# Holography of non-conformal field theory and entanglement entropy

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# Outline

- Gauge/gravity duality for finite N
- Exact holography for finite N
- Holographic entanglement entropy for the LLM
- Gauge/gravity duality and entanglement entropy
- Summary

# Gauge/gravity for finite N

- Duality properties of field theory and gravity theory

| Quantum field theory | duality ⟺ | Quantum gravity |
| --- | --- | --- |
| Strongly coupled gauge theory | Large N ⟺ | Weakly curved gravity (classical gravity) |

- **Very useful but difficult to check the duality!**
  For some BPS objects which have no quantum corrections, it was possible to check the duality using supersymmetry and **conformal symmetry** **in the large N limit.**

# Gauge/gravity for finite N

- Duality properties of field theory and gravity theory

| Quantum field theory | duality ⟺ | Quantum gravity |
|---|---|---|
| Strongly coupled gauge theory | Large N ⟺ | Weakly curved gravity (classical gravity) |

- To test Maldacena's conjecture, **large N is necessary!**

# Gauge/gravity for finite N

- AdS/CFT Conjecture by Maldacena:

We started with a quantum theory and saw that it includes gravity, so it is natural to think that this correspondence goes beyond the supergravity approximation. We are led to the conjecture that *type IIB string theory on* $(AdS_5 \times S^5)_N$ *plus some appropriate boundary conditions (and possibly also some boundary degrees of freedom) is dual to* $\mathcal{N} = 4$, $d = 3 + 1$, $U(N)$ *super-Yang–Mills.*

➔No restriction on $N, \; g_{YM}$ **Strong conjecture!**

➔ to support this conjecture he considered the supergravity limit

# Gauge/gravity for finite N

- finite N?

$$g_{YM}^2 = g_s \qquad \lambda = g_{YM}^2 N \qquad L^4 = 4\pi g_{YM}^2 N \alpha'^2 = 4\pi\lambda\alpha'^2,$$

**for limiting cases**

$\lambda \gg 1 \quad \Longleftrightarrow \quad g_s \gg 1 \quad \& \quad \alpha' \ll 1$ ➔ **String loop effects**

$\lambda \ll 1 \quad \Longleftrightarrow \quad g_s \ll 1 \quad \& \quad \alpha' \gg 1$ ➔ **Stringy effects**

➔ **difficult to test the gauge/gravity duality!**

# Gauge/gravity for finite N

**No example of exact gauge/gravity duality for finite N up to now!**

⬇

**One example of exact gauge/gravity duality:**

**Conformal symmetry ➔ non-conformal symmetry**

**Large N ➔ finite N**

# Gauge/gravity for finite N

**ABJM (CFT) + mass deformation = mABJM (non-CFT)**

$\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ ⟺ LLM geometry with $\mathbb{Z}_k$ orbifold

[Lin-Lunin-Maldacena, 2004]

$$\mathcal{M}_1^{(n)} = \begin{pmatrix} \sqrt{n} & 0 & & & & \\ & \sqrt{n-1} & 0 & & & \\ & & \ddots & \ddots & & \\ & & & \sqrt{2} & 0 & \\ & & & & 1 & 0 \end{pmatrix}$$

$$\mathcal{M}_2^{(n)} = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & \sqrt{2} & & & \\ & & \ddots & \ddots & & \\ & & & 0 & \sqrt{n-1} & \\ & & & & 0 & \sqrt{n} \end{pmatrix}$$

Discrete Higgs vacua
denoted by numerical values
➔ Quantitative comparison
for finite N

[Gomis et al 08] [Cheon-Kim-Kim 2011]

# Exact gauge/gravity for finite N

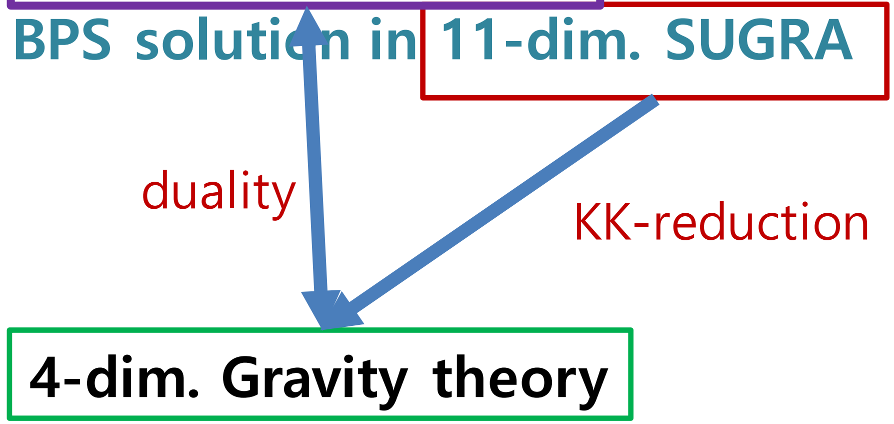
- To show the dual relation of two theories, correspondence between vacua of QFT and BPS solutions in gravity is not enough.
  - **mABJM theory: 3-dim. gauge theory**
  - **LLM geometry: BPS solution in 11-dim. SUGRA**

KK-reduction

**4-dim. Gravity theory**

# Exact gauge/gravity for finite N

- To show the dual relation of two theories, correspondence between vacua of QFT and BPS solutions in gravity is not enough.
  - **mABJM theory: 3-dim. gauge theory**
  - **LLM geometry: BPS solution in 11-dim. SUGRA**



# Exact gauge/gravity for finite N

- Vev of CPO with $\Delta = 1$ in field theory side

$$\langle \mathcal{O}^{(1)} \rangle_0 = \frac{k\mu}{4\sqrt{2}\,\pi} \sum_{n=0}^{2N_{\rm B}+1} n(n+1)(N_n - N'_n)$$

$$\mathcal{O}^{(1)} = \frac{1}{2\sqrt{2}} \mathrm{Tr}\left(Z^a Z_a^\dagger - W^{\dagger a} W_a\right)$$

- Gauge/gravity mapping (GKP-W relation)

$$\langle \mathcal{O}^{(1)} \rangle_0 = \mathbb{N}\psi_{(1)} = -6\sqrt{2}\mathbb{N}\beta_1\mu_0$$

$$\beta_1 = 2C_1^3 - 3C_1C_2 + C_3$$

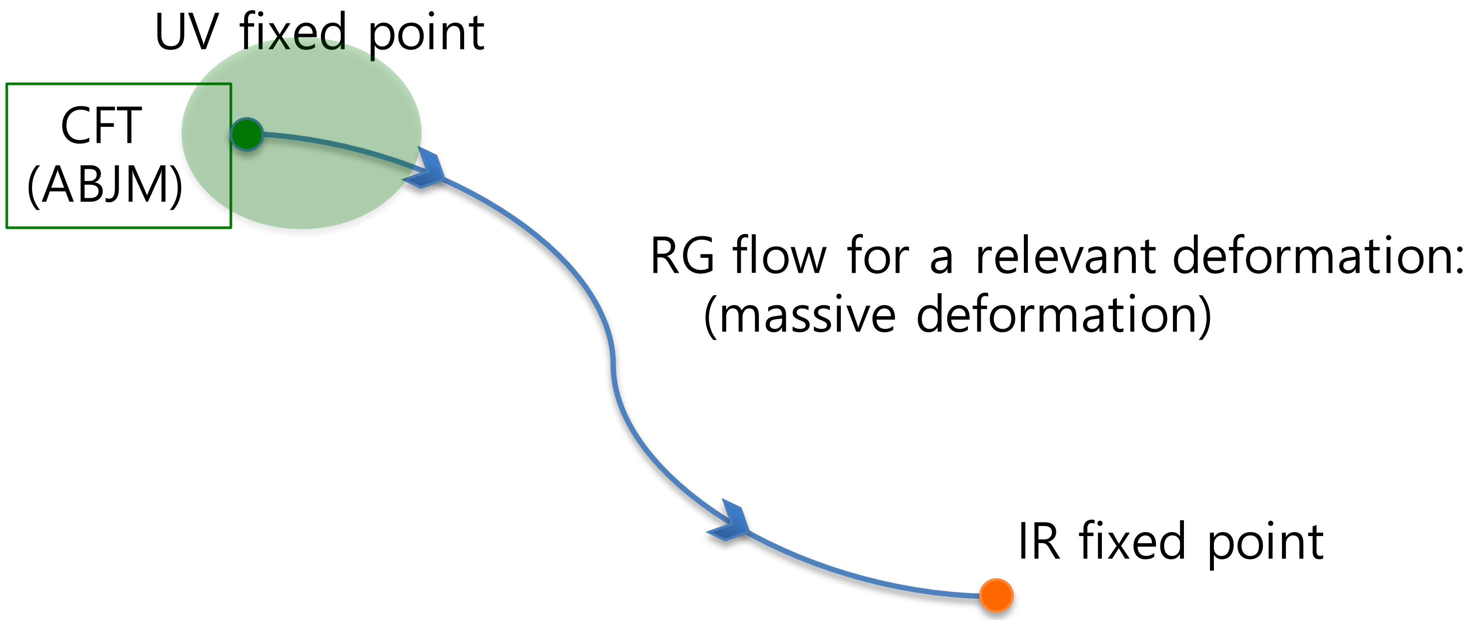
$$C_p = \sum_{i=1}^{2N_{\rm B}+1} (-1)^{i+1} \left( \frac{\tilde{x}_i}{2\pi l_{\rm P}^3 \mu_0 \sqrt{A}} \right)^p$$

- Fix the normalization factor (k=1):

$$\langle \mathcal{O}^{(1)} \rangle_0 = \frac{N^{\frac{3}{2}}\mu_0}{3\sqrt{2}\pi}\beta_1$$

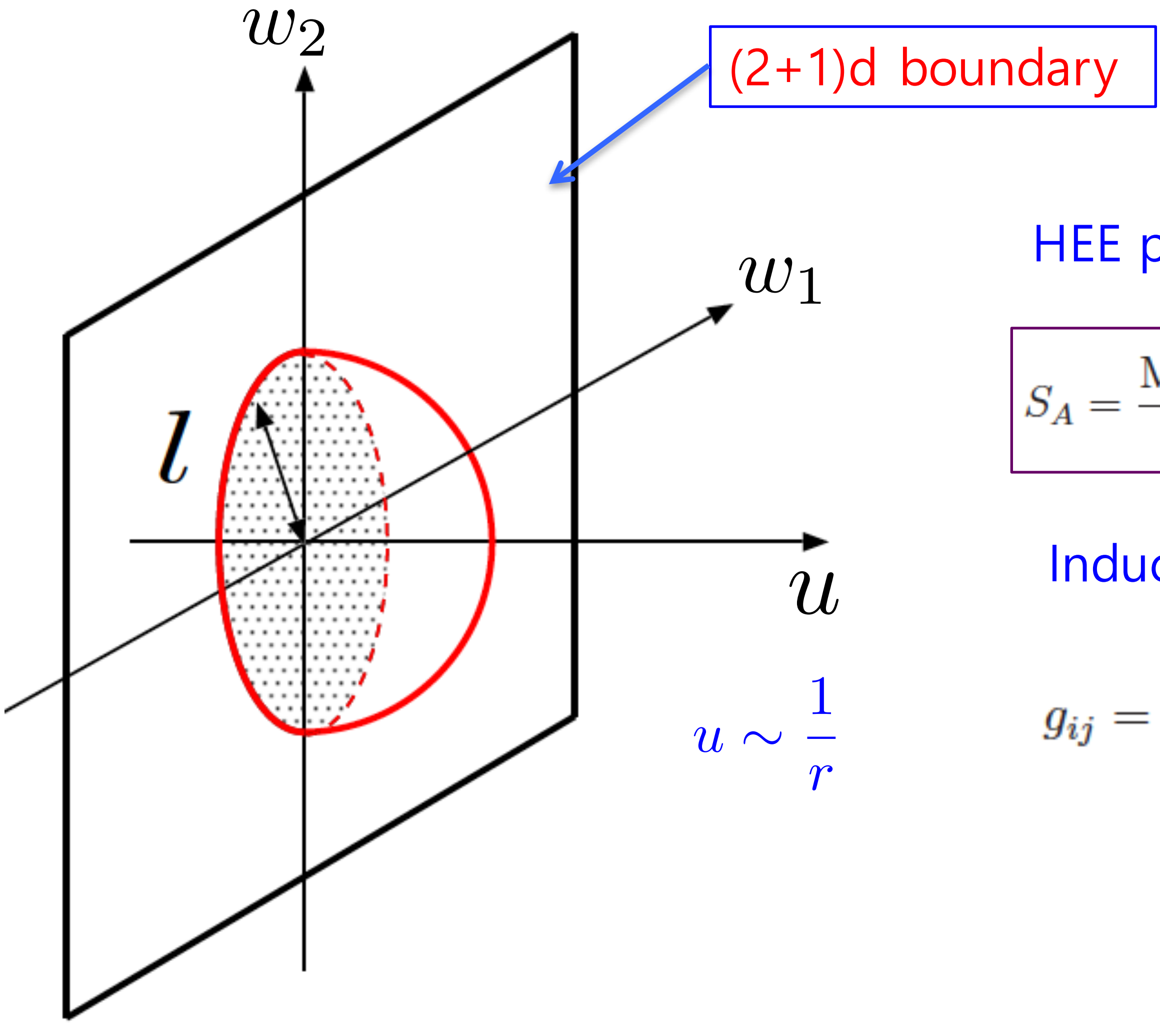
$$\mathbb{N} = -\frac{N^{\frac{3}{2}}}{36\pi}$$

$$N \geq 2$$




- **Exact holographic renormalization near the UV fixed point**
- **Scalar fields deformation induces the metric deformation**
  **➔ deformation of the entanglement entropy**
- **How can we relate vevs of CPOs in QFT with the holographic entanglement entropy?**

# Holographic Entanglement Entropy




$u \sim \frac{1}{r}$

HEE proposal

$$S_A = \frac{\mathrm{Min}(\gamma_A)}{4G_N}, \quad \gamma_A = \int d^9\sigma \sqrt{\det g_{ij}}$$

Induced metric $i = 1, 2, \cdots, 9$

$$g_{ij} = \frac{\partial X^M \partial X^N}{\partial \sigma^i \partial \sigma^j} G_{MN}$$

# HEE for the LLM geometry

- Half-BPS solutions with SO(2,1)XSO(4)XSO(4) isometry in 11-dimensional supergravity [04, Lin-Lunin-Maldacena]

$$ds^2 = -G_{tt}\left(-dt^2 + dw_1^2 + dw_2^2\right) + G_{xx}(dx^2 + dy^2) + G_{\theta\theta} ds^2_{S^3/\mathbb{Z}_k} + G_{\tilde{\theta}\tilde{\theta}} ds^2_{\tilde{S}^3/\mathbb{Z}_k}$$

$$f(x,y) = \sqrt{1 - 4z^2 - 4y^2V^2}$$

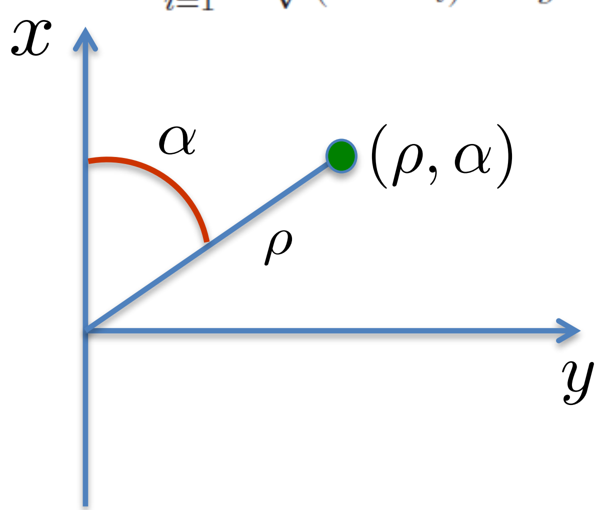
$$-G_{tt} = \left(\frac{4\mu_0^2 y\sqrt{\frac{1}{4} - z^2}}{f^2}\right)^{2/3},$$

$$G_{xx} = \left(\frac{f\sqrt{\frac{1}{4} - z^2}}{2\mu_0 y^2}\right)^{2/3},$$

$$G_{\theta\theta} = \left(\frac{fy\sqrt{\frac{1}{2} + z}}{2\mu_0\left(\frac{1}{2} - z\right)}\right)^{2/3},$$

$$G_{\tilde{\theta}\tilde{\theta}} = \left(\frac{fy\sqrt{\frac{1}{2} - z}}{2\mu_0\left(\frac{1}{2} + z\right)}\right)^{2/3}.$$

$$z(x,y) = \sum_{i=1}^{2m+1} \frac{(-1)^{i+1}(x - x_i)}{2\sqrt{(x - x_i)^2 + y^2}}, \qquad V(x,y) = \sum_{i=1}^{2m+1} \frac{(-1)^{i+1}}{2\sqrt{(x - x_i)^2 + y^2}}$$




**To calculate the HEE in the LLM geometry we have to solve PDE equations**

# HEE for the LLM geometry

- HEE proposal [Ryu-Takayanagi 06]

$$S_A = \frac{\mathrm{Min}(\gamma_A)}{4G_N}, \quad \gamma_A = \int d^9\sigma \sqrt{\det g_{ij}} \qquad i = 1, 2, \cdots, 9$$

- Induced metric

$$g_{ij} = \frac{\partial X^M \partial X^N}{\partial \sigma^i \partial \sigma^j} G_{MN}$$

- Mapping for the Disk case:

$$w_1 = \sigma_2 \cos \sigma_1, \quad w_2 = \sigma_2 \sin \sigma_1, \quad u = u(\sigma_2, \sigma_3), \quad \alpha = \alpha(\sigma_2, \sigma_3),$$
$$\sigma_4 \sim \sigma_9 = \theta_1 \sim \theta_6.$$

$$\sigma_2 = \rho, \ \sigma_3 = \phi$$

$$w_1 = \rho \cos \sigma^1, \ w_2 = \rho \sin \sigma^1, \ u = u(\rho, \phi), \ \alpha = \alpha(\rho, \phi)$$

[C.Kim, K.Kim, OK work in progress]

# HEE for the LLM geometry

$$u = u_0(\tilde{\rho}) + u_1(\tilde{\rho},\phi)\mu_0 l + u_2(\tilde{\rho},\phi)(\mu_0 l)^2 + \cdots,$$

$$\tilde{g}(\tilde{u},\phi) = 1 + g_1(\phi)u_0(\mu_0 l) + \left(g_1(\phi)u_1 + g_2(\phi)u_0^2\right)(\mu_0 l)^2 + \cdots$$

$$g_1(\phi) = D_1 \cos\phi,$$

$$g_2(\phi) = D_2 + D_3\cos(2\phi)$$

$$D_0 = \sqrt{2}\sqrt{C_2 - C_1^2}\,, \quad D_1 = -\frac{(C_1C_2 - C_3)}{\sqrt{2}}$$

$$C_k = \sum_{i=1}^{2N_b+1}(-1)^{i+1}\left(\frac{\hat{x}_i}{\sqrt{N}k}\right)^k$$

$$D_2 = \frac{1}{16}\left(-5C_2^2 - 2\left(C_3 - C_1C_2\right)^2 - 4C_1C_3 + 9C_4\right),$$

$$x_i = 2\pi l_p^3 \mu_0 \hat{x}_i.$$

$$D_3 = \frac{1}{16}\left(-3C_2^2 - 2\left(C_3 - C_1C_2\right)^2 - 12C_1C_3 + 15C_4\right)$$

$$z(x,y) = \sum_{i=1}^{2m+1}\frac{(-1)^{i+1}(x-x_i)}{2\sqrt{(x-x_i)^2+y^2}}, \qquad V(x,y) = \sum_{i=1}^{2m+1}\frac{(-1)^{i+1}}{2\sqrt{(x-x_i)^2+y^2}}.$$

# HEE for the LLM geometry

- Solution satisfying the boundary condition

$$\dot{u}(\rho = 0) = 0 \text{ and } u(\rho = 1) = 0$$

$$u_0(\rho) = \sqrt{1-\rho^2}$$

$$u_1(\phi, \phi) = -\frac{D_1}{2}\left(1-\rho^2\right)\cos\phi$$

$$\begin{aligned} u_2(\rho, \phi) = &-\frac{1}{6\sqrt{1-\rho^2}}\left(D_1^2 + 20D_2 - 12D_3\right)\log\left(1+\sqrt{1-\rho^2}\right) \\ &+ \frac{1}{48}\Big[\left(8 + (9-13\rho^2)\sqrt{1-\rho^2}\right)D_1^2 + 16\left(10 - (6-\rho^2)\sqrt{1-\rho^2}\right)D_2 \\ &- 48\left(2-\sqrt{1-\rho^2}\right)D_3\Big] + \frac{1}{48}(11D_1^2 - 16D_3)(1-\rho^2)^{3/2}\cos(2\phi). \end{aligned}$$

# HEE for the LLM geometry

- Entanglement entropy

$$S_{\rm disk} = \frac{\pi^5 R^9}{24 G_N k}\left\{\frac{l}{\epsilon} - 1 - \mu_0^2 l^2\left[\frac{4}{3} + \frac{1}{24}(C_3 - 3C_1C_2 + 2C_1^3)^2\right]\right\} + \mathcal{O}(\mu_0^3)$$

- Renormalized Entanglement entropy

$$\begin{aligned}\mathcal{F}_{\rm disk} &\equiv \left(l\frac{\partial}{\partial l} - 1\right) S_{\rm disk} \\ &= \frac{\pi^5 R^9}{24 G_N k}\left\{1 - \mu_0^2 l^2\left[\frac{4}{3} + \frac{1}{24}(C_3 - 3C_1C_2 + 2C_1^3)^2\right]\right\} + \mathcal{O}(\mu_0^3)\end{aligned}$$

# HEE for the LLM geometry

- Entanglement entropy

$$S_{\text{disk}} = \frac{\pi^5 R^9}{24 G_N k} \left\{ \frac{l}{\epsilon} - 1 - \mu_0^2 l^2 \left[ \frac{4}{3} + \frac{1}{24} (C_3 - 3C_1 C_2 + 2C_1^3)^2 \right] \right\} + \mathcal{O}(\mu_0^3)$$

- Renormalized Entanglement entropy

$$\mathcal{F}_{\text{disk}} \equiv \left( l \frac{\partial}{\partial l} - 1 \right) S_{\text{disk}}$$
$$= \frac{\pi^5 R^9}{24 G_N k} \left\{ 1 - \mu_0^2 l^2 \left[ \boxed{\frac{4}{3} + \frac{1}{24} (C_3 - 3C_1 C_2 + 2C_1^3)^2} \right] \right\} + \mathcal{O}(\mu_0^3)$$

**Positive for all LLM solutions**
**➔ Monotonically decreasing behavior of c-function**

# Gauge/gravity duality and EE

- Entanglement entropy of **the LLM geometry in 11d SUGRA**

  (RT formula)

$$S_{\text{disk}} = \frac{\pi^5 R^9}{24 G_N k}\left\{\frac{l}{\epsilon} - 1 - \mu_0^2 l^2 \left[\frac{4}{3} + \frac{1}{24}(C_3 - 3C_1C_2 + 2C_1^3)^2\right]\right\} + \mathcal{O}(\mu_0^3)$$

- Exact holography for the vev of CPO

$$\langle \mathcal{O}^{(1)} \rangle_0 = \frac{N^{\frac{3}{2}}\mu_0}{3\sqrt{2}\,\pi}\,(C_3 - 3C_1C_2 + 2C_1^3)$$

# Gauge/gravity duality and EE

- Entanglement entropy of **the vacua in mABJM theory**(RT formula)

$$S_{\text{disk}} = \frac{\pi^5 R^9}{24 G_N k}\left\{\frac{l}{\epsilon} - 1 - \mu_0^2 l^2\left[\frac{4}{3} + \frac{1}{24}(C_3 - 3C_1C_2 + 2C_1^3)^2\right]\right\} + \mathcal{O}(\mu_0^3)$$

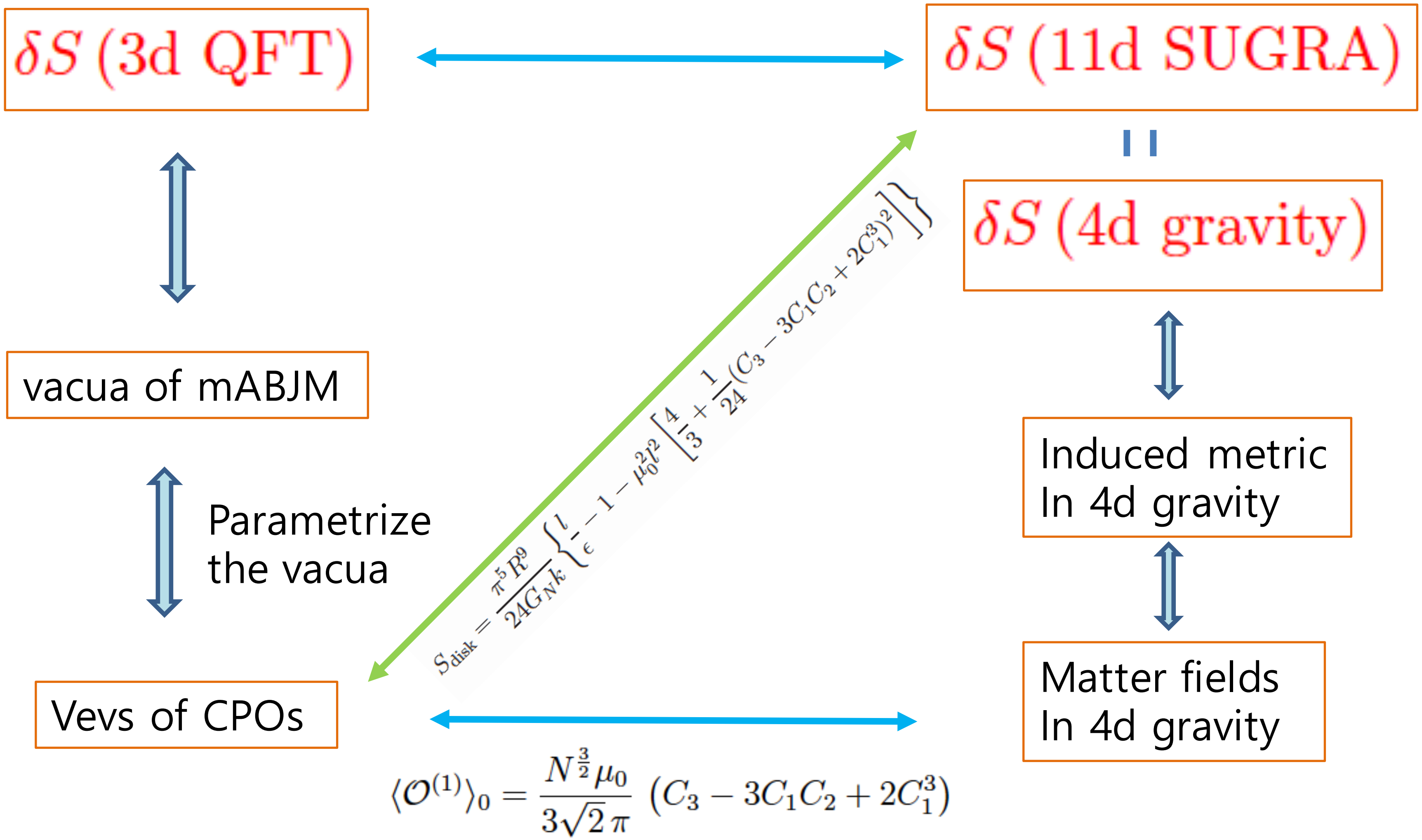
??

- Exact holography for the vev of CPO

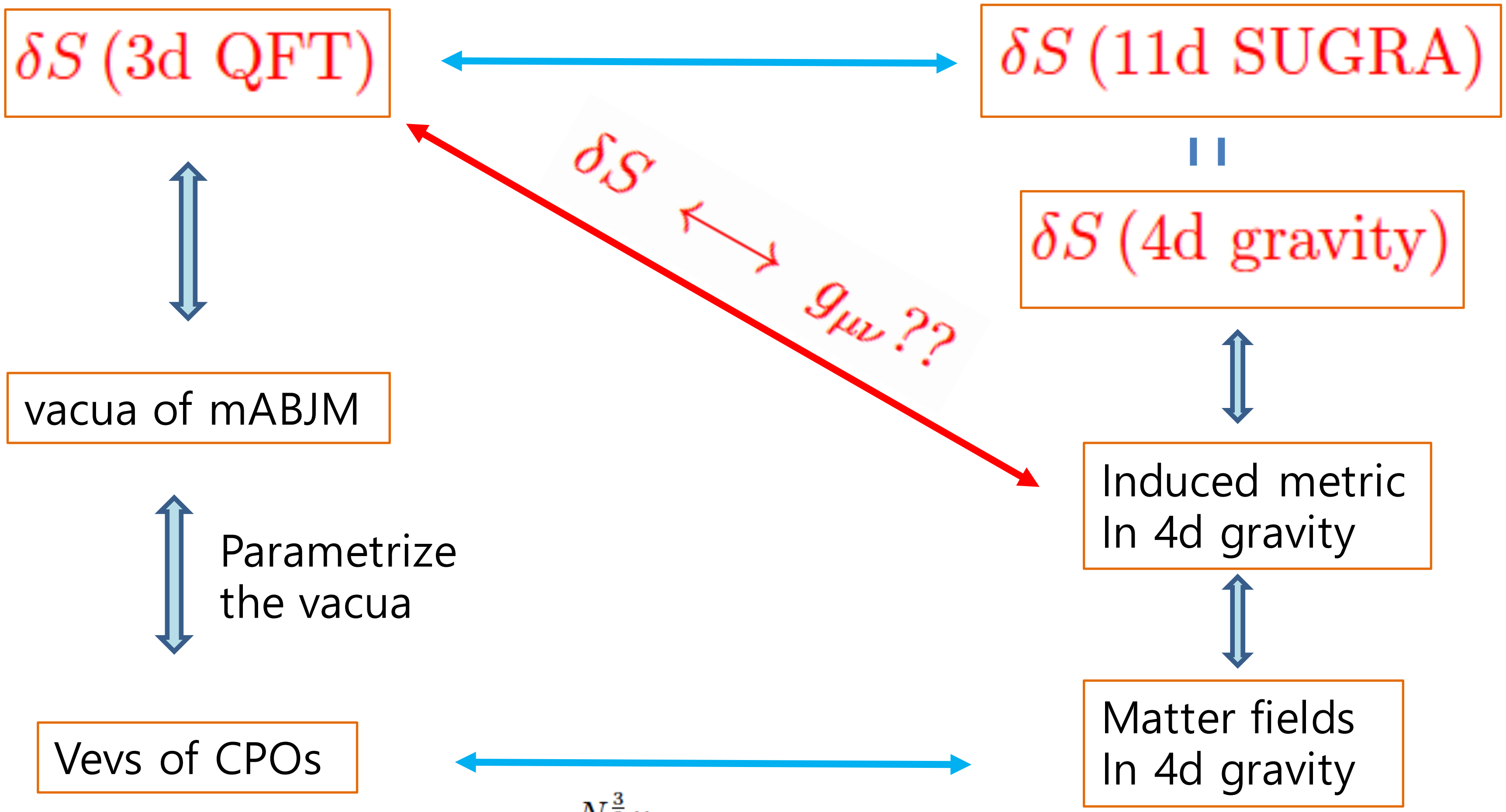
$$\langle \mathcal{O}^{(1)} \rangle_0 = \frac{N^{\frac{3}{2}}\mu_0}{3\sqrt{2}\,\pi}\left(C_3 - 3C_1C_2 + 2C_1^3\right)$$

- **How can we consider this correspondence?**

# Gauge/gravity duality and EE



# Gauge/gravity duality and EE




$$\langle \mathcal{O}^{(1)} \rangle_0 = \frac{N^{\frac{3}{2}} \mu_0}{3\sqrt{2}\,\pi} \left( C_3 - 3C_1C_2 + 2C_1^3 \right)$$

[D. Jang, Y. Kim, OK, D.Tolla, work in progress]