Lattice Universe with Rotating Black Holes

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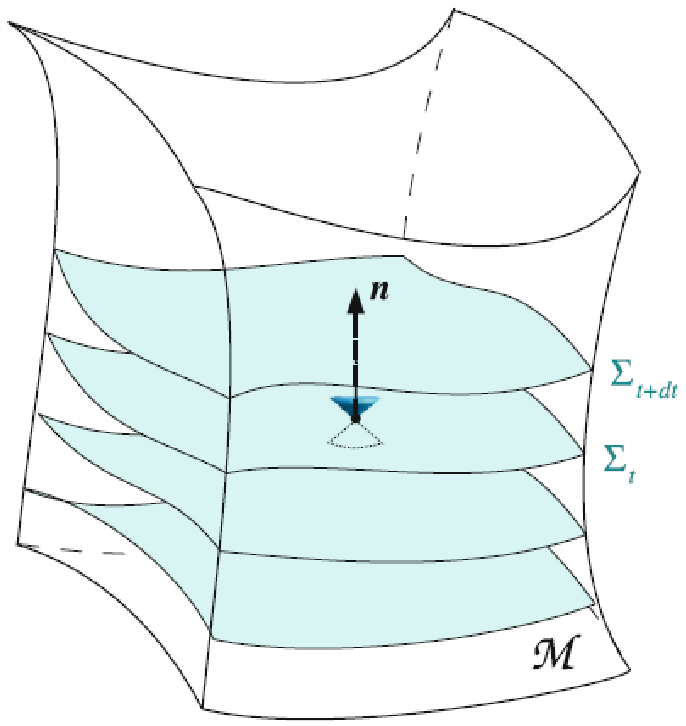
Contents

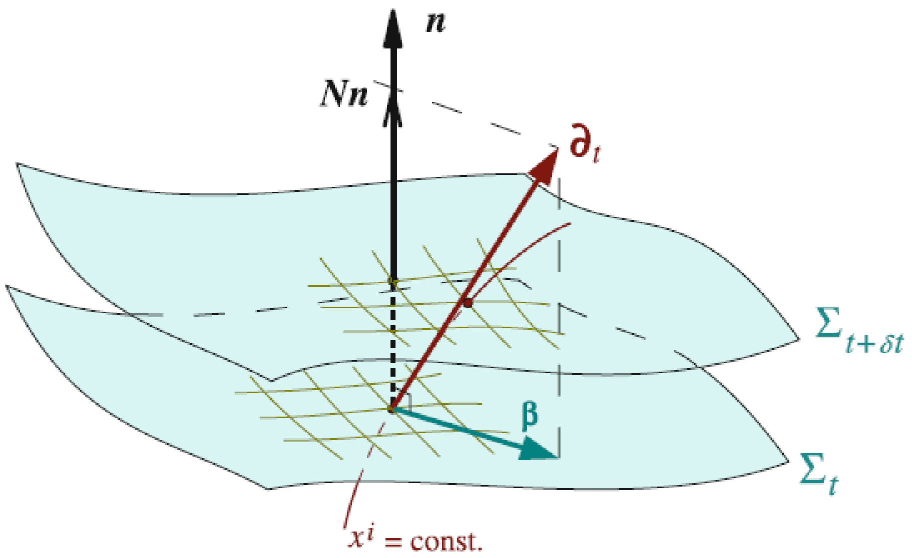
Numerical Relativity

Globally Hyperbolic Spacetime

- A spacelike hypersurface Σ is **Cauchy surface** if its domain of dependence is the whole spacetime $\mathcal{M}$.
- $\mathcal{M}$ is **globally hyperbolic** if it admit a Cauchy surface.
- A **global time function**, t, can be chosen on $\mathcal{M}$ such that each surface of constant t is a Cauchy surface.



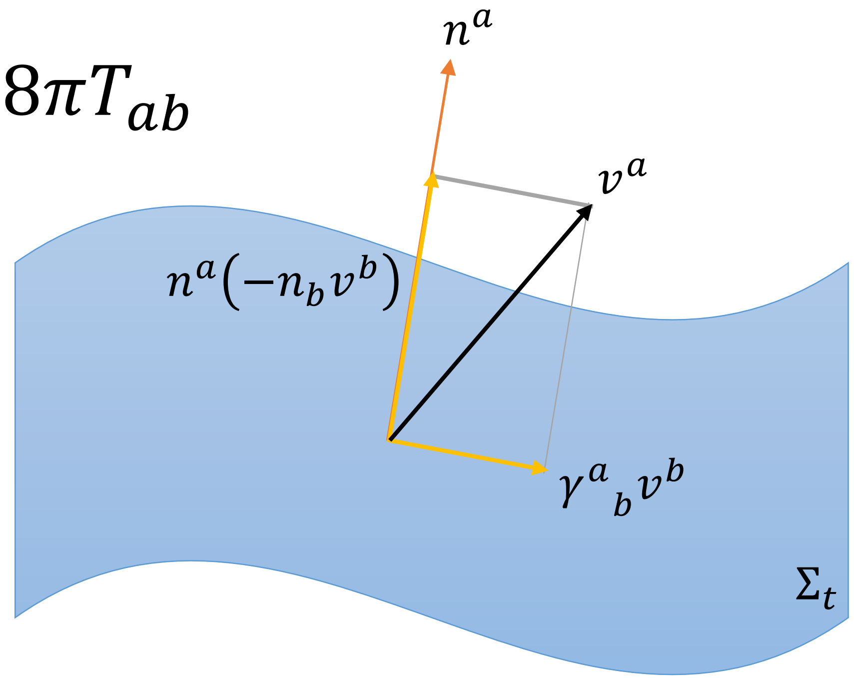
Geometry of Foliations



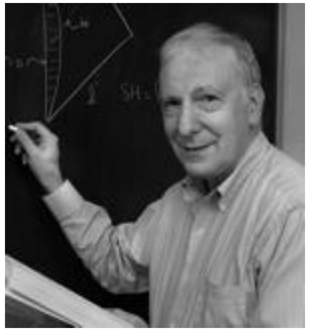
- Cauchy surface : Σ_t
- Lapse function : $N = 1/\sqrt{-\nabla_a t \nabla^a t}$
- Normal vector : $n^a = -N\nabla^a t$
- Normal evolution vector : $m^a = Nn^a \quad \rightarrow \quad \langle \nabla_a t, m^a \delta t \rangle = \delta t$
- Shift vector : $(\partial/\partial t)^a = m^a + \beta^a$
- Induced metric : $\gamma_{ab} = g_{ab} + n_a n_b$
- Extrinsic curvature : $K_{ab} = -\gamma^c{}_a \nabla_c n_b = -(2N)^{-1} \mathcal{L}_m \gamma_{ab}$
- Derivative operator : $D_c T_{ab} = \gamma^d{}_a \gamma^e{}_b \gamma^f{}_c \nabla_f T_{de}$

3+1 Decomposition

- Einstein equation : $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab}$
- Hamiltonian constraint $(n^a n^b)$
  - ${}^{(3)}R + K^2 - K_{ab}K^{ab} = 16\pi E$
- Momentum constraint $(n^a \gamma^b{}_c)$
  - $D_b K^b{}_c - D_c K = 8\pi p_c$
- Evolution equation $(\gamma^a{}_c \gamma^b{}_d)$
  - $\mathcal{L}_m K_{ab} + D_a D_b N - N\left[{}^{(3)}R_{ab} + KK_{ab} - 2K_{ac}K^c{}_b\right] = 4\pi N\{(S - E)\gamma_{ab} -$



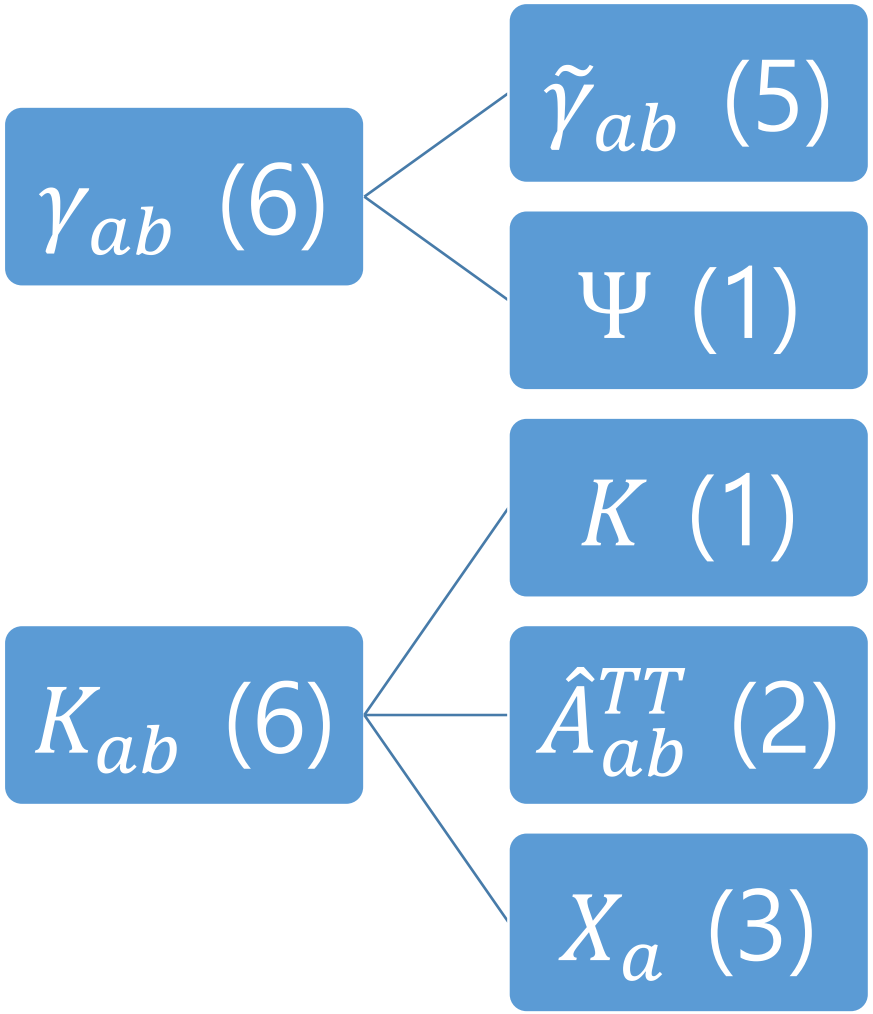
Initial Value Formulation



- Initial data : $(\gamma_{ab}, K_{ab}, E, p_a)$ on Σ which satisfy constraint equations.
- If the initial data are given once, we can always **uniquely develop** a spacetime, $\mathcal{M}$, which starts from the data and satisfies the Einstein equation. [Y. Choquet-Bruhat, R. Geroch (1969)]
- $\mathcal{M}$ is globally hyperbolic with Cauchy surface Σ.
- γ_{ab} and K_{ab} are induced metric and extrinsic curvature of Σ.

Conformal Transverse-Traceless Method

- Traceless decomposition
  - $K_{ab} = A_{ab} + \frac{1}{3}\gamma_{ab}K$
- Conformal decomposition
  - $\gamma_{ab} = \Psi^4\tilde{\gamma}_{ab}$, $A_{ab} = \Psi^{-2}\hat{A}_{ab}$
- Transverse/longitudinal decomposition
  - $\hat{A}_{ab} = \hat{A}^L_{ab} + \hat{A}^{TT}_{ab} \qquad (\tilde{D}^b\hat{A}^{TT}_{ab} = 0)$
- York decomposition
  - $\hat{A}^L_{ab} = (\tilde{L}X)_{ab} = \tilde{D}_a X_b + \tilde{D}_b X_a - \frac{2}{3}\tilde{\gamma}_{ab}\tilde{D}^c X_c$



Constraint Equation in CTT Form

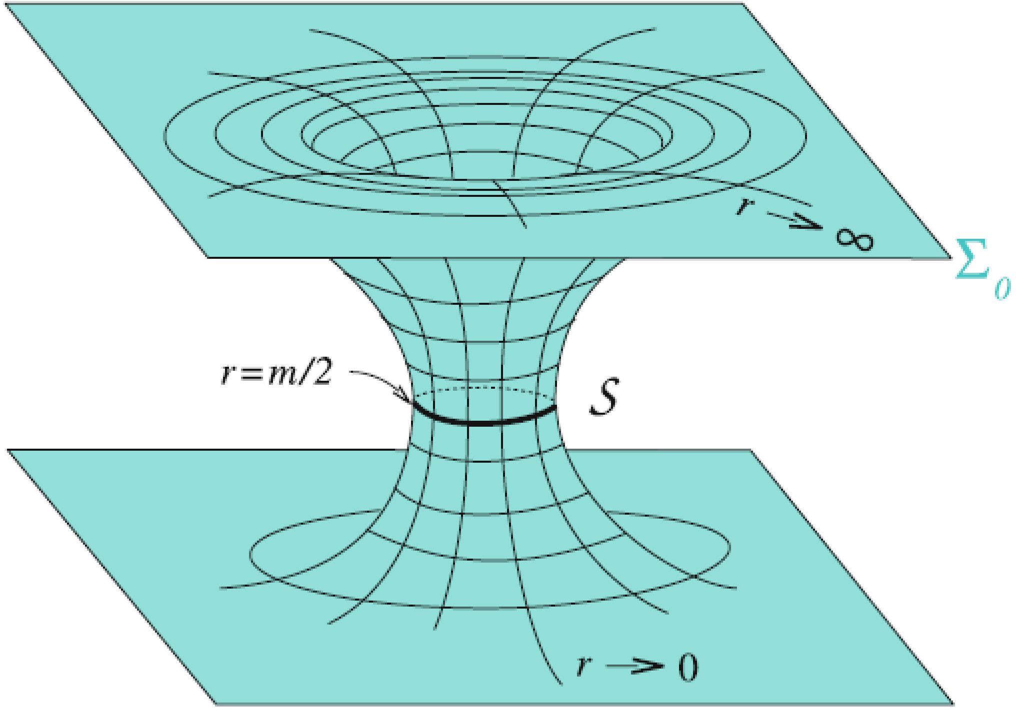
- $\widetilde{D}^a \widetilde{D}_a \Psi = \frac{1}{8}\Psi\tilde{R} - \frac{1}{8}\hat{A}_{ab}\hat{A}^{ab}\Psi^{-7} + \frac{1}{12}K^2\Psi^5 + 16\pi\tilde{E}\Psi^{-3}$
- $\widetilde{D}^b \widetilde{D}_b X_a + \frac{1}{3}\widetilde{D}_a \widetilde{D}^b X_b = -\tilde{R}_a{}^b X_b + \frac{2}{3}\Psi^6 \widetilde{D}_a K + 8\pi\tilde{p}_a$
- With conformal flatness ($\tilde{\gamma}_{ab} = f_{ab}$),
- $\Delta\Psi = -\frac{1}{8}\hat{A}_{ab}\hat{A}^{ab}\Psi^{-7} + \frac{1}{12}K^2\Psi^5 + 16\pi\tilde{E}\Psi^{-3}$
- $\Delta X_a + \frac{1}{3}\overline{D}_a \overline{D}^b X_b = \frac{2}{3}\Psi^6 \overline{D}_a K + 8\pi\tilde{p}_a$

ADM Quantities

- Asymptotic flatness (when $r \to \infty$)
  - $\gamma_{ij} = f_{ij} + O(r^{-1})$ $\qquad \frac{\partial \gamma_{ij}}{\partial x^k} = O(r^{-2})$
  - $K_{ij} = O(r^{-2})$ $\qquad \frac{\partial K_{ij}}{\partial x^k} = O(r^{-3})$
- $H_{ADM} = -2 \oint_S \left[N(\kappa - \kappa_0) + \beta^i \left(K_{ij} - K\gamma_{ij}\right) s^j\right] \sqrt{q} d^2 y$
- $M_{ADM} = -\frac{1}{8\pi} \lim_{S\to\infty} \oint_S (\kappa - \kappa_0) \sqrt{q} d^2 y$
- $P_k^{ADM} = \frac{1}{8\pi} \lim_{S\to\infty} \oint_S \left(\partial/\partial x^k\right)^i \left(K_{ij} - K\gamma_{ij}\right) s^j \sqrt{q} d^2 y$
- $J_k^{ADM} = \frac{1}{8\pi} \lim_{S\to\infty} \oint_S (\phi_k)^i \left(K_{ij} - K\gamma_{ij}\right) s^j \sqrt{q} d^2 y,$ $\qquad \phi_i = \epsilon_{ijk} x^j \left(\partial/\partial x^k\right)$

Elementary Black Hole Solution

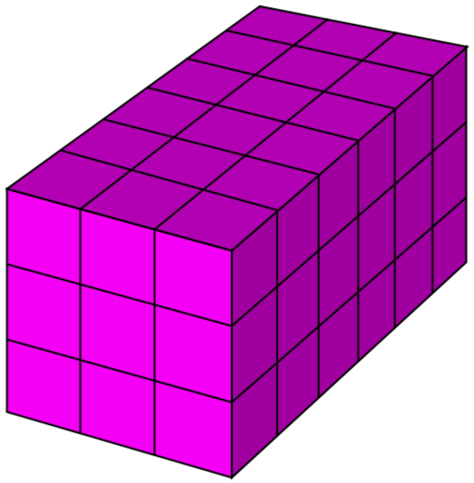
- Vacuum : $\tilde{E} = 0 = \tilde{p}_i$
- Conformal Flatness : $\tilde{\gamma}_{ij} = f_{ij}$
- Time-symmetric : $K_{ij} = 0$
- $\Delta\Psi = 0$
- $\rightarrow \Psi = 1 + \frac{M}{2r}$
- $\rightarrow dl^2 = \left(1 + \frac{M}{2r}\right)^4 (dr^2 + r^2 d\Omega^2)$



Bowen-York Solution

- Vacuum : $\tilde{E} = 0 = \tilde{p}_i$
- Conformal Flatness : $\tilde{\gamma}_{ij} = f_{ij}$
- Maximal slicing : $K = 0$
- No TT part : $\hat{A}_{ij}^{TT} = 0$
- $\Delta\Psi = -\frac{1}{8}\left(\tilde{L}X\right)_{ij}\left(\tilde{L}X\right)^{ij}\Psi^{-7}$
- $\Delta X_i + \frac{1}{3}\overline{D}_i\overline{D}^j X_j = 0$
- $\rightarrow X_i = -\frac{1}{4r}\left(7P_i + l_i l_j P^j\right) - \frac{1}{r^2}\epsilon_{ijk}S^j l^k - \frac{1}{r^4}\left(-Q_i + 3l_i l_j Q^j\right)$

Finite Difference Method



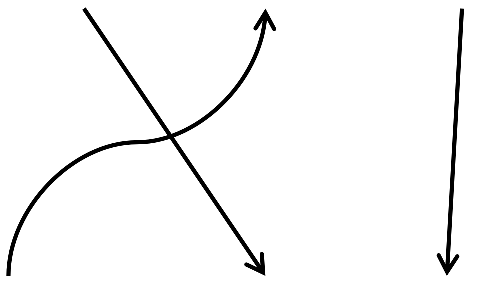
- $x_i = x_0 + ih,\ y_i = y_0 + ih,\ z_i = z_0 + ih$
- $u(x_i, y_i, z_i) \to u_{ijk}$
- $\partial_x u(x_i, y_i, z_i) \to \frac{u_{i+1,jk} - u_{i-1,jk}}{2h}$
- $\partial_x^2 u(x_i, y_i, z_i) \to \frac{u_{i+1,jk} + u_{i-1,jk} - 2u_{ijk}}{h^2}$
- $\Delta u(x_i, y_i, z_i) \to \frac{u_{i+1,jk} + u_{i-1,jk} + u_{i,j+1,k} + u_{i,j-1,k} + u_{ij,k+1} + u_{ij,k-1} - 6u_{ijk}}{h^2}$

Linear System

- $\Delta u = f \rightarrow \mathrm{A}\mathbf{u} = \mathbf{f}$
- $$\frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & & 0 & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{bmatrix} = \begin{bmatrix} f_1 - u_0/h^2 \\ f_2 \\ \vdots \\ f_{N-2} \\ f_{N-1} - u_N/h^2 \end{bmatrix}$$
- $\mathbf{u} = A^{-1}\mathbf{f}$

Gauss-Seidel Method

index	0	1	$\cdots$	$i-1$	i	$i+1$	$\cdots$	N
field	$u_0^{(k+1)}$	$u_1^{(k+1)}$	$\cdots$	$u_{i-1}^{(k+1)}$	$u_i^{(k)}$	$u_{i+1}^{(k)}$	$\cdots$	$u_N^{(k)}$

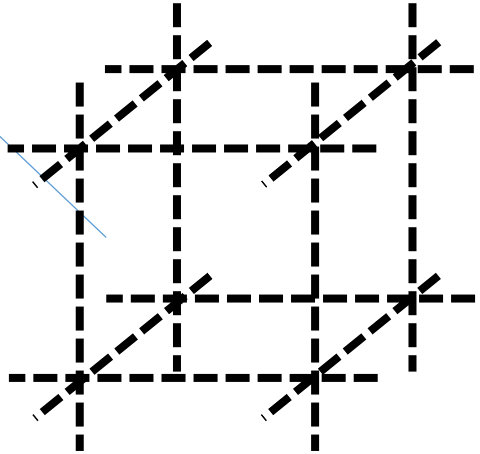


$$u_i^{(k+1)} \leftarrow \left(u_{i-1}^{(k+1)} + u_{i+1}^{(k)} - f_i h^2\right)/2$$

Lattice Universe with Rotating Black Holes

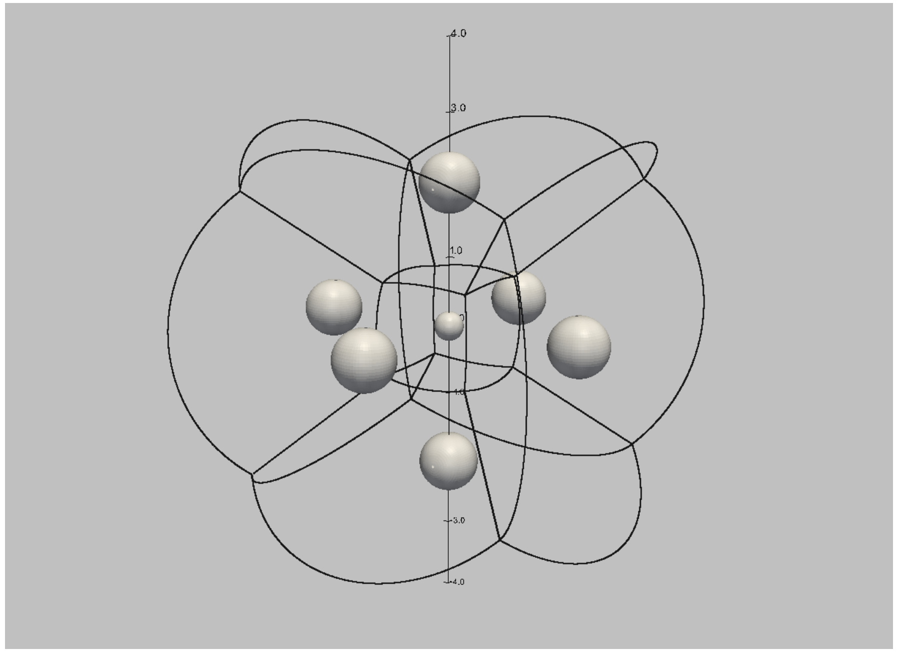
Lattice Universe

periodic boundary condition



- The toy model for inhomogeneous cosmology.
- The list of all possible methods that tile 3-spaces with identical regular polyhedral cells.

Cell shape	The number of adjacent cells at edge	Background curvature	Total number of cells
Tetrahedron	3	+	5
Cube	3	+	8
Tetrahedron	4	+	16
Octahedron	3	+	24
Dodecahedron	3	+	120
Tetrahedron	5	+	600
Cube	4	0	∞
Cube	5	-	∞
Dodecahedron	4	-	∞
Dodecahedron	5	-	∞
Icoshedron	3	-	∞



Tiling 8 cubic cells [E. Bentivegna, M. Korzynski (2012)]

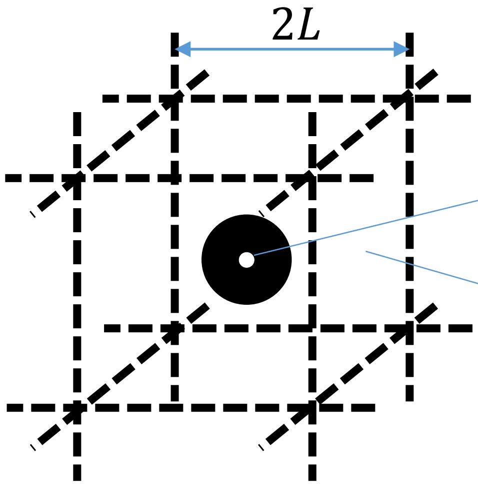
Black hole universe

- Black hole universe is lattice of black holes.
- The solutions of black hole universe up to now.

Method	Type of universe	Initial data	Evolution
Approximate	Closed	R. W. Lindquist, J. A. Wheeler (1959)	
	Flat / Open	T. Clifton, P. G. Ferreira (2011)	
Analytic	Closed (5 cells)	J. A. Wheeler (1983)	-
	Closed (all possible cases)	T. Clifton et al. (2012)	-
Numerical	Closed (8 cells)	E. Bentivegna, M. Korzynski (2012)	
	Flat	C. M. Yoo et al. (2012)	C. M. Yoo et al. (2013)
	Flat with Λ	C. M. Yoo, H. Okawa (2014)	
	Flat with rotating black holes	C. Park, G. W. Kang (2015)	-

Our Model

- Assumption : vacuum ($T_{ab} = 0$), conformal flatness ($\tilde{\gamma}_{ab} = f_{ab}$), minimum GW ($\hat{A}_{ab}^{TT} = 0$)
- Our choice : maximal slice near puncture ($K = 0$), asymptotic flatness at the puncture.



Puncture
$O = (0,0,0)$

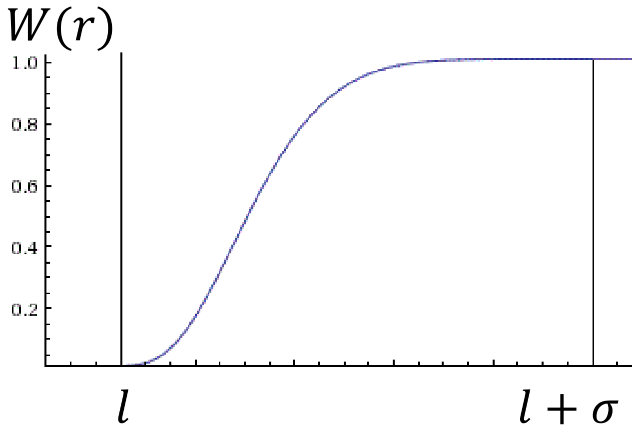
<numerical domain>
$\mathcal{D} = \{(x, y, z) | -L \leq x, y, z \leq L\} - \{O\}$

leading order solutions
of Bowen-York spinning black hole

$$\Psi \to \frac{M}{2r} + 1$$
$$X_i \to -\frac{1}{r^2}\epsilon_{ijk}S^j x^k$$
$$K = -3H_{eff}W(r)$$

periodic boundary condition

$$\phi(x + 2L, y, z) = \phi(x, y, z)$$
$$\phi(x, y + 2L, z) = \phi(x, y, z)$$
$$\phi(x, y, z + 2L) = \phi(x, y, z)$$



ADM Quantities

- Kelvin transformation
  - $r\bar{r} = a^2$, $\frac{x^i}{r} = \frac{\bar{x}^i}{r}$
  - $f_{ab} = \left(\frac{a}{\bar{r}}\right)^4 \bar{f}_{ab}$
  - $\gamma_{ab} = \bar{\Psi}^4 \bar{f}_{ab}$, $A_{ab} = \bar{\Psi}^{-2} \bar{A}_{ab}$
- $M_{ADM} = -\frac{1}{2\pi} \lim_{\bar{r}\to\infty} \oint \frac{\partial \bar{\Psi}}{\partial \bar{r}} \bar{r}^2 \sin\theta \, d\theta d\phi = M$
- $P^i_{ADM} = \frac{1}{8\pi} \lim_{\bar{r}\to\infty} \oint \bar{\Psi}^{-2} \delta^{ij} \bar{A}_{jk} \bar{l}^k \bar{r}^2 \sin\theta \, d\theta d\phi = 0$
- $J^i_{ADM} = \frac{1}{8\pi} \lim_{\bar{r}\to\infty} \oint \bar{\Psi}^{-2} \left\{ \bar{r} \left(-\epsilon^{jil} \right) \bar{l}_l \right\} \bar{A}_{jk} \bar{l}^k \bar{r}^2 \sin\theta \, d\theta d\phi = S^i$

Numerical Solutions

Integrable compatibility conditions

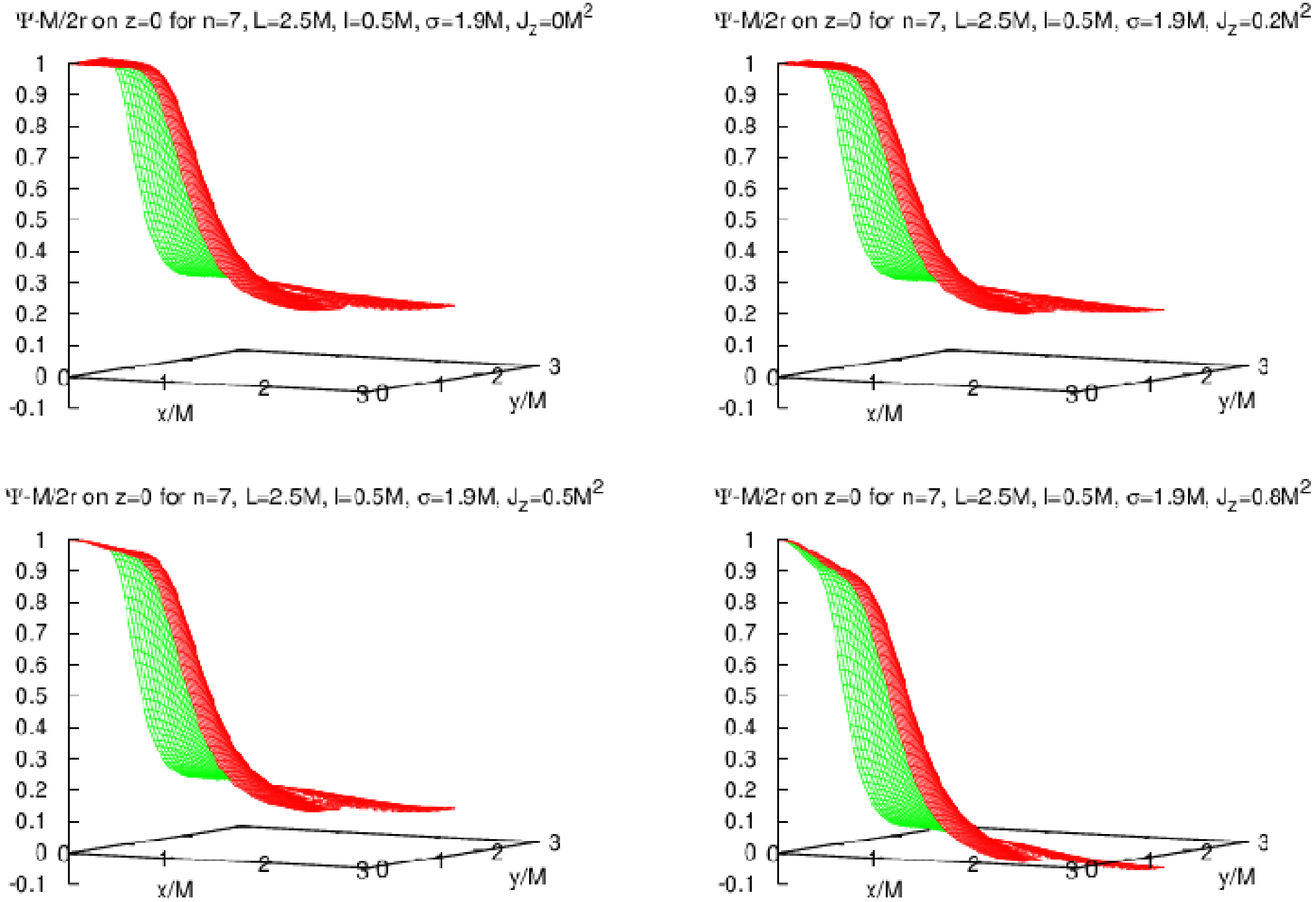
- Momentum constraints
  - $\int_{\mathcal{D}} \partial_j (\tilde{L}X)^{ij} dx^3 = \frac{2}{3} \int_{\mathcal{D}} \Psi^6 \partial^i K dx^3$
  - Both sides vanish automatically.
- Hamiltonian constraint
  - $\int_{\mathcal{D}} \Delta\Psi dx^3 = -2\pi \int_{\mathcal{D}} \frac{1}{16\pi} (\tilde{L}X)^{ij} (\tilde{L}X)_{ij} \Psi^{-7} dx^3 + \frac{3}{4} H_{eff}^2 \int_{\mathcal{D}} W^2 \Psi^5 dx^3$
  - $H_{eff}^2 = \frac{8\pi}{3}\left(\frac{M}{V} + \frac{M_K}{V}\right)$
    - $M_K = \int_{\mathcal{D}} \frac{1}{16\pi} (\tilde{L}X)^{ij} (\tilde{L}X)_{ij} \Psi^{-7} dx^3$
    - $V = \int_{\mathcal{D}} W^2 \Psi^5 dx^3$

effective Friedmann equation

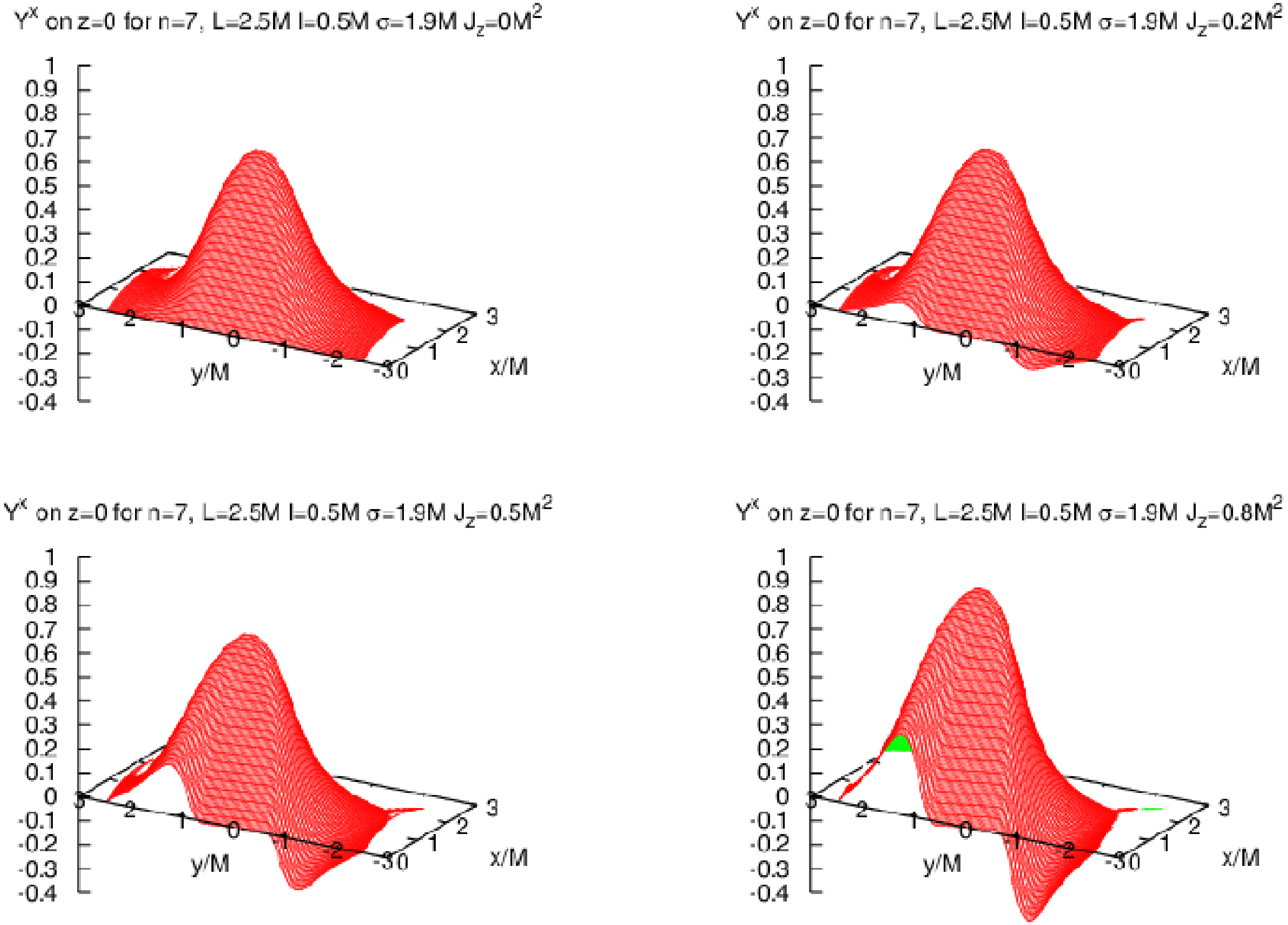
Parameters of the solutions

- L : coordinate length of the cube
- S^i : spin parameter of black hole
- H_{eff} : effect Hubble expansion rate near surfaces of box.
  - It is determined by effective Hubble equation: $H_{eff}^2 = \frac{8\pi}{3}\left(\frac{M}{V} + \frac{M_K}{V}\right)$
- M : ADM mass of black hole with respect to puncture.
  - It can be eliminated by
    - $x^i \to x^i/M$, $L \to L/M$, $l \to l/M$, $\sigma \to \sigma/M$
    - $H_{eff} \to MH_{eff}$, $M_K \to M_K/M$, $V \to V/M^3$
    - $S^i \to S^i/M^2$

$\Psi - M/2r$ on $z = 0$


Ψ-M/2r on z=0 for n=7, L=2.5M, l=0.5M, σ=1.9M, J_z=0M^2
Ψ-M/2r on z=0 for n=7, L=2.5M, l=0.5M, σ=1.9M, J_z=0.2M^2
Ψ-M/2r on z=0 for n=7, L=2.5M, l=0.5M, σ=1.9M, J_z=0.5M^2
Ψ-M/2r on z=0 for n=7, L=2.5M, l=0.5M, σ=1.9M, J_z=0.8M^2
x/M
y/M

$X^i + \epsilon^{ijk} S_j x_k / r^\wedge 3(1 - W)$ on $z = 0$

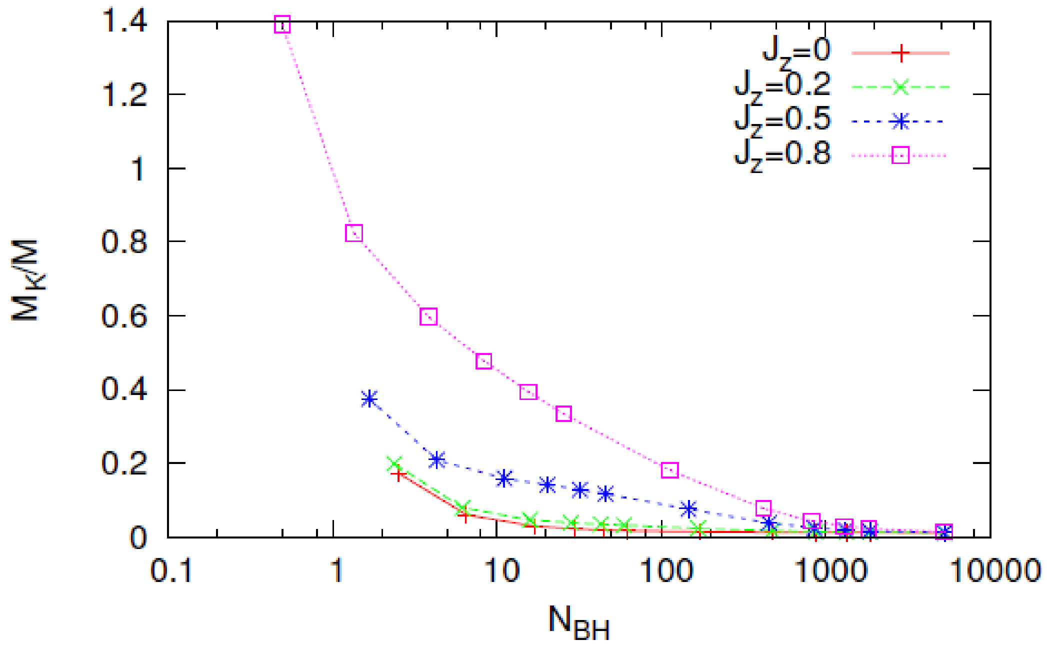


Analysis I

The cosmology from the Eulerian observers located at box sides.

Cosmology at the box surfaces

- We regard the Eulerian observers (n^a) at the box surfaces as cosmological comoving observers.
- H_{eff} : Hubble expansion of n^a
- By effective Friedmann equation.
- $H_{eff}^2 = \frac{8\pi}{3}\frac{M}{V}\left(1+\frac{M_K}{M}\right) \to \frac{8\pi}{3}\frac{M}{V}$ (Einstein de Sitter with M/V)



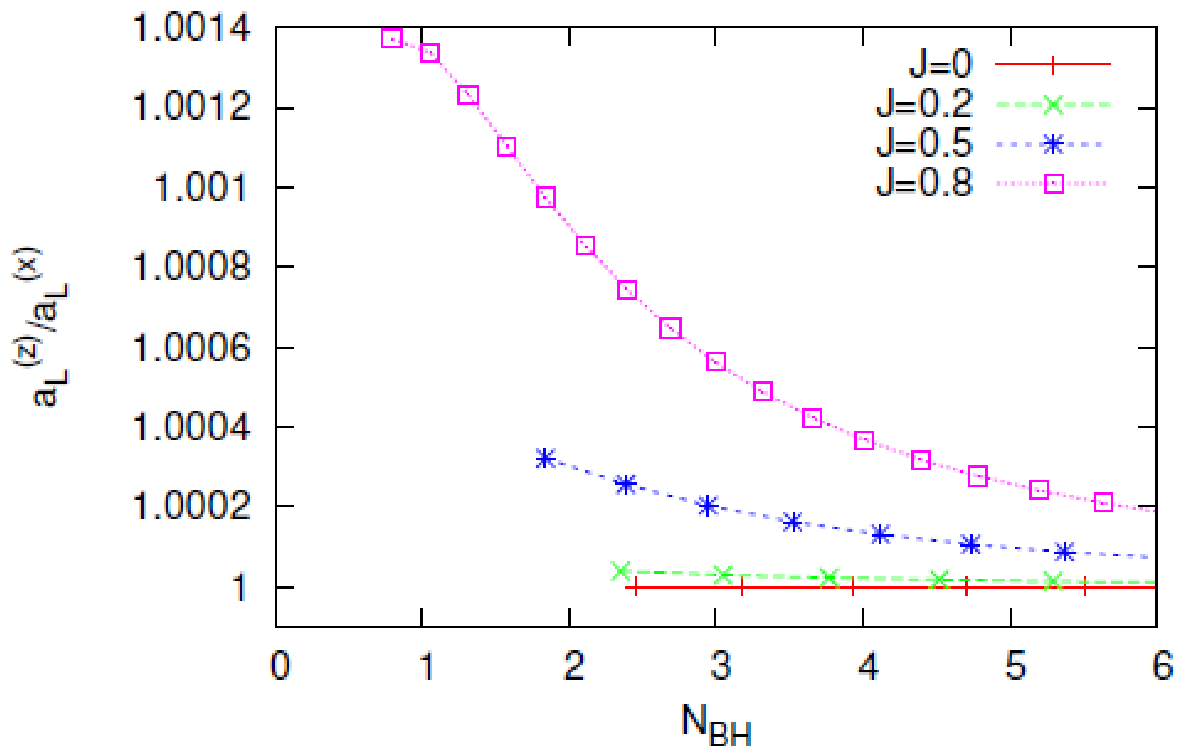
$$N_{BH} = \left(\frac{4\pi}{3}H_{eff}^{-3}\right)/V$$

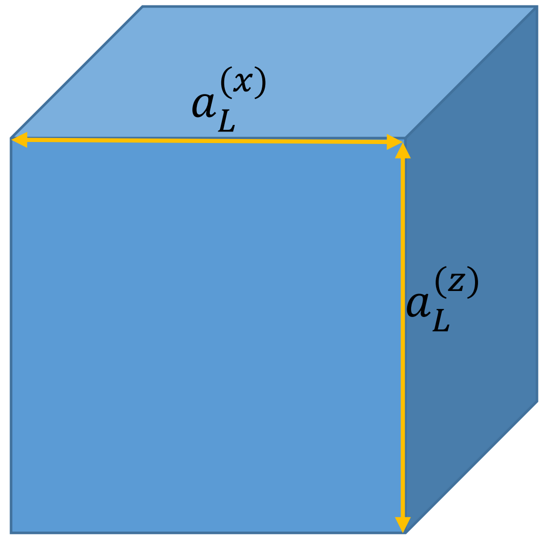
effective number of black holes in cosmological horizons

Analysis II

The cosmology from the Eulerian observers located at vertex.

Anisotropy of Physical Length of Edges





$$a_L^{(i)} = \int_{-L}^{L} \Psi^2\left(x^j = L, x^k = L\right) dx^i$$

Kasner Universe

- Bianchi Universe : Homogeneous, but anisotropic.
- Kasner Universe : Bianchi Type I
  - Spatially flat.
  - Admit global coordinate.
  - No vorticity of commoving observer.
  - $ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2$
  - $K = -\frac{p_1+p_2+p_3}{t}$

Averaged metric over the box surfaces

- For x^i, x^j, x^k such that $\epsilon_{ijk} \neq 0$,
- $\ell_i = \int_{-L}^{L} \Psi^2\left(x^j = L, x^k = L\right) dx^i$: physical length of $-L < x^i < L$
- $\mathcal{A}_{ij} = \int_{-L}^{L} \int_{-L}^{-L} \Psi^4\left(x^k = L\right) dx^i dx^j$: physical area of $-L < x^i, x^j < L$
- We define averaged metric $\bar{\gamma}_{ij}$ as
  - $\ell_i = \sqrt{\bar{\gamma}_{ii}}(2L)$
  - $\mathcal{A}_{ij} = \sqrt{\bar{\gamma}_{ii}\bar{\gamma}_{jj} - \bar{\gamma}_{ij}^2}(2L)^2$
- Eigenvalues of $\bar{\gamma}_{ij}$: λ_1, λ_2, λ_3

$\bar{\gamma}_{ij}$ mimics Kasner universe

- Let us we set
  - $t^{2p_i} = \lambda_i$
  - $K = -3H_{eff}$
- Then, we can get p_i, t.
- Numerical results : not yet.

Thank you for your listening