Entropy of Composite of Black Hole and Topological Soliton in Arbitrary Dimensions

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Quantum Gravity

4D Einstein-Hilbert action

- perturbation without matter field
  with matter field

Quantum Field Theory in Curved Spacetime

matter

quantum
(particle)

$M_{\text{matter}} \leq 1\text{TeV}$

gravity

classical
(background geometry)

$M_{\text{Pl}} \sim 10^{15}\text{TeV}$

Quantum Field Theory in Curved Spacetime

matter	gravity
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Hawking radiation

Quantum Field Theory in Curved Spacetime

Hawking radiation

$$\Downarrow$$

black hole thermodynamics

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Hawking radiation

$$\Downarrow$$

black hole thermodynamics

$$\Downarrow$$

$$S = \frac{A}{4}$$

(Hi)story of Area Law

- classical
  - (Einstein) equation $\Rightarrow$ Gibbons-Hawking
  - Nöther charge
- semiclassical
  (free) particles in BH background
  - brick-wall
  - entanglement approach
  - thermal atmosphere
  - induced gravity
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(Hi)story of Area Law

- string theory

counting of microstates

$\swarrow !$ $\searrow ?$

$$S = \frac{A}{4}$$

local physics near a black hole

Information Paradox?

$$\begin{pmatrix} \text{unitary} \\ \text{quantum mechanical} \\ \text{processes} \end{pmatrix} \qquad \begin{pmatrix} \text{black hole} \\ \text{thermodynamics} \\ \text{processes} \end{pmatrix}$$

$$\text{pure} \longrightarrow \text{pure} \qquad \text{vs.} \qquad \text{pure} \longrightarrow \text{mixed}$$

Introduction

Q. Static vacuum solution of the Einstein's equation with rotational symmetry?

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Schwarzschild solution

: characterized by the radius of horizon

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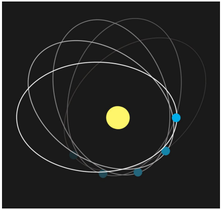
- size of an object $>$ radius of horizon
  $\longrightarrow$ Precession of the Mercury
- size of an object $<$ radius of horizon
  $\longrightarrow$ Black hole



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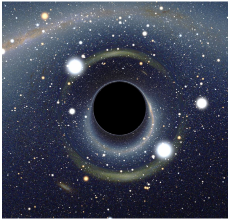

Introduction

Q. Resemblance between black hole mechanics & thermodynamics?

⋮

- Black hole entropy

$$\boxed{S_{\rm BH} \propto A_{\rm h}}$$

Introduction

- In 1972, Bekenstein,

$$S_{bh} = (\tfrac{1}{2}\ln 2/4\pi)\, k c^3 \hbar^{-1} G^{-1} A$$

$$= (1.46 \times 10^{48}\ \mathrm{erg}\ {}^\circ\mathrm{K}^{-1}\,\mathrm{cm}^{-2})A\ , \qquad (17)$$

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Black Holes and Entropy*

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(Received 2 November 1972)

There are a number of similarities between black-hole physics and thermodynamics. Most striking is the similarity in the behaviors of black-hole area and of entropy: Both quantities tend to increase irreversibly. In this paper we make this similarity the basis of a thermodynamic approach to black-hole physics. After a brief review of the elements of the theory of information, we discuss black-hole physics from the point of view of information theory. We show that it is natural to introduce the concept of black-hole entropy as the measure of information about a black-hole interior which is inaccessible to an exterior observer. Considerations of simplicity and consistency, and dimensional arguments indicate that the black-hole entropy is equal to the ratio of the black-hole area to the square of the Planck length times a dimensionless constant of order unity. A different approach making use of the specific properties of Kerr black holes and of concepts from information theory leads to the same conclusion, and suggests a definite value for the constant. The physical content of the concept of black-hole entropy derives from the following generalized version of the second law: When common entropy goes down a black hole, the common entropy in the black-hole exterior plus the black-hole entropy never decreases. The validity of this version of the second law is supported by an argument from information theory as well as by several examples.

Introduction

- In 1973, Bardeen, Carter, and Hawking,

  "The Four Laws of Black Hole Mechanics"

  1. 0th : Constant κ on the horizon $\Leftrightarrow$ (Constant T)
  2. 1st : $\mathrm{d}M = \frac{\kappa}{8\pi G}\mathrm{d}A + \Omega_{\mathrm{h}}\mathrm{d}J_{\mathrm{h}}$ $\Leftrightarrow$ ($dE = TdS$) $\longrightarrow T?\ S?$ Classically $T = 0$
  3. 2nd : $\mathrm{d}A \geq 0$ $\Leftrightarrow$ ($\mathrm{d}S \geq 0$)
  4. 3rd : $\kappa \not\to 0$ by a finite sequence of operations.

* surface gravity : for time-like Killing vector K^{μ}

$$\kappa \equiv -\frac{1}{2}(\nabla_{\mu}K_{\nu})(\nabla^{\mu}K^{\nu}),$$
$$= \lim Va \text{ (red-shifted four-acceleration)}$$

"In the case of a static black hole, Va is the force that must be exerted at infinity to hold a unit test mass in place." [Wald,1985]

Q. Resemblance between black hole mechanics & thermodynamics?

$$\vdots$$

- Black hole entropy

$$S_{\rm BH} \propto A_{\rm h}$$

- Temperature? $\longrightarrow$ Idea of quantum theory

$$\mathrm{d}M = \underbrace{\frac{\kappa}{8\pi G}}_{=T?}\overbrace{\mathrm{d}A}^{=\mathrm{d}S?} + \Omega_{\rm h}\mathrm{d}J$$

Introduction

- In 1973, Hawking fixed the proportionality between T and κ
- Hartle-Hawking's semi-classical approach (1976) :

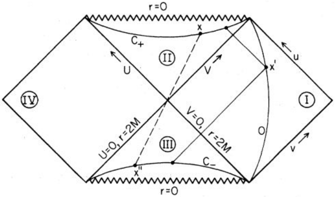
$$\mathrm{Prob}(\text{emission of particle by pair creations in a mode with energy } E)$$
$$= e^{-\beta E}\,\mathrm{Prob}(\text{absorption in the same mode})$$

$$\xrightarrow{\text{analytic continuation in the Euclidean path integral}}$$

$$\Longrightarrow \ \beta = \frac{2\pi}{\kappa}$$

"In equilbirum, the rate of emission particles by the black hole must exactly equal the rate of absorption."

Hawking Temperature : $\boxed{T_{\rm BH} = \frac{\kappa|_{r=r_{\rm h}}}{2\pi}}$



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$$\mathrm{d}M = \underbrace{\frac{\kappa}{8\pi G}}_{=T/4G} \overbrace{\mathrm{d}A}^{=\mathrm{d}S\times 4G} + \Omega_{\mathrm{h}} \mathrm{d}J$$

- **Area Law :**

$$\boxed{S_{\mathrm{BH}} = \frac{c^3 A_{\mathrm{h}}}{4G\hbar}}$$

Area Law

- d-dimensional curved spacetime ($d > 3$)

$$ds^2 = -e^{2\Psi(r)}A(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{A(r)} + r^2 d\Omega_{d-2}^2$$

: static + rotational symmetry

- Vacuum : $T^{\mu}_{\ \nu} = 0$
- Solution : $\Psi(r) = 0,\ A(r) = 1 - \frac{16\pi GM}{(d-2)\Omega_{d-2}} \frac{1}{r^{d-3}}$
- Radius of horizon : $r_{\mathrm{b}} = \left[\frac{16\pi GM}{(d-2)\Omega_{d-2}}\right]^{\frac{1}{d-3}}$

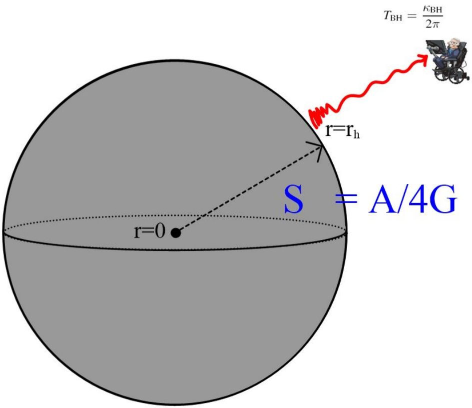
  $\rightarrow$ Area of the horizon : $\mathcal{A}_{\mathrm{b}} = r_{\mathrm{b}}^{d-2}\Omega_{d-2}$

- Temperature at the horizon : $T_{\mathrm{b}} = \frac{\kappa_{\mathrm{b}}}{2\pi} = \frac{d-3}{4\pi}\left[\frac{(d-2)\Omega_{d-2}}{16\pi GM}\right]^{\frac{1}{d-3}}$
- Thermodynamics law : $\mathrm{d}M = T\mathrm{d}S$ $\rightarrow$ Area law

$$S = \frac{\mathcal{A}_{\mathrm{b}}}{4G}$$

Area Law

- The entropy accounts for the hidden information behind the horizon.



Issue

Among various issues,

Universality?

- Dirts : $T^{\mu}_{\nu} \neq 0 \;\rightarrow$ spin, charge, $\cdots$
- Classical vs. quantum
- Methods
- $\cdots$

Matter distribution :

$$-T^t{}_t = -T^r{}_r = \cdots \sim \frac{d-2}{2}\frac{v^2}{r^2}$$

- Energy :

$$E(R) \sim \int^R drr^{d-2}(-T^t{}_t) \sim \int^R drr^{d-2}\frac{1}{r^2} \overset{d>3}{\sim} R^{d-3} \overset{R\to\infty}{\longrightarrow} \infty$$

- nonBPS

$$\boxed{-T^t{}_t = \frac{(d-2)(d-3)}{16\pi G}\frac{\delta}{r^2}}$$

Dirt

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Geometry

- Solution of the Einstein's equation :

$$A(r) = 1 - \delta - \frac{C}{r^{d-3}}, \quad \left(0 \le \delta < 1,\ C = \frac{16\pi GM}{(d-2)\Omega_{d-2}}\right)$$

: constant shift

- Geometry $\to$ deficit solid angle

$$\Delta_{d-2} \equiv \Omega_{d-2} - \Omega'_{d-2} = \boxed{\Omega_{d-2}\left[1 - (1-\delta)^{\frac{d-2}{2}}\right]}$$

- Black hole horizon :

$$r_{\mathrm{h}} = \left(\frac{1}{1-\delta}\right)^{\frac{1}{d-3}} r_{\mathrm{b}}$$

$$\mathcal{A}_{\mathrm{h}} = \left(\frac{1}{1-\delta}\right)^{\frac{d-2}{d-3}} \mathcal{A}_{\mathrm{b}}.$$

Thermodynamics

All thermodynamic quantities are changed :

- Temperature :

$$T_{\rm h} = \frac{\kappa_{\rm h}}{2\pi} = (1-\delta)^{\frac{d-2}{d-3}} T_{\rm b}$$

- Pressure :

$$P = T^r{}_r = -\frac{(d-2)(d-3)}{16\pi G}\frac{\delta}{r^2}$$

- Thermodynamic law :

$$T_{\rm h}{\rm d}S_{\rm h} = {\rm d}E_{\rm h} + P_{\rm h}{\rm d}V_{\rm h}$$

$\rightarrow$ Change of energy $\&$ work

$$E_{\rm h} = \Omega_{d-2}\int_0^{r_{\rm h}} dr r^{d-2}(-T^t{}_t) = \frac{\mathcal{A}_{\rm b}}{r_{\rm b}}\frac{d-2}{16\pi G}\frac{\delta}{1-\delta}$$

$$P_{\rm h}{\rm d}V_{\rm h} = -\frac{\mathcal{A}_{\rm b}}{r_{\rm b}}\frac{d-2}{16\pi G}\frac{\delta\,{\rm d}\delta}{(1-\delta)^2}$$

Area Law

Area Law

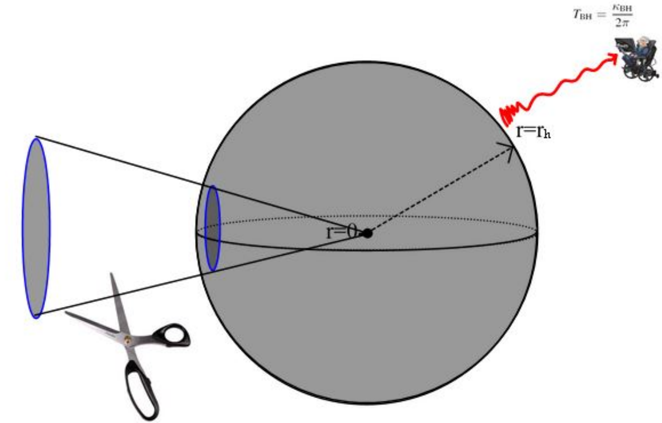
- Thermodynamic law $\longrightarrow$ area law

$$\boxed{S_{\delta \mathrm{BH}} = \frac{\mathcal{A}_{\mathrm{h}}}{4G}}$$

: exact

Area Law with Deficit Solid Angle

- The entropy accounts for the hidden information behind the horizon.



Source

- Global topological soliton of hedgehog ansatz :

$$\boxed{\phi^i = \hat{r}^i \phi(r)}$$

Model :

$$S_\phi = \int d^d x \sqrt{-g} \left[-\frac{g^{\mu\nu}}{2} \partial_\mu \phi^i \partial_\nu \phi^i - V(\phi) \right]$$

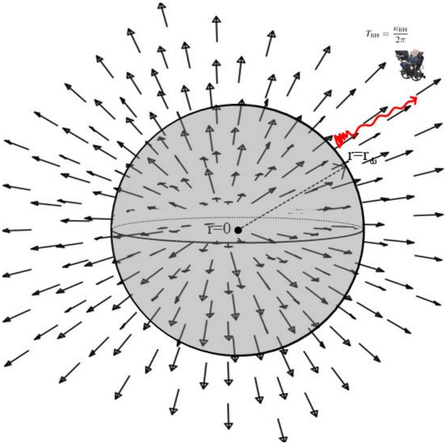
- $\boxed{r_{\rm h} \gg 1/\sqrt{\lambda} v}$ with $\delta = \dfrac{8\pi G v^2}{d-3}$

Einstein equation :

$$\frac{d-2}{r} \frac{d\Psi}{dr} = 8\pi G \left(\frac{d\phi}{dr} \right)^2 \underset{r \geq r_{\rm h}}{\overset{\phi \approx v}{\longrightarrow}} 0$$

$$\frac{1}{r^{d-2}} \frac{d}{dr} \left[r^{d-3} (1 - A) \right] = 8\pi G \left[\frac{d-2}{r^2} \phi^2 + A(\phi')^2 + 2V(\phi) \right] \underset{r \geq r_{\rm h}}{\overset{\phi \approx v}{\longrightarrow}} \frac{8\pi G v^2}{r^2}$$

Source



Hair

- Goldstone degree $\overset{\text{winding}}{\longrightarrow}$ long topological hair

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = (d-2)(d-3)\left[(d-1)(d-2)\left(\frac{C}{r^{d-1}}\right)^2 + 4\frac{C}{r^{d-1}}\frac{\delta}{r^2} + 2\left(\frac{\delta}{r^2}\right)^2\right]$$

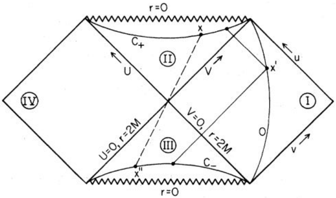
- Higgs degree $\longrightarrow$ short scalar hair

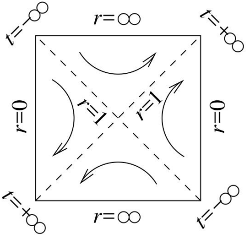
De Sitter Entropy

- In 1976, Gibbons and Hawking extended the area law to the cosmological horizon (the event horizon in the de Sitter spacetime).

$$d(-E) = \frac{\kappa_{dS}}{8\pi G} dA_{dS}, \quad T_{dS} = \frac{\kappa_{dS}}{2\pi}$$

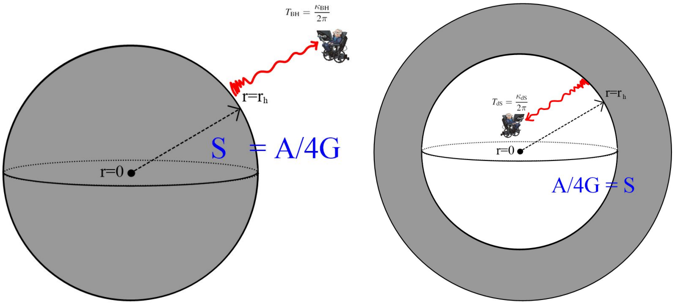
$$\boxed{S_{dS} = \frac{A_{dS}}{4G}}$$





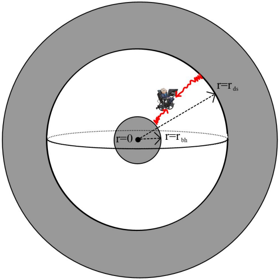
De Sitter Entropy

- The entropy accounts for the hidden information behind the horizon.



De Sitter Entropy

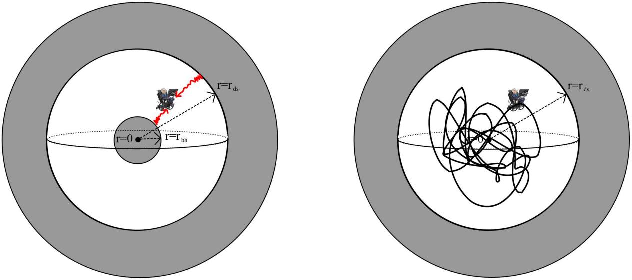
- The black hole is the simplest matter source which is parameterized with only global hairs (M,J,Q). Complicating information is hidden behind the horizon.



- There is no fixed temperature. It depends on the normalisation. (Standard normalisation, Bousso-Hawking normalisation...)

De Sitter Entropy

- Then, how about controlling more complicating matter source which is not hidden behind the black hole horizon? Can we see how the entropy behaves?
- If it deforms the geometry from the SdS or dS, the area law still holds?



Source

- First, we will consider a matter which energy density goes as $1/r^2$ which is the maximum order we can consider as a field theory model.
- Even though the field energy is divergent when the radius goes to infinity, it will not change the background's vacuum dominant behavior.

$$\text{Energy density behavior : } \{\Lambda, -T^t{}_t\} \xrightarrow{r\to r_H \gg 1} \{\Lambda \gg (d-2)\frac{v^2}{2r^2}\}$$

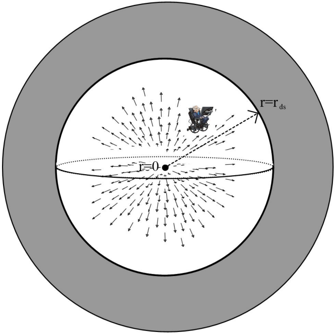
- Let's consider the field which has the same $O(N-1)$ rotation symmetry with the space. For example, we will consider a hedge hog shape.

$$\phi^i = \hat{r}^i \phi(r), \quad (i = 1, \cdots, d-2)$$

$\longrightarrow$ This leads to same energy behavior.
$\longrightarrow$ This scalar field will have divergent energy when r goes to infinity. Since this is not the finite energy case, there exists a topological soliton solution even in the higher dimension (Derrick-Hobart theorem).

Source

- Then how about the entropy changes from this deformation by the topological soliton?



Source

Now, let's see if the effect of matter appears on the horizon as a global effect, such as, a deficit angle, and see it is possible to derive the area law in this case. For this work, we will assume the following to derive the area law with the scalar field we prepared in the previous slide.

1. Dimension : $d > 3$
2. Gravity theory : minimal, Einstein-Hilbert action with a positive cosmological constant

$$S_{\rm EH} = \int d^d x \sqrt{-g}\,(R - 2\Lambda)$$

3. Matter source : spherically symmetric static scalar field

$$\phi^i \equiv \hat{\phi}^i \phi,\ \hat{\phi}^i \hat{\phi}^i = 1,\ O(d-1) \Rightarrow\ \phi^i = \hat{r}^i \phi(r), \quad (i = 1, \cdots, d-1)$$

4. Field potential : Higgs potential which is chosen in a minimal shape for supporting static global topological defect

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2$$

Source

- Action

$$S = \int d^d x \sqrt{-g} \left[\frac{1}{16\pi G}(R - 2\Lambda) - \left(\frac{g^{\mu\nu}}{2} \partial_\mu \phi^i \partial_\nu \phi^i + V(\phi) \right) \right]$$
$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v^2)^2$$

- Metric in the static coordinate

$$ds^2 = -e^{2\Omega(r)} A(r) dt^2 + \frac{dr^2}{A(r)} + r^2 d\Omega_{d-2}^2$$

where

$$d\Omega_{d-2}^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \cdots + \sin^2\theta_1 \cdots \sin^2\theta_{d-3} d\theta_{d-2}^2$$
$$A(r) \equiv 1 - \Delta_{\mathrm{dS}} - \left(\frac{r}{l}\right)^2 = 1 - \frac{2(\#)GM(r)}{r^{d-3}} - \left(\frac{r}{l}\right)^2$$
$$\Delta_{\mathrm{dS}} = \frac{16\pi GM(r)}{(d-2)\Omega_{d-2} r^{d-3}}, \quad (\#) = \frac{8\pi}{(d-2)\Omega_{d-2}}$$

Equations of Motion and our Strategy

The equations of motion is given by

$$A\phi'' + A\phi'\left[\ln(r^{d-2}Ae^{\Omega})\right]' - \frac{d-2}{r^2}\phi = \frac{dV}{d\phi} = \lambda\phi(\phi^2 - v^2)$$
$$\frac{d-2}{r^{d-2}}\left[r^{d-3}(1-A)\right]' - 2\Lambda = 8\pi G\left[\frac{d-2}{r^2}\phi^2 + A(\phi')^2 + 2V\right]$$
$$\frac{d-2}{r}\Omega' = 8\pi G(\phi')^2$$

By using the asymptotic solution and the first law of thermodynamics, we will derive the entropy of the deformed system.

$$d(-E_{\delta \mathrm{dS}}) + P_{\delta \mathrm{dS}} d(-V_{\delta \mathrm{dS}}) = T_{\delta \mathrm{dS}} dS_{\delta \mathrm{dS}} \;\rightarrow\; S_{\delta \mathrm{dS}} = \frac{A^{\mathrm{H}}_{\delta \mathrm{dS}}}{4G_d}$$

Note that since the system has the pressure, we should consider PdV term. [Padmanabhan, 2002].

Boundary Conditions and Analysis

To solve the equations, we need to consider boundary conditions.

1. $\phi(r \rightarrow 0)$ =0
2. $\phi(r \rightarrow r_{\mathrm{H}}) = v$
3. $M(r \rightarrow 0) = 0$
4. $\Omega(r \rightarrow r_{\mathrm{H}}) = 0$

(1) to have a well-defined field

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3. $M(r \to 0) = 0$
4. $\Omega(r \to r_{\rm H}) = 0$

(2) When the size of the horizon $(r_{\rm H} \sim l) \gg$ the core of a topological defect $(r_{\rm core} \sim 1/\sqrt{\lambda}v)$, ϕ has the value of the vacuum.

$\to \; E|_{\text{topological}}$

$$\int d^{d-1}x \left(T_{tt} = \frac{d-2}{r^2}\phi^2 + A(\phi')^2 + 2V \right)$$

$$\xrightarrow{r \to r_{\rm H}} \begin{cases} |\phi| \to v \\ |\phi'| \to 0 \end{cases}$$

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(3) No singularity, No black hole horizon

Boundary Conditions and Analysis

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(4) for the correct calculation of temperature at the de Sitter horizon. This boundary condition can differently be chosen since the above one will be achieved by a rescaling of the time variable from the new one.

Asymptotic Solutions - Near the Origin

By considering b.c., when $r \to 0$, the solutions for $\phi(r)$, $A(r)$, $\Omega(r)$ are given by

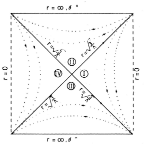
$$\frac{\phi(r)}{v} \approx \phi_0 r + \cdots ,$$

$$A(r) \approx 1 - \left[\frac{1}{\lambda v^2 l^2} + \delta \frac{d-3}{d-2}\left(\frac{1}{\lambda v^2}\phi_0^2 + \frac{1}{2(d-1)}\right)\right](\sqrt{\lambda} v r)^2 + \cdots ,$$

$$\Omega(r) \approx \Omega_0 + \frac{\delta}{2}\frac{d-3}{d-2}\frac{\phi_0^2}{(\sqrt{\lambda} v)^2}(\sqrt{\lambda} v r)^2 + \cdots ,$$

where we defined $\delta = 8\pi G v^2/(d-3)$, which will be used for deficit angles later. From this, we know that

- The geometry near the origin is Minkowski space time



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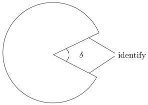
where we defined $\delta = 8\pi G v^2/(d-3)$, which will be used for deficit angles later. From this, we know that

- No deficit angle by the mild energy configuration



Asymptotic Solutions - Near the Horizon

By considering b.c., when $r \to r_{\rm H}$, the solutions for $\phi(r)$, $A(r)$, $\Omega(r)$ are given by

$$\frac{\phi(r)}{v} \approx 1 - \frac{d-2}{2\left(1+\frac{3-d}{\lambda v^2 l^2}\right)} \frac{1}{(\sqrt{\lambda} v r)^2} + \cdots ,$$

$$A(r) \approx -\left(\frac{r}{l}\right)^2 + 1 - \delta + \cdots ,$$

$$\Omega(r) \approx -\frac{(d-2)(d-3)}{4\left(1+\frac{3-d}{\lambda v^2 l^2}\right)^2} \delta \frac{1}{(\sqrt{\lambda} v r)^4} + \cdots .$$

From this result, we find that

$$ds^2 = -\left[1 - \delta - \left(\frac{r}{l}\right)^2\right] dt^2 + \frac{dr^2}{1-\delta-\left(\frac{r}{l}\right)^2} + r^2 d\Omega_{d-2}$$

$$\curvearrowleft \; t' = \sqrt{1-\delta}\, t, \; r' = r/\sqrt{1-\delta}$$

$$= -\left[1 - \left(\frac{r'}{l}\right)^2\right] dt'^2 + \frac{dr'^2}{1-\left(\frac{r'}{l}\right)^2} + r'^2 (1-\delta) d\Omega^2_{d-2}.$$

Asymptotic Solutions - Near the Horizon

$$
\begin{aligned}
ds^2 &= -\left[1-\delta-\left(\frac{r}{l}\right)^2\right]dt^2 + \frac{dr^2}{1-\delta-\left(\frac{r}{l}\right)^2} + r^2 d\Omega_{d-2} \\
&\curvearrowleft \; t' = \sqrt{1-\delta}\,t, \; r' = r/\sqrt{1-\delta} \\
&= -\left[1-\left(\frac{r'}{l}\right)^2\right]dt'^2 + \frac{dr'^2}{1-\left(\frac{r'}{l}\right)^2} + r'^2(1-\delta)d\Omega^2_{d-2}
\end{aligned}
$$

By looking at this asymptotic metric form, we see

- Geometry near the horizon
  $\rightarrow$ A deficit angle $\Delta_{\rm deficit}$ appeared in the dS_d (whole geometry:$\delta\mathrm{dS}$),

$$\Delta_{\rm deficit} = \Omega_{d-2}\left(1-(1-\delta)^{\frac{d-2}{2}}\right) \quad \left(\approx \Omega_{d-2}\frac{d-2}{2}\delta + \mathcal{O}(\delta^2) \text{ for small } \delta \ll 1\right)$$

  $\rightarrow$ Since $\delta = 8\pi G v^2/(d-3)$, the positive deficit angle grows as v^2.

Asymptotic Solutions - Near the Horizon

$$
\begin{aligned}
ds^2 &= -\left[1-\delta-\left(\frac{r}{l}\right)^2\right]dt^2+\frac{dr^2}{1-\delta-\left(\frac{r}{l}\right)^2}+r^2d\Omega_{d-2}\\
&\curvearrowleft\ t' = \sqrt{1-\delta}\,t,\ r' = r/\sqrt{1-\delta}\\
&= -\left[1-\left(\frac{r'}{l}\right)^2\right]dt'^2+\frac{dr'^2}{1-\left(\frac{r'}{l}\right)^2}+r'^2(1-\delta)d\Omega^2_{d-2}
\end{aligned}
$$

By looking at this asymptotic metric form, we see

- Horizon radius, $r_{\rm H}$, is shifted by

$$
r'_{\rm H} = l \ \rightarrow\ r_{\rm H} = \sqrt{1-\delta}\,l = \sqrt{1-\frac{8\pi G v^2}{d-3}}\,l
$$

Asymptotic Solutions - Near the Horizon

$$
\begin{aligned}
ds^2 &= -\left[1-\delta-\left(\frac{r}{l}\right)^2\right]dt^2 + \frac{dr^2}{1-\delta-\left(\frac{r}{l}\right)^2} + r^2 d\Omega_{d-2} \\
&\curvearrowleft\ t' = \sqrt{1-\delta}\,t,\ r' = r/\sqrt{1-\delta} \\
&= -\left[1-\left(\frac{r'}{l}\right)^2\right]dt'^2 + \frac{dr'^2}{1-\left(\frac{r'}{l}\right)^2} + r'^2(1-\delta)d\Omega^2_{d-2}
\end{aligned}
$$

By looking at this asymptotic metric form, we see

- Horizon area, $A^{\rm H}_{\delta{\rm dS}}$

$$A^{\rm H}_{\delta{\rm dS}} = r_{\rm H}^{d-2}\Omega_{d-2} = l^{d-2}(1-\delta)^{\frac{d-2}{2}}\Omega_{d-2}$$

Now let's calculate the thermodynamic quantities of our spacetime. First, let's see the temperature.

$$
\begin{aligned}
ds^2 &= -\left[1-\delta-\left(\frac{r}{l}\right)^2\right]dt^2+\frac{dr^2}{1-\delta-\left(\frac{r}{l}\right)^2}+r^2 d\Omega_{d-2} \\
&\curvearrowleft\ t' = \sqrt{1-\delta}\,t,\ r' = r/\sqrt{1-\delta} \\
&= -\left[1-\left(\frac{r'}{l}\right)^2\right]dt'^2+\frac{dr'^2}{1-\left(\frac{r'}{l}\right)^2}+r'^2(1-\delta)d\Omega_{d-2}^2
\end{aligned}
$$

Local Rindler Temperature near the horizon, $T_{\mathrm{dS}} = \dfrac{\kappa}{2\pi} = \dfrac{\sqrt{1-\delta}}{2\pi l}$

where the surface gravity, κ, is obtained by

$$
\kappa^2 = -\frac{1}{2}(\nabla_\mu K_\nu)(\nabla^\mu K^\nu) = \cdots = \frac{g^{rr}}{4}\frac{(\partial_r g_{tt})^2}{g_{tt}} \xrightarrow{r\to r_{\mathrm{H}}} \frac{(\partial_r g_{tt})^2}{4}
$$

$$
\Rightarrow\ \kappa(\kappa_{\mathrm{H}}) \approx \frac{r_{\mathrm{H}}}{l^2} = \frac{\sqrt{1-\delta}}{l}
$$

Entropy

Since we have T, dE, by using the thermodynamic law,

$$\boxed{d(-E) + Pd(-V) = TdS}$$

we can obtain the entropy corresponding the hidden degrees of freedom behind the horizon.

- Note that we should use the negative value of the energy which corresponds to degrees in the opposite pole.
- Since the system has pressure on the horizon, we should consider the PdV term in the first law. [Padmanabhan, 2002]
- Note that PdV has a minus sign here: The volume of the hidden area decreases when the volume inside the horizon increases.
- And we can not integrate dS from $S = 0$ value, since the closed-up spacetime breaks our approximation condition.
  $\longrightarrow$ We will integrate dS from the pure de Sitter entropy where $v = 0$.

Entropy

As in the previous points, we will calculate the entropy as,

$$S_{\delta\mathrm{dS}} = \Delta S_{\delta\mathrm{dS}} + S_{\mathrm{dS}}$$

where

$$S_{\mathrm{dS}} = \frac{A^{\mathrm{H}}_{\mathrm{dS}}}{4G} = \frac{l^{d-2}\Omega_{d-2}}{4G}$$

From $d(-E) + Pd(-V) = TdS$, we get $\Delta S_{\delta\mathrm{dS}}$ as,

$$E_{\delta\mathrm{dS}} \approx \Omega_{d-2}\frac{d-2}{d-3}\frac{v^2}{2}r_{\mathrm{H}}^{d-3} = \Omega_{d-2}\frac{d-2}{16\pi G}l^{d-3}\delta(1-\delta)^{\frac{d-3}{2}}$$

$$P_{\delta\mathrm{dS}} = T^r{}_r \approx -\frac{d-2}{2}\frac{v^2}{r^2}, \quad P_{\delta\mathrm{dS}}\mathrm{d}(-V_{\delta\mathrm{dS}}) = (d-2)\frac{v^2}{2r_{\mathrm{h}}^2}\Omega_{d-2}r_{\mathrm{h}}^{d-2}\mathrm{d}r_{\mathrm{h}}$$

$$\boxed{\Delta S_{\delta\mathrm{dS}} = \frac{A_{\mathrm{dS}}}{4G}\left(-\frac{d-2}{2}\right)(1-\delta)^{\frac{d-4}{2}}\mathrm{d}\delta}$$

Then the entropy for the deformed system is given by

$$S_{\delta \mathrm{dS}} = S_{\mathrm{dS}} + \Delta S_{\delta \mathrm{dS}} = S_{\mathrm{dS}} + \int_{S(\delta=0)}^{S(\delta)} dS_{\delta \mathrm{dS}}$$
$$= \frac{A^{\mathrm{h}}_{\mathrm{dS}}}{4G} (1-\delta)^{\frac{d-2}{2}}$$

$$\boxed{S_{\delta \mathrm{dS}} = \frac{A^{\mathrm{h}}_{\delta \mathrm{dS}}}{4G} = \frac{1}{4G} \ell^{d-2} \Omega_{d-2} (1-\delta)^{\frac{d-2}{2}}}$$

Therefore, the area law still holds in the deformed system. As we expected, putting the non-trivial matter distribution leads the negative contribution to the entropy and in the case of the topological soliton the entropy changes with a factor of the solid deficit angle.

Conclusion

An evidence is added for universality of the area law

dS entropy with matter is calculated without temperature ambiguity.