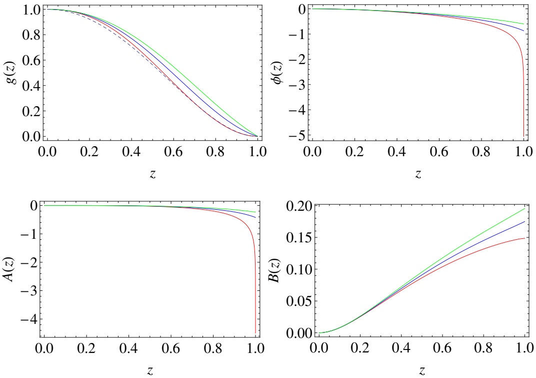


Figure: Fix $\alpha_1 = 2$ and $\alpha_2 = 2$, $\kappa = 1.39 \times 10^{-5}$ (Red), $\kappa = 0.5$ (Blue), and $\kappa = 1$ (Green)

Notice that the dashed line in $g(z)$ is RN-AdS solution without dilaton field.

Perturbation

$$A_\mu \to A_t(z)(dt) + \left[\tilde{A}_x(t,z)(dx) + \tilde{A}_y(t,z)(dy)\right]$$
$$\tilde{a}_j \to \alpha_j j + i\Omega e^{-i\Omega t}\chi_j(z)\,,$$

where $j = (x, y)$. Metric fluctuation given by

$$g_{\mu\nu} \to g_{\mu\nu} + 2\frac{L^2}{z^2}\left[\tilde{g}_{tx}(t,z)(dtdx) + \tilde{g}_{ty}(t,z)(dtdy)\right]\,.$$

Fluctuation fields are given in terms of frequency

$$\tilde{A}_j(t,z) = \int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}e^{-i\Omega t}A_j(z), \qquad \tilde{g}_{tj}(t,z) = \int_{-\infty}^{\infty}\frac{d\Omega}{2\pi}e^{-i\Omega t}g_{tj}(z),$$

where $j = (x, y)$.

Perturbation Equations

$$A_i'' + \left(\bar{B}_i{}' + \frac{g'}{g} - 2\phi'\right) A_i' + \left(\frac{\Omega^2}{g^2} - \frac{4z^2 e^{2\phi-2A}\rho_z^2}{g}\right) A_i - \alpha_i L e^{6\phi-A}\rho_z \chi_i' = 0\,,$$

$$\chi_i'' + \left(A' + \frac{g'}{g} - \frac{2}{z} + 4\phi'\right)\chi_i' + \frac{\Omega^2}{g^2}\chi_i - \frac{\alpha_i e^{\bar{B}_i - A}}{g^2} g_{\text{ti}} = 0\,,$$

$$g_{ti}' + \left(\bar{B}_i{}' - A'\right) g_{ti} - \frac{4e^{-A}z^2\rho_z}{L} A_i - \alpha_i g e^{4\phi}\chi_i' = 0\,,$$

where $i = (x, y)$, $\bar{B}_i = (-B, B)\,.$

Numerical Scheme

From prime to head

$$\hat{A}_i(z) \equiv g(z) A_i'(z)\,, \quad \hat{\chi}_i(z) \equiv g(z)\chi_i'(z)\,,$$

New variable

$$A_i(z) = (1-z)^{\lambda} a_i(z)\,, \quad \hat{A}_i(z) = (1-z)^{\lambda} \hat{a}_i(z)\,, \quad g_{ti}(z) = (1-z)^{\lambda}\zeta_{ti}(z)$$
$$\chi_i(z) = (1-z)^{\lambda}\eta_i(z)\,, \quad \hat{\chi}_i(z) = (1-z)^{\lambda}\hat{\eta}_i(z)\,,$$

First-order Differential Equations

$$\hat{a}_i' + \left(\bar{B}_i{}' - \frac{\lambda}{1-z} - 2\phi'\right)\hat{a}_i + \left(\frac{\Omega^2}{g} - 4z^2 e^{2\phi-2A}\rho_z^2\right)a_i - \alpha_i\rho_z L e^{6\phi-A}\hat{\eta}_i = 0\,,$$

$$a_i' - \frac{\lambda}{1-z}a_i - \frac{\hat{a}_i}{g} = 0\,,$$

$$\hat{\eta}_i' + \left(A' - \frac{\lambda}{1-z} - \frac{2}{z} + 4\phi'\right)\hat{\eta}_i + \frac{\Omega^2}{g}\eta_i - \frac{\alpha_i e^{-A+\bar{B}_i}\zeta_{ti}}{g} = 0\,,$$

$$\eta_i' - \frac{\lambda}{1-z}\eta_i - \frac{\hat{\eta}_i}{g} = 0\,,$$

$$\zeta_{ti}' + \left(-A' + \bar{B}_i{}' - \frac{\lambda}{1-z}\right)\zeta_{ti} - \alpha_i e^{4\phi}\hat{\eta}_i - \frac{4\rho_z e^{-A}z^2 a_i}{L} = 0\,,$$

Eigenvalue equations at horizon

$$\begin{pmatrix} 0 & -\frac{\Omega^2}{g'(1)} & 0 & 0 & 0 \\ \frac{1}{g'(1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\Omega^2}{g'(1)} & \frac{\alpha_i e^{-A(1)-\bar{B}_i(1)}}{g'(1)} \\ 0 & 0 & \frac{1}{g'(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{a}_i \\ a_i \\ \hat{\eta}_i \\ \eta_i \\ \zeta_{ti} \end{pmatrix} = \lambda \begin{pmatrix} \hat{a}_i \\ a_i \\ \hat{\eta}_i \\ \eta_i \\ \zeta_{ti} \end{pmatrix}.$$

Eigenvectors corresponding to $\lambda = 0, \lambda = (i\Omega)/(g'(1))$ (Degeneracy)

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\alpha_i e^{-A(1)+\bar{B}_i(1)}}{\Omega^2} \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ i\Omega \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} i\Omega \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Expansion of boundary fields

$$\Phi = \sum_{n=0}^{\infty} \Phi^{(n)} z^n \,.$$

Expanding Solutions of Background fields

$$A^{(0)} = B^{(0)} = \phi^{(0)} = 0, \qquad g^{(0)} = 1, \qquad B^{(1)} = \phi^{(1)} = A^{(1)} = g^{(1)} = 0,$$
$$A^{(2)} = A^{(3)} = 0, \qquad \phi^{(2)} = g^{(2)} = -\frac{1}{4}(\alpha + \alpha_2), \qquad B^{(2)} = \frac{1}{4}(\alpha_1^2 - \alpha_2^2),$$
$$A^{(4)} = -\frac{1}{96}(5\alpha_1^4 + 6\alpha_1^2\alpha_2^2 + 5\alpha_1^4), \qquad \phi^{(4)} = -\frac{1}{16}(3\alpha_1^4 + 4\alpha_1^2\alpha_2^2 + 3\alpha_2^4 - 4\rho_z^2),$$
$$g^{(4)} = \frac{1}{24}(24\rho_z^2 - 5\alpha_1^4 - 6\alpha_1^2\alpha_2^2 - 5\alpha_2^4)\,.$$

Expanding Solutions of fluctuation fields

$$\chi_i^{(1)} = g_{ti}^{(1)} = 0\,, \quad \chi_i^{(2)} = \frac{1}{2}\left(\chi_i^{(0)}\Omega^2 - \alpha_i g_{ti}^{(0)}\right)\,, A_i^{(2)} = \frac{-\Omega A_i^{(0)}}{2}\,,$$

$$g_{ti}^{(2)} = \frac{1}{4}\left(2\alpha_i\chi_i^{(0)}\Omega^2 - \alpha_1^2 g_{ti}^{(0)} - \alpha_2^2 g_{ti}^{(0)}\right)\,,$$

$$A_i^{(3)} = \frac{1}{6}\left(-\alpha_j^2 A_i^{(1)} - \Omega^2 A_i^{(1)} - \alpha_i^2 L\rho_z g_{ti}^{(0)} + \alpha_i L\Omega^2\rho_z\chi_i^{(0)}\right)\,,$$

$$g_{ti}^{(3)} = \frac{4A_i^{(0)}\rho_z + 3\alpha_i L\chi_i^{(3)}}{3L}\,,$$

$$A_i^{(4)} = \frac{1}{24}\Big((-1)^i(\alpha_1^2 - \alpha_2^2)A_i^{(0)}\Omega^2 + 8A_i^{(0)}\rho_z^2 - 6A_i^{(1)}g^{(3)} + A_i^{(0)}\Omega^4 + 6\alpha_i\rho_z L\chi_i^{(3)}\Big)$$

Quadratic On-shell Action

$$S_0^{(2)} = \int d^3x \Bigg[g_{ti} \left(-4\rho_z L e^{-A+\bar{B}_i} A_i + \frac{2e^{\bar{B}_i} L^2 g'_{\rm ti}}{z^2} - \frac{\alpha_i g L^2 e^{4\phi+\bar{B}_i}\chi'_i}{2z^2} \right) + g_{\rm ti}^2 \left(-\frac{e^{\bar{B}_i} L^2 A'}{z^2} + \frac{e^{\bar{B}_i} L^2 \bar{B}'_i}{z^2} - \frac{e^{\bar{B}'_i} L^2 g'}{2gz^2} - \frac{e^{\bar{B}_i} L^2}{z^3} \right) - 2e^{\bar{B}_i - 2\phi} g A_i A'_i - \frac{gL^2 e^{A+4\phi}\dot{\chi}_i \chi'_i}{2z^2} \Bigg] \Bigg|_{z\to 0}$$

Counter term

$$S_{ct} = \int d^3x \sqrt{-\gamma} \left[-\frac{4}{L} + \frac{L}{2}\left(1 + \frac{I}{2}\gamma^{tl}\delta\gamma_{tl}\right) \sum_{i=1}^{2} \gamma^{mn} \partial_m a_i \partial_n a_i \right] \Bigg|_{z\to 0}$$

,

where $l, m, n = (t, x, y)$ and $\delta\gamma_{tl} = e^{-i\Omega t} g_{tl}(z)$. We make $I = -1$ when $l \neq m$ or n, otherwise, $I = 1$.

Renormalized Action

$$
\begin{aligned}
S^{(2)}_{re} &= \lim_{z\to 0}\left(S^{(2)}_0 + S^{(2)}_{GH} + S^{(2)}_{ct}\right)\\
&= 2V_2\int_0^\infty \frac{d\Omega}{2\pi}\left(\bar{A}^{(0)}_i A^{(1)}_i + \frac{1}{2}L^2(\epsilon_{ij}B^{(3)} + g^{(3)})\bar{g}^{(0)}_{ti} g^{(0)}_{ti} - 2L\rho_z\bar{A}^{(0)}_i g^{(0)}_{ti}\right.\\
&\qquad\left. + \frac{3}{4}L^2\Omega^2\bar{\chi}^{(0)}_i\chi^{(3)}_i - \frac{3}{4}L^2\alpha_i\bar{g}^{(0)}_{ti}\chi^{(3)}_i\right).
\end{aligned}
$$

where $\epsilon_{xy} = -\epsilon_{yx} = 1$

Electric and heat currents ($g^{(0)}_{ti} = \chi^{(0)}_i = 0$)

$$
J_i = A^{(1)}_i\,, \qquad Q_i = T_{ti} - \mu J_i = -L\rho_z A^{(0)}_i - \mu A^{(1)}_i\,.
$$

Electric and thermal transport

$$\begin{pmatrix} J_i \\ Q_i \end{pmatrix} = \begin{pmatrix} \sigma & \bar{\alpha}T \\ \bar{\alpha}T & \bar{\kappa}T \end{pmatrix} \begin{pmatrix} E_i \\ -(\nabla_i T)/T \end{pmatrix} .$$

Electric and thermal transport

$$\sigma_j = \frac{-iA_j^{(1)}}{\Omega A_j^{(0)}} , \qquad T\bar{\alpha} = \frac{iL\rho_z}{\Omega} - \mu\sigma_j$$

Thermal conductivity

$$Q_j = \frac{1}{2}L^2(\epsilon_{ji}B^{(3)} + g^{(3)})g_{tj}^{(0)} , \quad g_{tj}^{(0)} = -\frac{\nabla_j T}{i\Omega T} , \quad \bar{\kappa} = \frac{-2iL^2(\epsilon_{ji}B^{(3)} + g^{(3)})}{\Omega T} ,$$

$\mathrm{Re}[\sigma_{ii}(\bar{\omega})]$

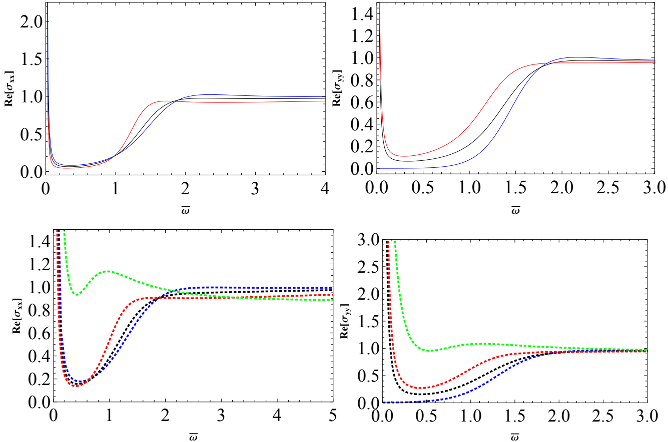
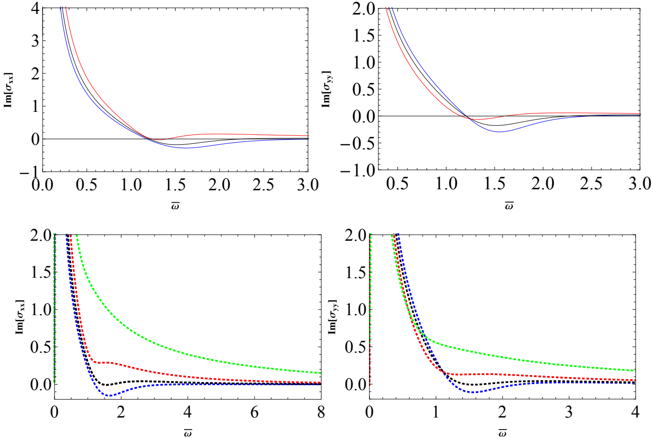


Figure: Fix $\alpha_1 = 2$ with $\kappa = 0.5$ (Line), $\kappa = 1$ (Dotted) and $\alpha_2 = 0$ (Blue), $\alpha_2 = 2$ (Black), $\alpha_2 = 4$ (Red), and $\alpha_2 = 6$ (Green)

$\mathrm{Im}[\sigma_{ii}(\bar{\omega})]$



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$\mathrm{Re}[\sigma_{DC\,ii}(\beta)]$

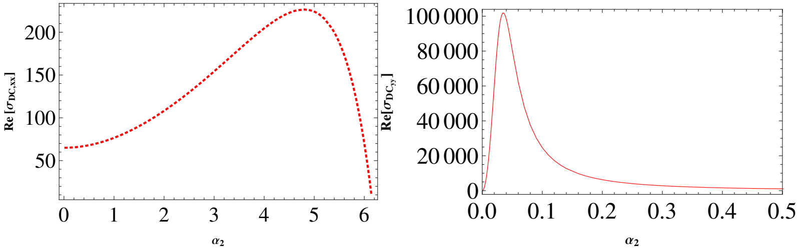
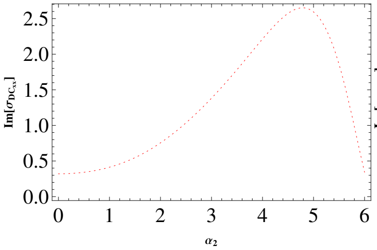


Figure: Fix $\alpha_1 = 2$ with $\kappa = 1$

$\text{Im}[\sigma_{DC\,ii}(\beta)]$



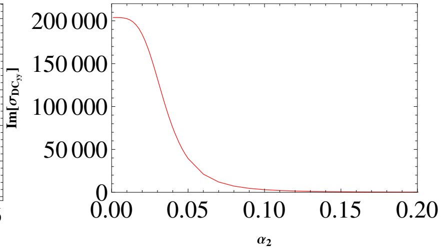


Figure: Fix $\alpha_1 = 2$ with $\kappa = 1$

Source and operator

$$\Phi_i^a(k,z) \to \mathbb{S}_i^a + \cdots + \mathbb{O}_i^a z^{\delta_a} + \cdots ,$$

where a is a field index and i is solution index corresponding to in-going boundary conditions. Also,

Superposition

$$\begin{aligned}
\Phi_i^a(k,z) &= \phi_i^a c^i = J^a + \cdots + \Pi^a z^{\delta_a} + \cdots \\
J^a &= \mathbb{S}_i^a c^i , \\
\Pi^a &= \mathbb{O}_i^a c^i = \mathbb{O}_i^a (\mathbb{S}^{-1})_b^i J^b
\end{aligned}$$

On-shell Action

$$S_{re}^{(2)} = 2V \int \frac{d\Omega}{2\pi} \left[\bar{J}^a \mathbb{A}_{ab}(\Omega) J^b + \bar{J}^a \mathbb{B}_{ab}(\Omega) \Pi^b \right] ,$$

where

$$J^a = \begin{pmatrix} A_x^{(0)} \\ A_y^{(0)} \\ g_{tx}^{(0)} \\ g_{ty}^{(0)} \\ \chi_x^{(0)} \\ \chi_y^{(0)} \end{pmatrix} , \quad \Pi^a = \begin{pmatrix} A_x^{(1)} \\ A_y^{(1)} \\ g_{tx}^{(3)} \\ g_{ty}^{(3)} \\ \chi_x^{(3)} \\ \chi_y^{(3)} \end{pmatrix} ,$$

$$\mathbb{A} = \begin{pmatrix} 0 & -L\rho_z & 0 \\ -L\rho_z & \frac{1}{4}L^2(g^{(3)} + \epsilon_{ij}B^{(3)}) & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \mathbb{I}_{2\times 2} ,$$

$$\mathbb{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\frac{3L^2\alpha_i}{4} \\ 0 & 0 & \frac{3L^2\Omega^2}{4} \end{pmatrix} \mathbb{I}_{2\times 2} ,$$

Retarded Green Function

$$G^R_{ab} = \mathbb{A}_{ab}(k) + \mathbb{B}_{ac}\mathbb{O}^c_i(\mathbb{S}^{-1})^i_b(k)$$

Response function and the sources

$$\begin{pmatrix} J_i \\ T_{ti} \end{pmatrix} = \begin{pmatrix} G^{ij}_{JJ} & G^{ij}_{JT} \\ G^{ij}_{TJ} & G^{ij}_{TT} \end{pmatrix} \begin{pmatrix} A^{(0)}_j \\ g^{(0)}_{tj} \end{pmatrix}$$

Conductivities is defined as

$$\begin{pmatrix} J_i \\ Q_i \end{pmatrix} = \begin{pmatrix} \sigma^{ij}_{JJ} & \tilde{\alpha}^{ij}T \\ \bar{\alpha}^{ij}T & \bar{\kappa}^{ij}T \end{pmatrix} \begin{pmatrix} E^{(0)}_j \\ -(\nabla_j T)/T \end{pmatrix}$$

where $Q_j = T_{tj} - \mu J_j, \quad E_j = i\Omega(a^{(0)}_j + \mu h^{(0)}_{tj}) \quad g^{(0)}_{tj} = -\frac{\nabla_j T}{i\Omega T}$.

Transport Coefficients

Electric, thermoelectric, and thermo conductivity can be written as

$$\begin{pmatrix} \sigma^{ij}_{JJ} & \tilde{\alpha}^{ij}T \\ \bar{\alpha}^{ij}T & \bar{\kappa}^{ij}T \end{pmatrix} = \begin{pmatrix} -\frac{iG^{ij}_{JJ}}{\Omega} & \frac{i(\mu G^{ij}_{JJ}-G^{ij}_{JT})}{\Omega} \\ \frac{i(\mu G^{ij}_{JJ}-G^{ij}_{TJ})}{\Omega} & -\frac{i(G^{ij}_{TT}-G^{ij}_{TT}(\Omega=0)-\mu(G^{ij}_{JT}+G^{ij}_{TJ}-\mu G^{ij}_{JJ}))}{\Omega} \end{pmatrix}$$

D. T. Son and A. O. Starinets, JHEP **0209**, 042 (2002) doi:10.1088/1126-6708/2002/09/042 [hep-th/0205051].

S. A. Hartnoll, Class. Quant. Grav. **26**, 224002 (2009) doi:10.1088/0264-9381/26/22/224002 [arXiv:0903.3246 [hep-th]].

K. Y. Kim, K. K. Kim, Y. Seo and S. J. Sin, JHEP **1507**, 027 (2015) doi:10.1007/JHEP07(2015)027 [arXiv:1502.05386 [hep-th]].

$\mathrm{Re}[\sigma_{ii}(\bar{\omega})]$

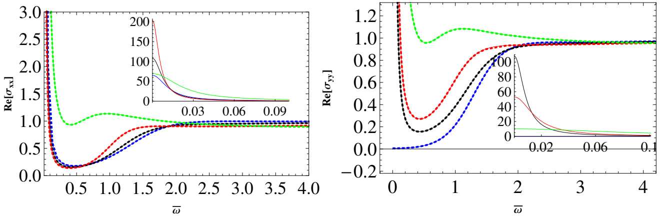


Figure: Fix $\alpha = 2$ with $\kappa = 1$, and $\beta = 0$ (Blue), $\beta = 2$ (Black), $\beta = 4$ (Red), and $\beta = 6$ (Green). Note: Dotted line ($g^{(0)}_{ti} = 0$), while solid line (Thermal Gradient is on)

Thermoelectric and Thermo Conductivity

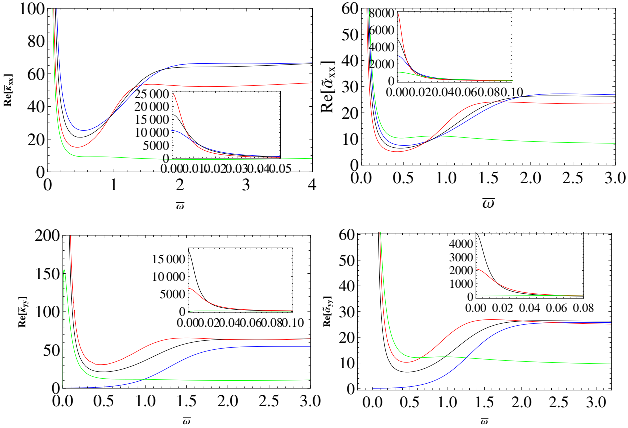


Figure: Fix $\alpha = 2$ with $\kappa = 1$, and $\beta = 0$ (Blue), $\beta = 2$ (Black), $\beta = 4$ (Red), and $\beta = 6$ (Green)

Drude-like Behavior

Drude Formula

$$\Gamma = \frac{k\tau}{1 - i\omega\tau},$$

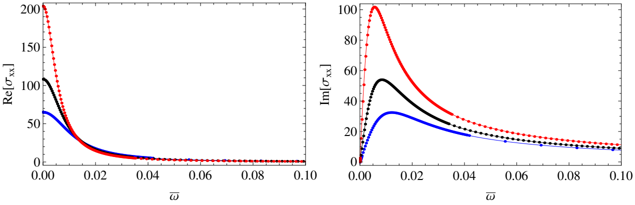
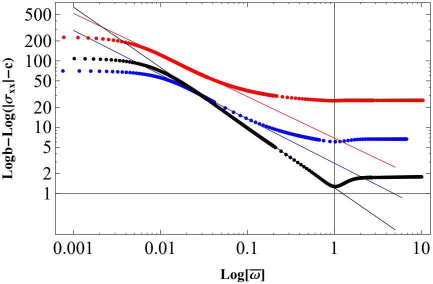


Figure: Fix $\alpha_1 = 2$ with $\kappa = 1$, and $\alpha_2 = 0$ (Blue), $\alpha_2 = 2$ (Black), $\alpha_2 = 4$ (Red)

Power Laws and Scaling Behavior

$$|\sigma| = \frac{b}{\omega^\gamma} + c\,,$$

σ_{xx}	γ	b	c	k	τ
$\alpha_2 = 0$	2/3	2.88	-5.65	0.785	82.7
$\alpha_2 = 2$	0.89	1.3	-0.8	0.9	120
$\alpha_2 = 4$	0.626	6.87	-24.5	1.13	180



Compare $\mathrm{Re}[\sigma_{xx}(\alpha_2)]$ and $\mathrm{Re}[\sigma_{yy}(\alpha_2)]$

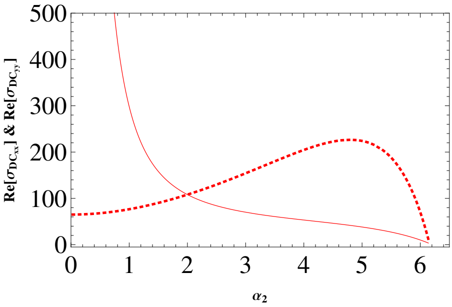


Figure: Fix $\alpha = 2$ with $\kappa = 1$ (Red), σ_x (Dotted) and σ_y (Line)

Summary

- We study anisotropic black brane solution turned on by Axion fields.
- We compute fluctuation equations using eigenvalue problem at horizon to obtain in-going boundary conditions.
- The eigenvalue solutions at horizon allow one to compute the Retarded Green Function without mismatch in degree of freedoms between horizon and boundary in case of multi-interacting fields in order to obtain transport coefficients.
- We study AC conductivity and finite DC conductivity in asymmetric system when source from gravity is turned on. we compute electric, Thermoelectric and thermal conductivity.

Results

- When a momentum relaxation is turned on, we numerically showed that conductivities in x- as well as in y-directions become finite as expected.
- We found that the y-direction momentum relaxation can affect both x- and y- direction linear responses, and same to x-direction due to background geometry.
- There exists a critical momentum relaxation at which the DC conductivity has maximum value.

Results

- There seems to be an upper bound of the anisotropy above which the dual geometry does not exist. This upper bound does not allow the sign change of the DC conductivity.
- In the low frequency regime, the electric conductivity shows a Drude peak. When the x-direction momentum relaxation is fixed to be $\alpha_1 = 2$, the Drude peak becomes broader as the y-direction momentum relaxation increases.
- In the intermediate frequency regime, magnitude of the electric conductivity shows a specific scaling behavior. Comparing with the power law behavior, our results show that the critical exponent becomes smaller as the anisotropy increases.

Thank You

EMDA model and Massive gravity

Action with massive term

$$S = \int d^4x\sqrt{-g}\left(R + \frac{6}{L^2} - 2(\nabla\phi)^2 - \frac{1}{2}e^{4\phi}(\nabla\tilde{a})^2 - e^{-2\phi}F^2 - \tilde{a}F\tilde{F} + \mathcal{L}_m\right) ,$$

where $\mathcal{L}_m = p_1(\phi)[\mathcal{K}] + p_2(\phi)([\mathcal{K}]^2 - [\mathcal{K}^2])$, and $\mathcal{K}^{\mu}{}_{\sigma}\mathcal{K}^{\sigma}{}_{\nu} \equiv g^{\mu\sigma}f_{\sigma\nu}$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{3}{L^2}g_{\mu\nu} = T_{\mu\nu},$$

Energy-momentum tensor

$$\begin{aligned} T_{\mu\nu} = \quad & 2\nabla_\mu\phi\nabla_\nu\phi + \frac{1}{2}e^{4\phi}(\nabla_\mu\tilde{a}\nabla_\nu\tilde{a}) + 2e^{-2\phi}F_{\mu\rho}F_\nu{}^\rho \\ & - \frac{1}{2}g_{\mu\nu}\left(2(\nabla\phi)^2 + \frac{1}{2}e^{4\phi}(\nabla\tilde{a})^2 + e^{-2\phi}F^2 - p([\mathcal{K}]^2 - [\mathcal{K}^2])\right) \\ & - \frac{1}{2}p_1\mathcal{K}_{\mu\nu} - p_2\left([\mathcal{K}]\mathcal{K}_{\mu\nu} - (\mathcal{K}^2)_{\mu\nu}\right) . \end{aligned}$$

Full and reference metric

$$ds^2 = \frac{L^2}{z^2}\left(-g(z)dt^2 + g(z)^{-1}dz^2 + e^{2U_1(z)}dx^2 + e^{2U_2(z)}dy^2\right)$$

$$f_{\mu\nu} = diag\left(0, 0, H(z)k_1^2, H(z)k_2^2\right) \,,$$

with above metric, the last two terms in stress-energy tensor become

$$p_1\mathcal{K}_{xx} + p_2\left([\mathcal{K}]\mathcal{K}_{xx} - (\mathcal{K}^2)_{xx}\right) = \frac{e^{U_1-U_2}}{4z}\left(p_1 H k_2 e^{U_1} + 3p_1 H k_1 e^{U_2} + 4p_2 H^2 k_1 k_2 z\right)$$

$$p_1\mathcal{K}_{yy} + p_2\left([\mathcal{K}]\mathcal{K}_{yy} - (\mathcal{K}^2)_{yy}\right) = \frac{e^{U_2-U_1}}{4z}\left(p_1 H k_1 e^{U_2} + 3p_1 H k_2 e^{U_1} + 4p_2 H^2 k_1 k_2 z\right)$$

Anisotropic Property

$$T_{xx} \neq T_{yy}$$

EOM

$$R_{\mu\nu} = -\frac{3}{L^2}g_{\mu\nu} + 2\nabla_\mu\phi\nabla_\nu\phi + \frac{1}{2}e^{4\phi}(\nabla_\mu\tilde{a}\nabla_\nu\tilde{a}) + 2e^{-2\phi}F_{\mu\rho}F_\nu{}^\rho$$
$$-\frac{1}{2}g_{\mu\nu}e^{-2\phi}F^2 + \frac{1}{2}p_1\mathcal{K}_{\mu\nu} - p_2\left([\mathcal{K}]\mathcal{K}_{\mu\nu} - (\mathcal{K}^2)_{\mu\nu}\right) ,$$
$$\nabla_\mu(e^{-2\phi}F^{\mu\nu} + \tilde{a}\tilde{F}^{\mu\nu}) = 0\,,$$
$$\Box\phi - \frac{1}{2}e^{4\phi}(\nabla\tilde{a})^2 + \frac{1}{2}e^{-2\phi}F^2 + \frac{1}{4}\left[p_1'(\phi)[\mathcal{K}] + p_2'(\phi)([\mathcal{K}]^2 - [\mathcal{K}^2])\right] = 0\,,$$
$$\Box\tilde{a} + 4\nabla_\mu\phi\nabla^\mu\tilde{a} - F_{\mu\nu}\tilde{F}^{\mu\nu} = 0\,.$$

We check $SL(2,R)$ transformation by defining

$$\lambda = \lambda_1 + i\lambda_2 \equiv \tilde{a} + ie^{-2\phi}\,, \quad F_{\pm} = F \pm i\tilde{F}\,,$$

so that above equations of motion can be rewritten as

$$\begin{aligned}
R_{\mu\nu} &= -\frac{3}{L^2}g_{\mu\nu} + \frac{1}{4\lambda_2^2}\left(\partial_\mu\bar{\lambda}\partial_\nu\lambda + \partial_\nu\bar{\lambda}\partial_\mu\lambda\right) + 2\lambda_2 F_{\mu\sigma}F_\nu{}^\sigma - \frac{1}{2}\lambda_2 g_{\mu\nu}F_{\mu\nu}F^{\mu\nu} \\
&\quad + \frac{1}{2}p_1\mathcal{K}_{\mu\nu} - p_2\left([\mathcal{K}]\mathcal{K}_{\mu\nu} - (\mathcal{K}^2)_{\mu\nu}\right)\,, \\
&\nabla_\mu(\lambda F_+^{\mu\nu} - \bar{\lambda}F_-^{\mu\nu}) = 0\,, \\
&\frac{\nabla_\mu\partial^\mu\lambda}{\lambda_2^2} + i\frac{\partial_\mu\lambda\partial^\mu\lambda}{\lambda_2^3} - \frac{i}{2}F_-^2 + \frac{i}{4}\left[p_1'(\phi)[\mathcal{K}] + p_2'(\phi)([\mathcal{K}]^2 - [\mathcal{K}^2])\right] = 0\,,
\end{aligned}$$

Transformation Rules

$$\lambda \to \frac{a\lambda + b}{c\lambda + d}\,, \qquad ad - bc = 1$$

$$F_{\mu\nu} \to (c\lambda_1 + d)F_{\mu\nu} - c\lambda_2 \tilde{F}_{\mu\nu}$$

Shift symmetry

$$\mathbb{T}_b \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \Longrightarrow \lambda_1 \to \lambda_1 + b$$

and Rotation symmetry

$$\mathbb{S} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Longrightarrow \begin{cases} \lambda \to -1/\lambda\,, \quad \bar{\lambda} \to -1/\bar{\lambda}\,, \\ F_+ \to -\lambda F_+\,, \quad F_- \to -\bar{\lambda} F_-\,, \end{cases}$$

After transformation, Einstein equation has additional term

$$-\frac{\lambda_1 \lambda_2^2}{|\lambda|^2}(2F_{\mu\sigma}\tilde{F}_{\nu}{}^{\sigma} + 2F_{\nu\sigma}\tilde{F}_{\mu}{}^{\sigma} - g_{\mu\nu}F_{\mu\nu}\tilde{F}^{\mu\nu})\,.$$

but this term vanishes identically in 4 dimensions starting from purely charged solution. The rest would be invariant if we choose p_1 and p_2 to be constant.

Gauge field for purely charged solution

$$A_\mu dx^\mu = A_t(z)dt\,,$$

so that field strength

$$F = e^{-U_1-U_2}(\lambda_2)^{-1}Q\,dt\wedge dz\,.$$

for both electrically and magnetically charged solution after transformation

$$\bar{F} = e^{-U_1-U_2}(\bar{\lambda}_2)^{-1}(\bar{Q}_e-\bar{\lambda}_1\bar{Q}_m)\,dt\wedge dz+\bar{Q}_m dx\wedge dy\,.$$

Conductivity with broken rotational symmetry

Generalizing broken rotational symmetry of $(x, y) \to (G^1 y, -G^{-1} x)$; defining $\sigma_1 = \sigma_{yx}/4$, $\sigma_2 = \sigma_{xx}/4$; and using AdS/CFT dictionary, we obtain

$$J_x\Big|_{z\to 0} = 4\left(\sigma_2 F_{tx} - \sigma_1 F_{ty}\right) = 4\left(\lambda_2 F_{zx} - \lambda_1 F_{ty}\right)$$

$$J_y\Big|_{z\to 0} = 4\left(\sigma_1 F_{tx} + G^{-2}\sigma_2 F_{ty}\right) = 4\left(\lambda_2 F_{zy} + \lambda_1 F_{tx}\right)$$

Let's define

$$\sigma_\pm = \sigma_1 \pm iG^{-2}\sigma_2$$

and with this definition, we assume that $\sigma_\pm$ transforms as

$$\sigma_\pm \to \frac{a\sigma_\pm + b}{c\sigma_\pm + d}\,.$$

$$\lambda_2 F_{zx} - \lambda_1 Fty \equiv -\frac{\lambda}{2}(F_+)_{ty} - \frac{\bar{\lambda}}{2}(F_-)_{ty}$$
$$\to -F_{ty}$$

$$\sigma_2 F_{tx} - \sigma_1 F_{ty} \equiv -\frac{1}{2}\sigma_+(F_{ty} + iF_{tx}) - \frac{1}{2}\sigma_-(F_{ty} - iF_{tx})$$
$$\to \frac{1}{\sigma_+\sigma_-}\left[\sigma_1(\lambda_2 F_{zx} - \lambda_1 F_{ty}) - G^{-2}\sigma_2(\lambda_2 F_{zy} + \lambda_1 F_{tx})\right] = -F_{ty}\,,$$

Transformation shows that $4\left(\bar{\sigma}_2 \bar{F}_{tx} - \bar{\sigma}_1 \bar{F}_{ty}\right) = 4\left(\bar{\lambda}_2 \bar{F}_{zx} - \bar{\lambda}_1 \bar{F}_{ty}\right)$. Same to terms associated with J_y.

Starting from purely electric solution where $\sigma_{xy} = \sigma_{yx} = 0$ we obtain conductivity in new configuration after transformation

$$\bar{\sigma}_{xx} = \frac{\sigma_{xx}}{d^2 + c^2 G^{-4}(\sigma_{xx}/4)^2} = \frac{\sigma_{xx}}{d^2 + c^2(\sigma_{yy}/4)^2}\,,$$
$$\bar{\sigma}_{yx} = 4\frac{ac\,G^{-4}(\sigma_{xx}/4)^2 + bd}{d^2 + c^2 G^{-4}(\sigma_{xx}/4)^2} = 4\frac{ac\,(\sigma_{yy}/4)^2 + bd}{d^2 + c^2(\sigma_{yy}/4)^2}\,,$$

where we use $G^{-2} = \sigma_{yy}/\sigma_{xx}$. Then, it also follows that

$$\bar{\sigma}_{yy} = \frac{G^{-2}\sigma_{xx}}{d^2 + c^2 G^{-4}(\sigma_{xx}/4)^2} = \frac{\sigma_{yy}}{d^2 + c^2(\sigma_{yy}/4)^2}\,,$$
$$\bar{\sigma}_{yx} = -4\frac{ac\,G^{-4}(\sigma_{xx}/4)^2 + bd}{d^2 + c^2 G^{-4}(\sigma_{xx}/4)^2} = -4\frac{ac\,(\sigma_{yy}/4)^2 + bd}{d^2 + c^2(\sigma_{yy}/4)^2}\,.$$

Then we also have $\bar{\sigma}_{yy} = G^{-2}\bar{\sigma}_{xx}$ and $\bar{\sigma}_{xy} = -\bar{\sigma}_{yx}$. Also $SL(2,R)$ elements are determined by conserved quantities such as $\bar{Q}_e, \bar{Q}_m, \bar{\lambda_1}|_{z\to 0}, \bar{\lambda_2}|_{z\to 0}$ using the the fact that effective potential is invariant under transformation.

Conductivity in external Magnetic Field in General Class

Conserved Current

$$J_x = -4ge^{-U_a-2\phi}A'_x + 4z^2Qe^{-2U_1}\delta g_{tx} - 4gz^2Be^{-U_t-2\phi}\delta g_{ry}$$
$$J_y = -4ge^{U_a-2\phi}A'_y + 4z^2Qe^{-2U_2}\delta g_{ty} + 4gz^2Be^{-U_t-2\phi}\delta g_{rx}$$

where $U_t = U_1 + U_2$ and $U_a = U_1 - U_2$, assuming $H \equiv 1$ and $p_1 = 0$ and $p_2 = p$

$$\bar{\sigma}_{xx}\Big|_{z\to 1} = -\frac{4pk_1k_2e^{U_t-U_a}\left(2Q_e^2 + 2B^2e^{-4\phi} - pk_1k_2e^{U_t-2\phi}\right)}{4B^2\left(Q_e^2 - pk_1k_2e^{U_t-2\phi}\right) + 4B^4e^{-4\phi} + p^2k_1^2k_2^2e^{2U_t}}$$

$$\bar{\sigma}_{yy}\Big|_{z\to 1} = -\frac{4pk_1k_2e^{U_t+U_a}\left(2Q_e^2 + 2B^2e^{-4\phi} - pk_1k_2e^{U_t-2\phi}\right)}{4B^2\left(Q_e^2 - pk_1k_2e^{U_t-2\phi}\right) + 4B^4e^{-4\phi} + p^2k_1^2k_2^2e^{2U_t}}$$

$$\bar{\sigma}_{yx}\Big|_{z\to 1} = \frac{16BQ_e\left(Q_e^2 + B^2e^{-4\phi} - pk_1k_2e^{U_t-2\phi}\right)}{4B^2\left(Q_e^2 - pk_1k_2e^{U_t-2\phi}\right) + 4B^4e^{-4\phi} + p^2k_1^2k_2^2e^{2U_t}}$$

$$\bar{\sigma}_{xy}\Big|_{z\to 1} = -\frac{16BQ_e\left(Q_e^2 + B^2e^{-4\phi} - pk_1k_2e^{U_t-2\phi}\right)}{4B^2\left(Q_e^2 - pk_1k_2e^{U_t-2\phi}\right) + 4B^4e^{-4\phi} + p^2k_1^2k_2^2e^{2U_t}}$$

assuming $H \equiv 1$ and p_1, p_2 are non-zero

$$\bar{\sigma}_{xx}\Big|_{z\to 1} = \frac{e^{-U_a}N}{D}\,, \qquad \bar{\sigma}_{yy}\Big|_{z\to 1} = \frac{e^{U_a}N}{D}$$

$$\bar{\sigma}_{yx}\Big|_{z\to 1} = \frac{\tilde{N}}{D}\,, \qquad \bar{\sigma}_{xy}\Big|_{z\to 1} = -\frac{\tilde{N}}{D}$$

where

$$\begin{aligned}
N =& 4e^{U_t}\left(k_1\left(4k_2p_2 + p_1e^{U_2}\right) + 3k_2p_1e^{U_1}\right)\left(e^{U_t}\left(e^{2\phi}\left(k_1\left(4k_2p_2\right.\right.\right.\right.\\
&\left.\left.\left.\left. +3p_1e^{U_2}\right) + k_2p_1e^{U_1}\right) - 8e^{U_t}a'^2\right) - 8B^2\right)\,,\\
D =& e^{2(U_t+2\phi)}\left(k_1^2\left(16k_2p_2p_1e^{U_2} + 16k_2^2p_2^2 + 3p_1^2e^{2U_2}\right)\right.\\
&\left. +2k_2k_1p_1e^{U_1}\left(8k_2p_2 + 5p_1e^{U_2}\right) + 3k_2^2p_1^2e^{2U_1}\right)\\
& - 32B^2e^{U_t}\left(e^{2\phi}\left(k_1\left(2k_2p_2 + p_1e^{U_2}\right) + k_2p_1e^{U_1}\right) - 2e^{U_t}a'^2\right) + 64B^4\,,\\
\tilde{N} =& 128Ba'e^{U_t-2\phi}\left(e^{U_t}\left(e^{2\phi}\left(k_1\left(2k_2p_2 + p_1e^{U_2}\right) + k_2p_1e^{U_1}\right) - 2e^{U_t}a'^2\right) - 2B^2\right)\,.
\end{aligned}$$

AAdS Case

Equation of motion

$$0 = \phi'^2 + \frac{1}{4}e^{4\phi}\tilde{a}'^2 U_A'^2 + U_B'^2 + U_A'' \,,$$

$$0 = \frac{1}{4}p_1 H e^{-U_A - U_B}\left(k_2 e^{2U_B} - k_1\right) + \left(2g\left(zU_A' - 1\right) + zg'\right)U_B' + gzU_B'' \,,$$

$$0 = -4z^4 Q^2 e^{2\phi - 4U_A} + \left(-4z^2 U_A'^2 + 16zU_A' + 4z^2 U_B'^2 + 4z^2\phi'^2 + z^2 e^{4\phi}\tilde{a}'^2 - 12\right)g$$
$$+ 12 + 2p_1 zHe^{-U_A}(k_1 e^{-U_B} + k_2 e^{U_B}) + 4k_1 k_2 H^2 z^2 p_2 e^{-2U_A} + 4z\left(1 - zU_A'\right)g' \,,$$

$$0 = p_1'(\phi)He^{-U_A}(k_1 e^{-U_B} + k_2 e^{U_B}) + 2k_1 k_2 p_2'(\phi)H^2 e^{-2U_A}$$
$$+ Q^2 z^3 e^{2\phi - 4U_A} - \frac{1}{2}gze^{4\phi}\tilde{a}'^2 + \left(2gzU_A' + zg' - 2g\right)\phi' + gz\phi'' \,,$$

$$0 = \left(2g\left(zU_A' + 2z\phi' - 1\right) + zg'\right)\tilde{a}' + gz\tilde{a}'' \,,$$

where we define $U_1 = U_A + U_B$ and $U_2 = U_A - U_B$

Isotropic Background Solutions when H, p_1, p_2 is constant

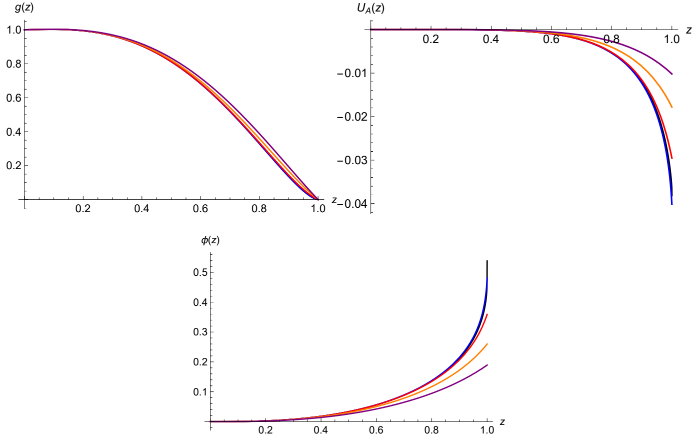


Figure: Fix $k_1 = k_2 = 0.1$ with $\kappa = 0.001$ (Black), and $\kappa = 0.15$ (Blue), $\kappa = 0.75$ (Red), $\kappa = 1.35$ (Orange), and $\kappa = 1.81$ (Purple)

Anisotropic Background Solutions $H = z^n$ with $n = 1$, and p_1, p_2 is constant

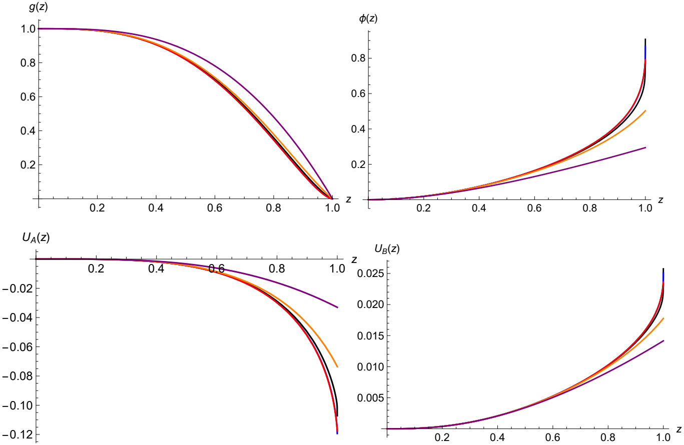


Figure: Fix $k_1 = 0.1, k_2 = 0.4$ with $\kappa = 0.0003$ (Black), and $\kappa = 0.008$ (Blue), $\kappa = 0.165$ (Red), $\kappa = 1.5$ (Orange), and $\kappa = 2.9$ (Purple)

Thanks Again