Area Law in de Sitter Spacetime with Topological Solition

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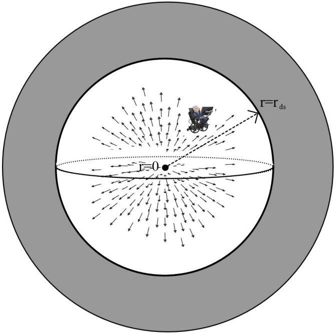
Goal

We want to examine the area law
for the general-dimensional de Sitter space time
deformed by a nontrivial matter source,
and estimate the change of the entropy.

- What kind of matter source?
- How?

How

- like this....



Why Area Law?

Area Law :

$$S = \frac{k_B c^3}{\hbar} \frac{A}{4G}$$

- quantum gravitational equation
- quantum gravitational states?
- area dependence $\longrightarrow$ holographic principle
- problem of universality :
  different approaches to QG (string theory, LQG, induced gravity, $\cdots$
  different microstates
  $\longrightarrow$ but same area law : why this result is universal?
  (check with nontrivial matter source?)
- information loss paradox : thermal radiation, evolution to mixed states.
  violates unitarity of evolution, forbidden in ordinary QM.

Black Hole Thermodynamics - Historical Review

- **In 1972, Bekenstein** proposed that the black hole area is proportional to the black hole entropy.

$$S_{\mathrm{BH}} = \frac{\ln 2}{2}\frac{A_{\mathrm{BH}}}{4G}$$

- **In 1973, Bardeen, Carter, and Hawking** suggested four laws of black hole thermodynamics.
  1. 0th : Constant κ (Constant T)
  2. 1st : $\mathrm{d}M = \frac{\kappa}{8\pi G}\mathrm{d}A + \Omega_{\mathrm{h}}\mathrm{d}J_{\mathrm{h}}$ ($dE = TdS$) $\to T?\ S?$ Classically $T = 0$
  3. 2nd : $\mathrm{d}A \geq 0$ ($\mathrm{d}S \geq 0$)
  4. 3rd : $\kappa \not\to 0$ by a finite sequence of operations.
- **In 1973, Hawking** fixed the proportionality between T and κ

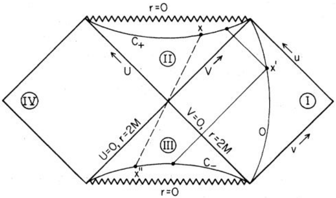
$$T_{\mathrm{BH}} = \frac{\kappa|_{r=r_{\mathrm{h}}}}{2\pi} \tag{2}$$

- **Area law (Bekenstein-Hawking Entropy)** :

$$\boxed{S_{\mathrm{BH}} = \frac{A_{\mathrm{BH}}}{4G}}$$

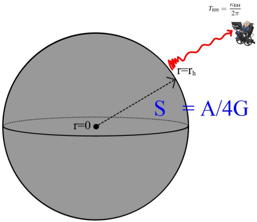
Black Hole Thermodynamics Extension (Schwarzschild BH)

- Hawking's semi-classical approach



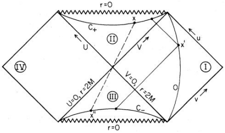
$$T_{\mathrm{BH}} = \frac{\kappa_{\mathrm{BH}}}{2\pi}$$
$$S_{\mathrm{BH}} = \frac{A_{\mathrm{BH}}}{4G}$$

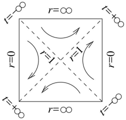
- The entropy accounts for the hidden information behind the horizon.



Black Hole Thermodynamics Extension (de Sitter spacetime)

- In 1976, Gibbons and Hawking extended the area law was extended to the cosmological horizon (the event horizon in the de Sitter spacetime).

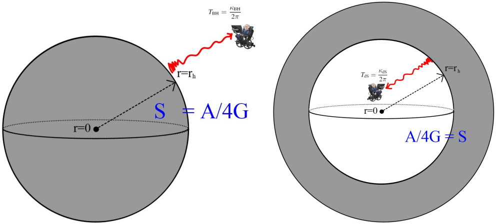




$$d(-E) = \frac{\kappa_{dS}}{8\pi G} dA_{dS}$$

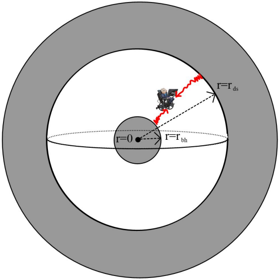
$$T_{dS} = \frac{\kappa_{dS}}{2\pi} \longrightarrow S_{dS} = \frac{A_{dS}}{4G}$$

- The entropy accounts for the hidden information behind the horizon.



Black Hole Thermodynamics Extension (SdS spacetime)

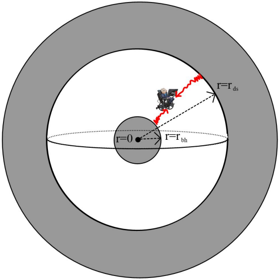
- The black hole is the simplest matter source which is parameterized with only global hairs (M,J,Q). Complicating information is hidden behind the horizon.

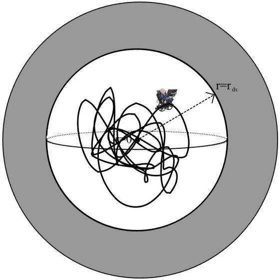


- There is no fixed temperature. It depends on the normalisation. (Standard normalisation, Bousso-Hawking normalisation...)

Black Hole Thermodynamics Extension (SdS spacetime)

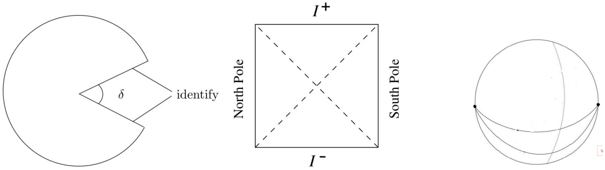
- Then, how about controlling more complicating matter source which is not hidden behind the black hole horizon? Can we see how the entropy behaves?
- If it deforms the geometry from the SdS or dS, the area law still holds?





Ex. Matter Distribution Without Horizon (SdS^3)

- An interesting and easy case : SdS_3, 3 dimensional Schwarzschild de Sitter spacetime.
- In 3 dimensional de Sitter space time, the degrees of freedom in the gravity side is same with that in the matter side.
- Localized matter at r=0 behaves like a point-like source rather than black hole with a horizon.
- Then the matter affects on the area law as a global effect, such as, a deficit angle. The geometry changes with the deficit angle.



- In this case, the area law could be derived easily, as Spradlin(2001) showed.

Ex. Matter Distribution Without Horizon (SdS^3)

- The temperature and the area law in SdS^3 are given by

$$T_{SdS^3} = \frac{\sqrt{1-8GE}}{2\pi}, \quad S_{SdS^3} = \frac{A^{H}_{SdS^3}}{4G} = \frac{\pi}{2G}\sqrt{1-8GE}, \quad (l \equiv 1)$$

- The result is obtained by taking the integration from the boundary where the spacetime is closed up with the 2π deficit angle.
- Note that this integration could be taken easily because in this model there is only the trivial mass parameter, E.
- This indicates that
  '**If any matter leads to the global effect on the horizon, we might calculate its entropy even in the spacetime with dimensions higher than three.**"

Our Model

- First, we will consider a matter which energy density goes as $1/r^2$ which is the maximum order we can consider as a field theory model.
- Even though the field energy is divergent when the radius goes to infinity, it will not change the background's vacuum dominant behavior.

$$\text{Energy density behavior : } \{\Lambda, -T^t{}_t\} \xrightarrow{r\to r_H \gg 1} \{\Lambda \gg (d-2)\frac{v^2}{2r^2}\}$$

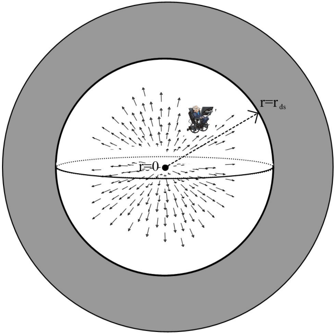
- For this consideration, let's choose a proper field configuration.
- Let's consider the field which has the same $O(N-1)$ rotation symmetry with the space. For example, we will consider a hedge hog shape.

$$\phi^i = \hat{r}^i \phi(r), \quad (i = 1, \cdots, d-2)$$

$\longrightarrow$ This leads to same energy behavior.
$\longrightarrow$ This scalar field will have divergent energy when r goes to infinity. Since this is not the finite energy case, there exists a topological soliton solution even in the higher dimension (Derrick-Hobart theorem).

Our Model

- Then how about the entropy changes from this deformation by the topological soliton?



Assumptions in our Model

Now, let's see if the effect of matter appears on the horizon as a global effect, such as, a deficit angle, and see it is possible to derive the area law in this case. For this work, we will assume the following to derive the area law with the scalar field we prepared in the previous slide.

1. Dimension : $d > 3$
2. Gravity theory : minimal, Einstein-Hilbert action with a positive cosmological constant

$$S_{\rm EH} = \int d^d x \sqrt{-g}\,(R - 2\Lambda)$$

3. Matter source : spherically symmetric static scalar field

$$\phi^i \equiv \hat{\phi}^i \phi,\ \hat{\phi}^i \hat{\phi}^i = 1,\ O(d-1) \Rightarrow\ \phi^i = \hat{r}^i \phi(r), \quad (i = 1, \cdots, d-1)$$

4. Field potential : Higgs potential which is chosen in a minimal shape for supporting static global topological defect

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2$$

Our Model (Action and Metric Ansatz)

- Action

$$S = \int d^d x \sqrt{-g} \left[\frac{1}{16\pi G}(R - 2\Lambda) - \left(\frac{g^{\mu\nu}}{2} \partial_\mu \phi^i \partial_\nu \phi^i + V(\phi) \right) \right]$$

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2$$

- Metric in the static coordinate

$$ds^2 = -e^{2\Omega(r)} A(r) dt^2 + \frac{dr^2}{A(r)} + r^2 d\Omega_{d-2}^2$$

where

$$d\Omega_{d-2}^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \cdots + \sin^2\theta_1 \cdots \sin^2\theta_{d-3} d\theta_{d-2}^2$$

$$A(r) \equiv 1 - \Delta_{\mathrm{dS}} - \left(\frac{r}{l}\right)^2 = 1 - \frac{2(\#)GM(r)}{r^{d-3}} - \left(\frac{r}{l}\right)^2$$

$$\Delta_{\mathrm{dS}} = \frac{16\pi GM(r)}{(d-2)\Omega_{d-2} r^{d-3}}, \quad (\#) = \frac{8\pi}{(d-2)\Omega_{d-2}}$$

Equations of Motion and our Strategy

The equations of motion is given by

$$A\phi'' + A\phi'\left[\ln(r^{d-2}Ae^{\Omega})\right]' - \frac{d-2}{r^2}\phi = \frac{dV}{d\phi} = \lambda\phi(\phi^2 - v^2)$$

$$\frac{d-2}{r^{d-2}}\left[r^{d-3}(1-A)\right]' - 2\Lambda = 8\pi G\left[\frac{d-2}{r^2}\phi^2 + A(\phi')^2 + 2V\right]$$

$$\frac{d-2}{r}\Omega' = 8\pi G(\phi')^2$$

By using the asymptotic solution and the first law of thermodynamics, we will derive the entropy of the deformed system.

$$d(-E_{\delta\mathrm{dS}}) + P_{\delta\mathrm{dS}}d(-V_{\delta\mathrm{dS}}) = T_{\delta\mathrm{dS}}dS_{\delta\mathrm{dS}} \;\rightarrow\; S_{\delta\mathrm{dS}} = \frac{A^{\mathrm{H}}_{\delta\mathrm{dS}}}{4G_d}$$

Note that since the system has the pressure, we should consider PdV term. [Padmanabhan, 2002].

Boundary Conditions and Analysis

To solve the equations, we need to consider boundary conditions.

1. $\phi(r \to 0)$ =0
2. $\phi(r \to r_{\rm H}) = v$
3. $M(r \to 0) = 0$
4. $\Omega(r \to r_{\rm H}) = 0$

(1) to have a well-defined field

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4. $\Omega(r \to r_{\mathrm{H}}) = 0$

(2) When the size of the horizon $(r_{\mathrm{H}} \sim l) \gg$ the core of a topological defect $(r_{\mathrm{core}} \sim 1/\sqrt{\lambda}v)$, ϕ has the value of the vacuum.

$\to$ $E|_{\text{topological}}$

$$\int d^{d-1}x \left(T_{tt} = \frac{d-2}{r^2}\phi^2 + A(\phi')^2 + 2V \right)$$

$$\xrightarrow{r \to r_{\mathrm{H}}} \begin{cases} |\phi| \to v \\ |\phi'| \to 0 \end{cases}$$

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(3) No singularity, No black hole horizon

Boundary Conditions and Analysis

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3. $M(r \to 0) = 0$
4. $\Omega(r \to r_{\mathrm{H}}) = 0$

(4) for the correct calculation of temperature at the de Sitter horizon. This boundary condition can differently be chosen since the above one will be achieved by a rescaling of the time variable from the new one.

Asymptotic Solutions - Near the Origin

By considering b.c., when $r \to 0$, the solutions for $\phi(r)$, $A(r)$, $\Omega(r)$ are given by

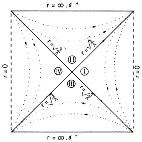
$$\frac{\phi(r)}{v} \approx \phi_0 r + \cdots ,$$

$$A(r) \approx 1 - \left[\frac{1}{\lambda v^2 l^2} + \delta \frac{d-3}{d-2} \left(\frac{1}{\lambda v^2} \phi_0^2 + \frac{1}{2(d-1)} \right) \right] (\sqrt{\lambda} v r)^2 + \cdots ,$$

$$\Omega(r) \approx \Omega_0 + \frac{\delta}{2} \frac{d-3}{d-2} \frac{\phi_0^2}{(\sqrt{\lambda} v)^2} (\sqrt{\lambda} v r)^2 + \cdots ,$$

where we defined $\delta = 8\pi G v^2/(d-3)$, which will be used for deficit angles later. From this, we know that

- The geometry near the origin is Minkowski space time



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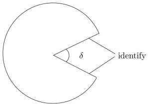
$$\Omega(r) \approx \Omega_0 + \frac{\delta}{2}\frac{d-3}{d-2}\frac{\phi_0^2}{(\sqrt{\lambda} v)^2}(\sqrt{\lambda} v r)^2 + \cdots ,$$

where we defined $\delta = 8\pi G v^2/(d-3)$, which will be used for deficit angles later. From this, we know that

- No deficit angle by the mild energy configuration



Asymptotic Solutions - Near the Horizon

By considering b.c., when $r \to r_{\rm H}$, the solutions for $\phi(r)$, $A(r)$, $\Omega(r)$ are given by

$$\frac{\phi(r)}{v} \approx 1 - \frac{d-2}{2\left(1+\frac{3-d}{\lambda v^2 l^2}\right)} \frac{1}{(\sqrt{\lambda} v r)^2} + \cdots ,$$
$$A(r) \approx -\left(\frac{r}{l}\right)^2 + 1 - \delta + \cdots ,$$
$$\Omega(r) \approx -\frac{(d-2)(d-3)}{4\left(1+\frac{3-d}{\lambda v^2 l^2}\right)^2} \delta \frac{1}{(\sqrt{\lambda} v r)^4} + \cdots .$$

From this result, we find that

$$ds^2 = -\left[1 - \delta - \left(\frac{r}{l}\right)^2\right] dt^2 + \frac{dr^2}{1-\delta-\left(\frac{r}{l}\right)^2} + r^2 d\Omega_{d-2}$$
$$\curvearrowleft \; t' = \sqrt{1-\delta}\, t, \; r' = r/\sqrt{1-\delta}$$
$$= -\left[1 - \left(\frac{r'}{l}\right)^2\right] dt'^2 + \frac{dr'^2}{1-\left(\frac{r'}{l}\right)^2} + r'^2 (1-\delta) d\Omega_{d-2}^2 .$$

Asymptotic Solutions - Near the Horizon

$$
\begin{aligned}
ds^2 &= -\left[1-\delta-\left(\frac{r}{l}\right)^2\right]dt^2 + \frac{dr^2}{1-\delta-\left(\frac{r}{l}\right)^2} + r^2 d\Omega_{d-2} \\
&\curvearrowleft \; t' = \sqrt{1-\delta}\,t, \; r' = r/\sqrt{1-\delta} \\
&= -\left[1-\left(\frac{r'}{l}\right)^2\right]dt'^2 + \frac{dr'^2}{1-\left(\frac{r'}{l}\right)^2} + r'^2(1-\delta)d\Omega^2_{d-2}
\end{aligned}
$$

By looking at this asymptotic metric form, we see

- Geometry near the horizon
  $\rightarrow$ A deficit angle Δ_{deficit} appeared in the dS_d (whole geometry:$\delta\mathrm{dS}$),

$$
\Delta_{\text{deficit}} = \Omega_{d-2}\left(1-(1-\delta)^{\frac{d-2}{2}}\right) \quad \left(\approx \Omega_{d-2}\frac{d-2}{2}\delta + \mathcal{O}(\delta^2)\right. \text{ for small } \delta \ll 1)
$$

$\rightarrow$ Since $\delta = 8\pi G v^2/(d-3)$, the positive deficit angle grows as v^2.

Asymptotic Solutions - Near the Horizon

$$
\begin{aligned}
ds^2 &= -\left[1-\delta-\left(\frac{r}{l}\right)^2\right]dt^2 + \frac{dr^2}{1-\delta-\left(\frac{r}{l}\right)^2} + r^2 d\Omega_{d-2} \\
&\curvearrowleft\ t' = \sqrt{1-\delta}\,t,\ r' = r/\sqrt{1-\delta} \\
&= -\left[1-\left(\frac{r'}{l}\right)^2\right]dt'^2 + \frac{dr'^2}{1-\left(\frac{r'}{l}\right)^2} + r'^2(1-\delta)d\Omega_{d-2}^2
\end{aligned}
$$

By looking at this asymptotic metric form, we see

- Horizon radius, $r_{\rm H}$, is shifted by

$$
r'_{\rm H} = l \ \rightarrow\ r_{\rm H} = \sqrt{1-\delta}\,l = \sqrt{1-\frac{8\pi G v^2}{d-3}}\,l \qquad (17)
$$

Asymptotic Solutions - Near the Horizon

$$
\begin{aligned}
ds^2 &= -\left[1-\delta-\left(\frac{r}{l}\right)^2\right]dt^2 + \frac{dr^2}{1-\delta-\left(\frac{r}{l}\right)^2} + r^2 d\Omega_{d-2} \\
&\curvearrowleft\ t' = \sqrt{1-\delta}\,t,\ \ r' = r/\sqrt{1-\delta} \\
&= -\left[1-\left(\frac{r'}{l}\right)^2\right]dt'^2 + \frac{dr'^2}{1-\left(\frac{r'}{l}\right)^2} + r'^2(1-\delta)d\Omega_{d-2}^2
\end{aligned}
$$

By looking at this asymptotic metric form, we see

- Horizon area, $A^{\mathrm{H}}_{\delta\mathrm{dS}}$

$$A^{\mathrm{H}}_{\delta\mathrm{dS}} = r_{\mathrm{H}}^{d-2}\Omega_{d-2} = l^{d-2}(1-\delta)^{\frac{d-2}{2}}\Omega_{d-2} \tag{17}$$

Temperature of the Horizon

Now let's calculate the thermodynamic quantities of our spacetime. First, let's see the temperature.

$$ds^2 = -\left[1-\delta-\left(\frac{r}{l}\right)^2\right]dt^2 + \frac{dr^2}{1-\delta-\left(\frac{r}{l}\right)^2} + r^2 d\Omega_{d-2}$$

$$\curvearrowleft \; t' = \sqrt{1-\delta}\,t, \; r' = r/\sqrt{1-\delta}$$

$$= -\left[1-\left(\frac{r'}{l}\right)^2\right]dt'^2 + \frac{dr'^2}{1-\left(\frac{r'}{l}\right)^2} + r'^2(1-\delta)d\Omega^2_{d-2}$$

Local Rindler Temperature near the horizon, $T_{\mathrm{dS}} = \frac{\kappa}{2\pi} = \frac{\sqrt{1-\delta}}{2\pi l}$ where the surface gravity, κ, is obtained by

$$\kappa^2 = -\frac{1}{2}(\nabla_\mu K_\nu)(\nabla^\mu K^\nu) = \cdots = \frac{g^{rr}}{4}\frac{(\partial_r g_{tt})^2}{g_{tt}} \xrightarrow{r\to r_{\mathrm{H}}} \frac{(\partial_r g_{tt})^2}{4}$$

$$\Rightarrow \kappa(\kappa_{\mathrm{H}}) \approx \frac{r_{\mathrm{H}}}{l^2} = \frac{\sqrt{1-\delta}}{l}$$

Entropy Calculation - Subtle Points

Since we have T, dE, by using the thermodynamic law,

$$\boxed{d(-E) + Pd(-V) = TdS}$$

we can obtain the entropy corresponding the hidden degrees of freedom behind the horizon.

- Note that we should use the negative value of the energy which corresponds to degrees in the opposite pole.
- Since the system has pressure on the horizon, we should consider the PdV term in the first law. [Padmanabhan, 2002]
- Note that PdV has a minus sign here: The volume of the hidden area decreases when the volume inside the horizon increases.
- And we can not integrate dS from $S = 0$ value, since the closed-up spacetime breaks our approximation condition.
  $\rightarrow$ We will integrate dS from the pure de Sitter entropy where $v = 0$.

Entropy Calculation with dS boundary condition

As in the previous points, we will calculate the entropy as,

$$S_{\delta \mathrm{dS}} = \Delta S_{\delta \mathrm{dS}} + S_{\mathrm{dS}}$$

where

$$S_{\mathrm{dS}} = \frac{A_{\mathrm{dS}}^{\mathrm{H}}}{4G} = \frac{l^{d-2}\Omega_{d-2}}{4G}$$

From $d(-E) + Pd(-V) = TdS$, we get $\Delta S_{\delta \mathrm{dS}}$ as,

$$E_{\delta \mathrm{dS}} \approx \Omega_{d-2}\frac{d-2}{d-3}\frac{v^2}{2}r_{\mathrm{H}}^{d-3} = \Omega_{d-2}\frac{d-2}{16\pi G}l^{d-3}\delta(1-\delta)^{\frac{d-3}{2}}$$

$$P_{\delta \mathrm{dS}} = T^r{}_r \approx -\frac{d-2}{2}\frac{v^2}{r^2}, \quad P_{\delta \mathrm{dS}}\mathrm{d}(-V_{\delta \mathrm{dS}}) = (d-2)\frac{v^2}{2r_{\mathrm{h}}^2}\Omega_{d-2}r_{\mathrm{h}}^{d-2}\mathrm{d}r_{\mathrm{h}}$$

$$\boxed{\Delta S_{\delta \mathrm{dS}} = \frac{A_{\mathrm{dS}}}{4G}\left(-\frac{d-2}{2}\right)(1-\delta)^{\frac{d-4}{2}}\mathrm{d}\delta}$$

Result : Area Law for the Distorted dS w/ Topological Defects

Then the entropy for the deformed system is given by

$$S_{\delta \mathrm{dS}} = S_{\mathrm{dS}} + \Delta S_{\delta \mathrm{dS}} = S_{\mathrm{dS}} + \int_{S(\delta=0)}^{S(\delta)} dS_{\delta \mathrm{dS}}$$
$$= \frac{A^{\mathrm{h}}_{\mathrm{dS}}}{4G}(1-\delta)^{\frac{d-2}{2}} \qquad (21)$$

$$\boxed{S_{\delta \mathrm{dS}} = \frac{A^{\mathrm{h}}_{\delta \mathrm{dS}}}{4G} = \frac{1}{4G}\ell^{d-2}\Omega_{d-2}(1-\delta)^{\frac{d-2}{2}}}$$

Therefore, the area law still holds in the deformed system. As we expected, putting the non-trivial matter distribution leads the negative contribution to the entropy and in the case of the topological soliton the entropy changes with a factor of the solid deficit angle.

Conclusion

- When $\Lambda \neq 0$, especially, when $\Lambda > 0$, in the general dimensional spacetime, by adding a nontrivial matter source, we examined the entropy change.
- Since we have the non-trivial matter distribution example which has the exact expression for the entropy behavior in the classical(or semi-classical level), we could investigate more about its quantum origin in the subsequent research.