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Phase transition for black holes: the contribution from the boundary term

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based on
Sunly Khimphun, Bum−Hoon Lee, Wonwoo Lee, arXiv:1605.07377
(To be appeared in PRD)

The plan of this talk

1. Motivation
2. Black Hole in Einstein and EGB theory
3. Phase transition
4. Hairy black holes in DEGB theory
5. Summary and Discussion

1. Motivations

(1) Black hole no hair theorem :

Stationary black holes are characterized by 3 quantities.
(M, Q, J) mass, charge, and angular momentum

Hairy black hole solution is possible in the dilatonic Einstein-Gauss-Bonnnet theory of gravitation.

(2) Black hole stability

perturbative way or non-perturbative way

Thermal stability(instability)

Specific heat (heat capacity): local stability
Free energy : global stability

(3) How can we distinguish the difference between Einstein theory and the Einstein-Gauss-Bonnet theory of gravitation (EGB theory) in four dimensions?

(4) The contribution coming from the topological term

The Gauss-Bonnet (GB) term in four dimensions causes the additional constant entropy. This quantity is related to the information on the topology of the spacetime manifold, which means that there exists a discontinuous jump when the topology of the manifold is changed or the black hole mass vanishes away, unlike the limiting case in Einstein theory.

We think that the topological information of the spacetime manifold could be additionally stored in the thermodynamic quantities for a black hole.

2. Black Hole in Einstein and EGB theory

2.1 Schwarzschild BH in Einstein theory

We consider the action

$$I = \int_{\mathcal{M}} \sqrt{-g} d^4x \left[\frac{R}{2\kappa} \right] - \oint_{\partial\mathcal{M}} \sqrt{-h} d^3x \frac{K - K_o}{\kappa}$$

Euclidean action

$$I_E = \int_{\mathcal{M}} \sqrt{g_E} d^4x_E \left[\frac{-R}{2\kappa} \right] + \oint_{\partial\mathcal{M}} \sqrt{h_E} d^3x_E \frac{K - K_o}{\kappa}$$

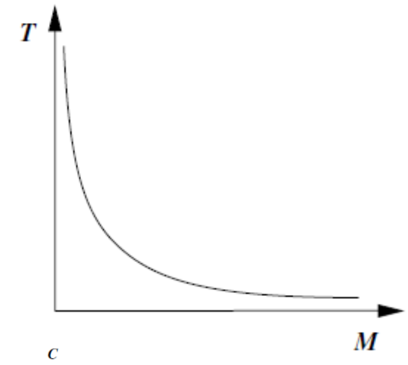
$$ds_E^2 = \left(1 - \frac{2GM}{r}\right) d\tau^2 + \frac{dr^2}{\left(1 - \frac{2GM}{r}\right)} + r^2 d\theta^2 + r^2 \sin\theta^2 d\phi^2$$

Gibbons and Hawking, PRD 15, 2752 (1977)

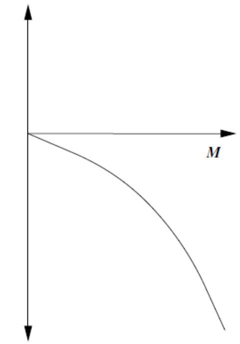
$$\sqrt{h_E} d^3 x_E = \sqrt{g_{\tau\tau}} 4\pi r^2 d\tau = \sqrt{g_{\tau\tau}} 32\pi^2 G M r^2$$

extrinsic curvature $K = -\frac{\sqrt{g_{rr}}}{2} \partial_r g_{\tau\tau} - \frac{1}{\sqrt{g_{rr}}} \frac{2}{r}$

temperature $T = \frac{1}{8\pi G M}$ **,** $\beta = 8\pi G M$



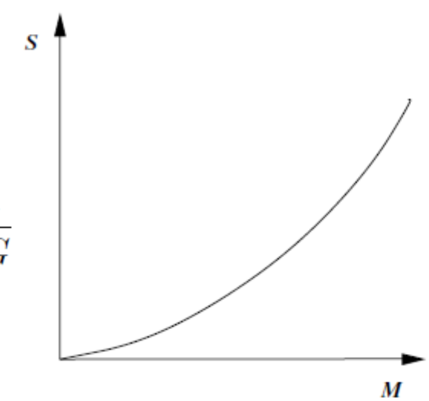
specific heat $C = \frac{\partial M}{\partial T_H} < 0.$



$$I_E = \oint_{\partial \mathcal{M}} \sqrt{h_E} d^3 x_E \frac{K - K_o}{\kappa} = 4\pi G M^2 = \frac{4\pi r_h^2}{4G} = \frac{A}{4G}$$

entropy

$$S = E\beta + \ln Z = E\beta - I_E = M\beta - I_E = 8\pi G M^2 - 4\pi G M^2 = \frac{4\pi r_h^2}{4G} = \frac{A}{4G}$$



flat Minkowski $I_E = S = 0$

2.2 Schwarzschild BH in EGB theory

We consider the action

$$I_E = \int_{\mathcal{M}} \sqrt{g_E} d^4 x_E \left[\frac{-R}{2\kappa} - \alpha (R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) \right] + \oint_{\partial\mathcal{M}} \sqrt{h_E} d^3 x_E \frac{K - K_o}{\kappa} + I_{\rm GBb}$$

where $$I_{\rm GBb} = 4\alpha \oint_{\partial\mathcal{M}} \sqrt{h_E} d^3 x_E (J - 2G_{ab}K^{ab})$$

R. Myers, Phys. Rev. D 36, 392 (1987),
Davis, Phys. Rev. D 67, 024030 (2003)

$$J_{ab} = \frac{1}{3}(2KK_{ac}K^c_b + K_{cd}K^{cd}K_{ab} - 2K_{ac}K^{cd}K_{db} - K^2 K_{ab})$$

from the variational principle

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + 16\pi G\alpha H_{\mu\nu} = 0$$

$$H_{\mu\nu} = 2(R_{\mu\sigma\kappa\tau}R_{\nu}{}^{\sigma\kappa\tau} - 2R_{\mu\rho\nu\sigma}R^{\rho\sigma} - 2R_{\mu\sigma}R^{\sigma}{}_{\nu} + RR_{\mu\nu}) - \frac{1}{2}L_{\rm GB}g_{\mu\nu}$$

boundary terms

$$\begin{aligned}
&\int d^4x\sqrt{-g}\{-\nabla_\mu(R\nabla_\nu\delta g^{\mu\nu}) + \nabla_\nu(\delta g^{\mu\nu}\nabla_\mu R) + g_{\mu\nu}\nabla_\lambda(R\nabla^\lambda\delta g^{\mu\nu}) - g_{\mu\nu}\nabla^\lambda(\delta g^{\mu\nu}\nabla_\lambda R) \\
&-4[-2\nabla_\mu(R_{\nu\alpha}\nabla^\alpha\delta g^{\mu\nu}) + 2\nabla^\alpha(\delta g^{\mu\nu}\nabla_\mu R_{\nu\alpha}) + \nabla_\lambda(R_{\mu\nu}\nabla^\lambda\delta g^{\mu\nu}) - \nabla^\lambda(\delta g^{\mu\nu}\nabla_\lambda R_{\mu\nu}) \\
&+g_{\mu\nu}\nabla^\beta(R_{\alpha\beta}\nabla^\alpha\delta g^{\mu\nu}) - g_{\mu\nu}\nabla^\alpha(\delta g^{\mu\nu}\nabla^\beta R_{\alpha\beta})] \\
&+4\nabla^\sigma(R_{\mu\rho\nu\sigma}\nabla^\rho\delta g^{\mu\nu}) - 4\nabla^\rho(\delta g^{\mu\nu}\nabla^\sigma R_{\mu\rho\nu\sigma})\} \qquad (
\end{aligned}$$

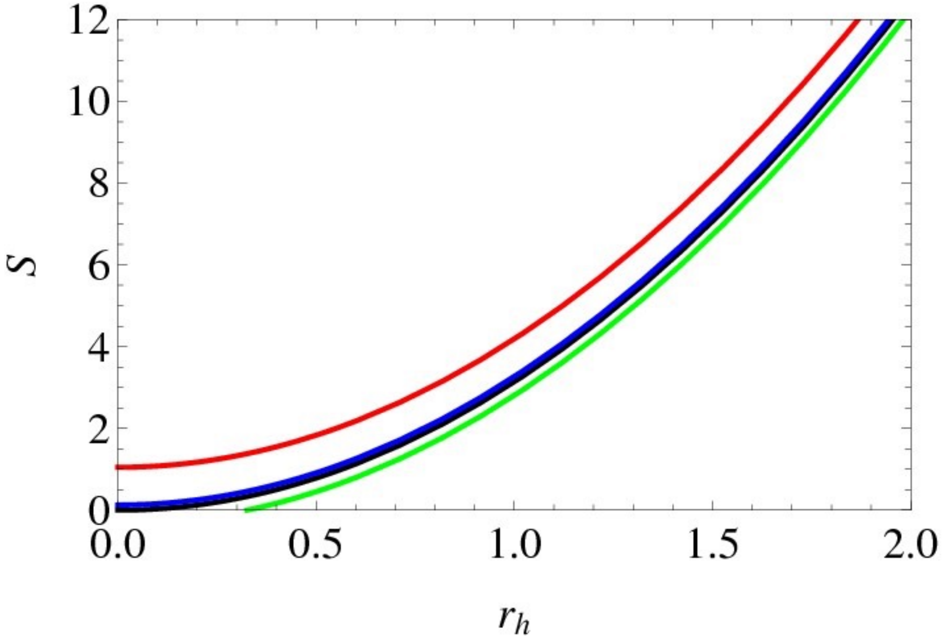
where

$$R = 0,\ R_{\mu\nu} = 0,\ R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6}$$

$$I_E = -64\alpha\pi^2 + \frac{A}{4G}$$

entropy

$$S = \frac{A}{4G} + \frac{8\alpha\pi\kappa}{G}$$



flat Minkowski $I_E = S = 0$

2.3 AdS BH in Einstein theory

Euclidean action

$$I_E = \int_{\mathcal{M}} \sqrt{g_E} d^4x_E \left[\frac{-(R+2\Lambda)}{2\kappa} \right] + \oint_{\partial\mathcal{M}} \sqrt{h_E} d^3x_E \frac{K}{\kappa} \; + I_{\text{ct}}$$

$$= \int_{\mathcal{M}} \sqrt{g_E} d^4x_E \left[\frac{\Lambda}{\kappa} \right] + \oint_{\partial\mathcal{M}} \sqrt{h_E} d^3x_E \frac{K}{\kappa} \; + I_{\text{ct}}$$

AdS BH In Einstein and EGB theory

$$ds^2 = -(1 - \frac{2GM}{r} + \frac{\Lambda}{3}r^2)dt^2 + \frac{dr^2}{(1 - \frac{2GM}{r} + \frac{\Lambda}{3}r^2)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad \text{for } k = 1$$

$$ds^2 = -(-\frac{2GM}{r} + \frac{\Lambda}{3}r^2)dt^2 + \frac{dr^2}{(-\frac{2GM}{r} + \frac{\Lambda}{3}r^2)} + r^2(dx^2 + dy^2), \quad \text{for } k = 0$$

$$ds^2 = -(-1 - \frac{2GM}{r} + \frac{\Lambda}{3}r^2)dt^2 + \frac{dr^2}{(-1 - \frac{2GM}{r} + \frac{\Lambda}{3}r^2)} + r^2(d\psi^2 + \sinh^2\psi d\phi^2), \quad \text{for } k = -1$$

$$\int_{\mathcal{M}} \sqrt{g_E} d^4 x_E \left[\frac{\Lambda}{\kappa}\right] = \frac{\Lambda\beta}{6G}(r^3(\infty) - r_+^3) \qquad \beta = \frac{4\pi r_+}{1+\Lambda r_+^2}$$

$$\oint_{\partial\mathcal{M}} \sqrt{h_E} d^3 x_E \frac{K}{\kappa} = -\frac{\beta}{2G}(2r - 3GM + \Lambda r^3)$$

$$\int_{\mathcal{M}} \sqrt{g_E} d^4 x_E \left[\frac{\Lambda}{\kappa}\right] + \oint_{\partial\mathcal{M}} \sqrt{h_E} d^3 x_E \frac{K}{\kappa} = -\frac{\Lambda\beta}{3G} r^3 - \frac{\Lambda\beta}{6G} r_+^3 - \frac{\beta r}{G} + \frac{3\beta M}{2}$$

The counterterm for AdS, Balasubramanian and Kraus, CMP 208, 413 (1999)

$$I_{ct} = \frac{1}{\kappa}\oint_{\partial\mathcal{M}} \sqrt{h_E} d^3x_E 2\sqrt{\frac{\Lambda}{3}}\left(1 + \frac{3}{4\Lambda}R\right) \qquad R = -4\Lambda - 4\sqrt{\tfrac{\Lambda}{3}}K$$

$$\int_{\mathcal{M}} \sqrt{g_E} d^4x_E \left[\frac{\Lambda}{\kappa}\right] + \oint_{\partial\mathcal{M}} \sqrt{h_E} d^3x_E \frac{K}{\kappa} + \frac{1}{\kappa}\oint_{\partial\mathcal{M}} \sqrt{h_E} d^3x_E \left[-4\sqrt{\frac{\Lambda}{3}} - 2K\right]$$

$$= \frac{\beta M}{2} - \frac{\Lambda\beta}{6G} r_+^3$$

entropy

$$S = M\beta - I_E = \frac{\beta M}{2} + \frac{\Lambda\beta}{6G} r_+^3 = \left[\frac{r_+}{4G}(1 + \frac{\Lambda}{3} r_+^2) + \frac{\Lambda r_+^3}{6G}\right] \frac{4\pi r_+}{(1 + \Lambda r_+^2)} = \frac{4\pi r_+^2}{4G} = \frac{A}{4G}$$

pure AdS $\quad I_E = S = 0$

2.4 AdS BH in EGB theory

Euclidean action

$$I_E = \int_{\mathcal{M}} \sqrt{g_E} d^4 x_E \left[\frac{-(R+2\Lambda)}{2\kappa} - \alpha(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) \right] + \oint_{\partial\mathcal{M}} \sqrt{h_E} d^3 x_E \frac{K}{\kappa} + I_{\rm GBb} + I_{\rm ct}$$

where

$$R = -4\Lambda,\ R_{\mu\nu}R^{\mu\nu} = 4\Lambda^2,\ R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6} + \frac{8\Lambda^2}{3}$$

$$\int_{\mathcal{M}} \sqrt{g_E} d^4 x_E \left[-\alpha(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) \right] = x \left(-32k\pi^2\alpha + \frac{\alpha\beta_H 16\pi\Lambda}{3}(GM - \frac{\Lambda}{3} r_{co}^3) \right)$$

$$I_E(\mathrm{GBb}) = -x\alpha\beta_H 16\pi \left[\frac{2G^2M^2}{r_{co}^3} + \frac{\Lambda}{3}\left(GM - \frac{\Lambda}{3} r_{co}^3 \right) \right]$$

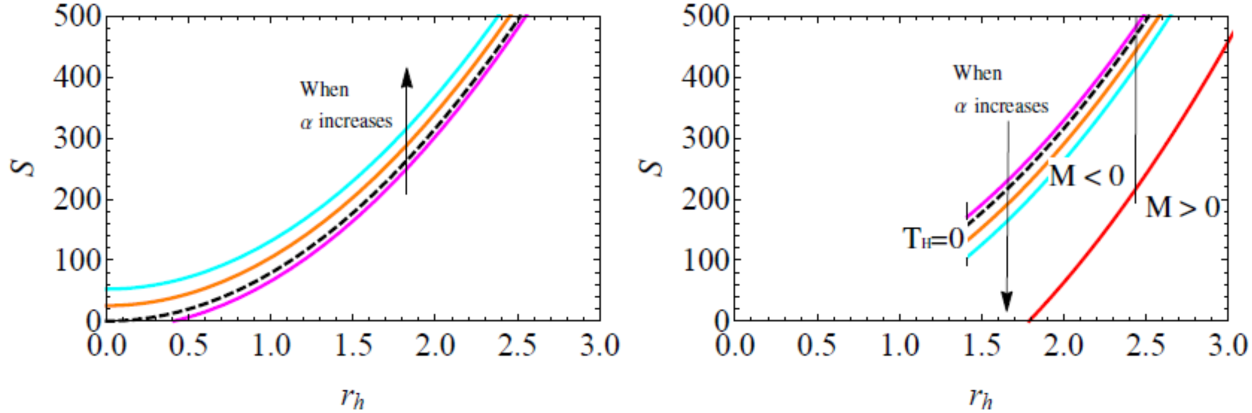
Euclidean action

$$I_E^{\mathrm{BEGB}} = x\left(\frac{\beta_H M}{4} - \frac{\beta_H \Lambda}{12G} r_h^3 - 32k\pi^2\alpha\right)$$

entropy

$$S^{\mathrm{EGB}} = \frac{A}{4G}\left(1 + \frac{8k\alpha\kappa}{r_h^2}\right)$$

pure AdS $I_E = S = 0$



(a) Entropy for $k = 1$

(b) Entropy for $k = -1$

Figure 5: (color online). The entropy as a function of the horizon radius r_h in EGB theory with $\Lambda = 1/2$, and $\kappa = 1$. $\alpha = 0$ for the dashed line in Einstein theory. $\alpha = 0.040$ for the orange line, $\alpha = 0.083$ for the cyan line, $\alpha = 0.400$ for the red line, and $\alpha = -0.021$ for the magenta line in EGB theory.

3. Phase transition

We investigate black hole thermodynamics and HP phase transition in both EGB and DEGB theories. The study of black hole thermodynamics becomes interesting subject after discovering the natural temperature and the intrinsic entropy of a black hole, in which the temperature is proportional to black hole's surface gravity and the entropy to one quarter of the area of its event horizon in Planck units. We explore how higher-order curvature terms can affect black hole thermodynamics and the phase transition The Euclidean path integral approach could be one of important tools to explore in this arena.

3.1 Thermodynamics of AdS black holes

The partition function of this system is defined by a functional integral over all metrics and matter fields on the AdS background

$$Z = \int \mathcal{D}[g]\mathcal{D}[\Phi] \exp(-I_E[g,\Phi])\,, \qquad (15)$$

where $I_E[g,\Phi]$ **is the Euclidean action of gravitational fields and matter fields. It can be evaluated by** $\ln Z = -I_E$.

The dominant contribution to the path integral comes from classical solutions to the equations of motion, in the semiclassical approximation.

Brown, Creighton, Mann, PRD50, 6394

free energy : $F = M - T_H S$ **internal energy :** $E = M \times \text{redshift factor}$

$$F = \frac{I_E}{\beta_H} \qquad E/T = M/T_H \qquad T = T_H \times \text{redshift factor}$$

specific heat : $C = \frac{\partial M}{\partial T_H}$

entropy : $S = M\beta_H - I_E$

The specific heat determines the thermodynamic stability. If a black hole has a negative specific heat, the black hole is thermodynamically unstable. This property can be expected from the behavior with a monotonically decreasing temperature. In the asymptotically AdS spacetime, a black hole has both a positive and negative specific heats depending on the size or the mass of a black hole.

The entropy formula should be modified to have the contribution from the GB term. From the first law of black hole thermodynamics, the entropy can have a constant after the integration. The constant can be determined in EGB theory. When the analytic form of the solution is known, we can obtain straightforwardly all thermodynamic quantities analytically after evaluating the Euclidean action. In EGB theory, we evaluate all thermodynamic quantities analytically. When the numerical solution is only possible, we numerically compute all thermodynamic quantities using thermodynamic relations. In DEGB theory, we employ numerical computation.

We begin by computing the Hawking temperature.

$$T_H = \frac{\kappa_h}{2\pi} = \frac{B'(r_h)e^{-\delta(r_h)}}{4\pi} \,. \tag{16}$$

If $\delta(r)$ is vanishing as in both Einstein and EGB theories, it takes the form

$$T_H = \frac{1}{2\pi}\left(\frac{GM}{r_h^2} + \frac{\Lambda}{3}r_h\right) = \frac{(k+\Lambda r_h^2)}{4\pi r_h}$$

by using $M(r_h) = \frac{r_h}{2G}\left(k + \frac{\Lambda}{3}r_h^2\right)$ and then $\beta = \frac{4\pi r_h}{k+\Lambda r_h^2}$

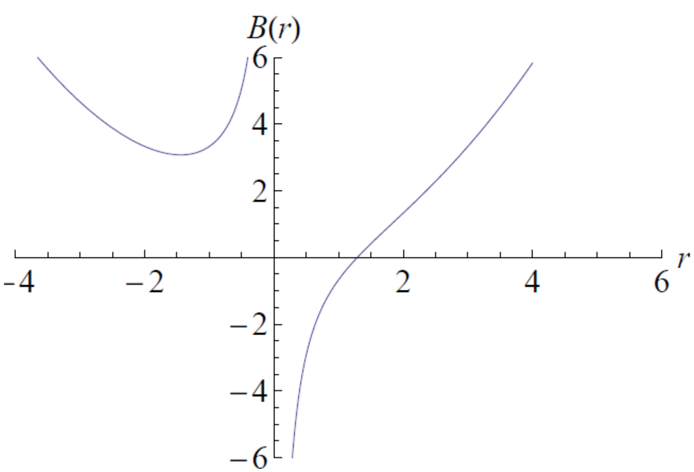
In Einstein and EGB theory

For a black hole with k=1

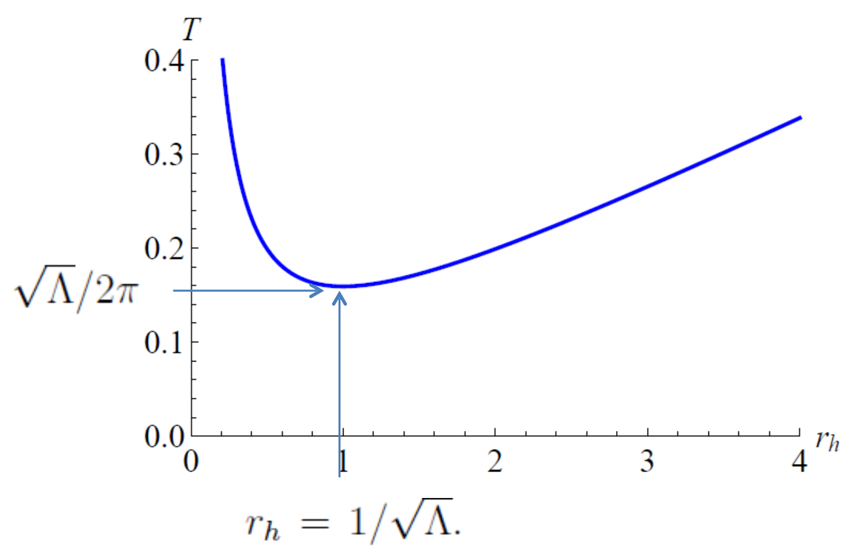
$$ds^2 = -(1 - \frac{2GM}{r} + \frac{\Lambda}{3}r^2)dt^2 + \frac{dr^2}{(1 - \frac{2GM}{r} + \frac{\Lambda}{3}r^2)} + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

We consider only the positive mass, because there is a naked singularity for the negative mass black hole.

metric function



temperature



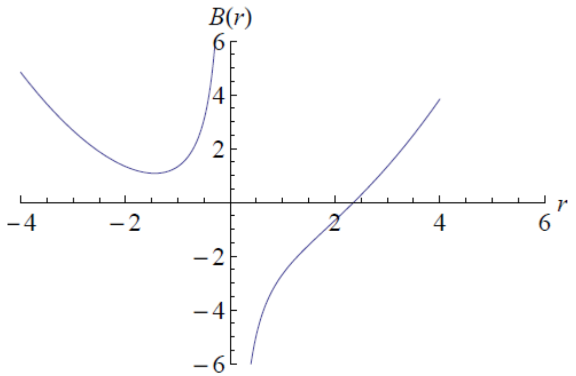
For a black hole with k=-1

$$ds^2 = -(-1 - \frac{2GM}{r} + \frac{\Lambda}{3}r^2)dt^2 + \frac{dr^2}{(-1 - \frac{2GM}{r} + \frac{\Lambda}{3}r^2)} + r^2(d\psi^2 + \sinh^2\psi d\phi^2),$$

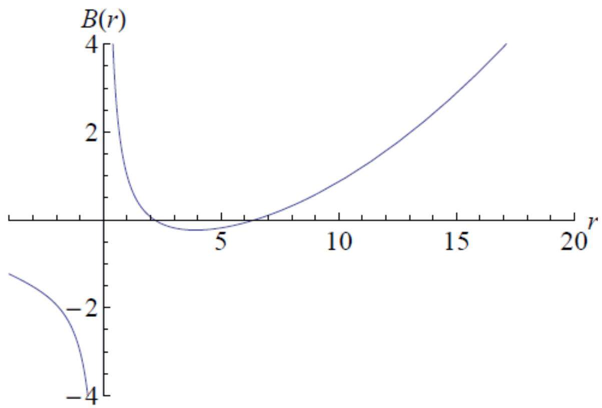
There is the limitation for the horizon radius $r_h > \sqrt{3/\Lambda}$ for the positive mass case.

There is the limitation for the horizon radius $r_h < \sqrt{3/\Lambda}$. for the negative mass case. There are two horizons. When $1/\Lambda = 9G^2M^2$, two horizons are coincide as $r_h = 1/\sqrt{\Lambda}$. This is the extremal case with the zero temperature.

metric function

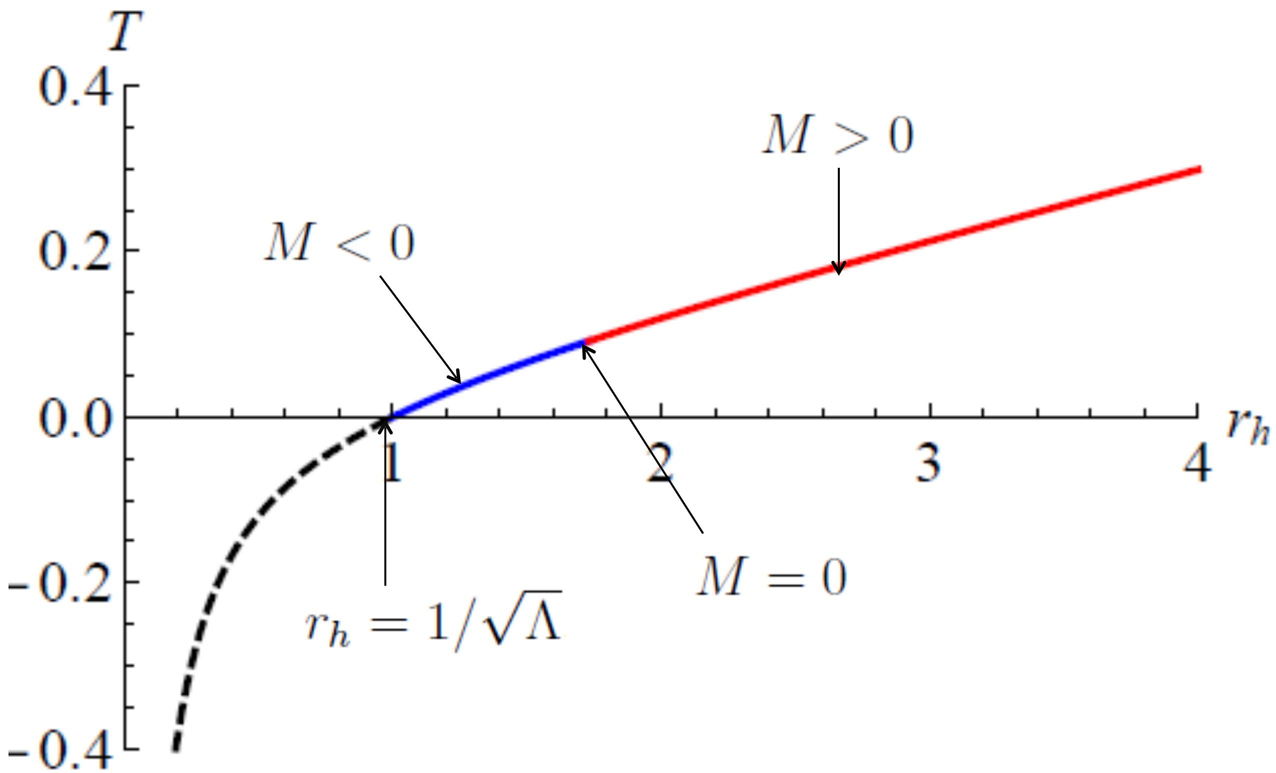


$M > 0$



$M < 0$

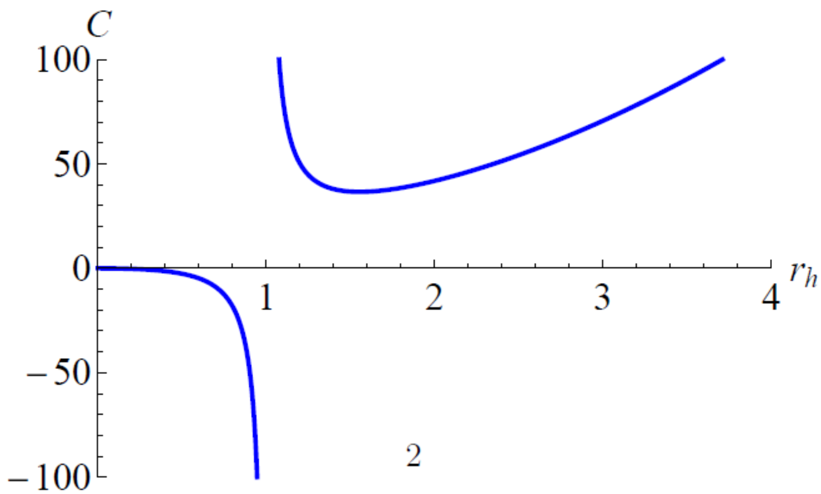
temperature



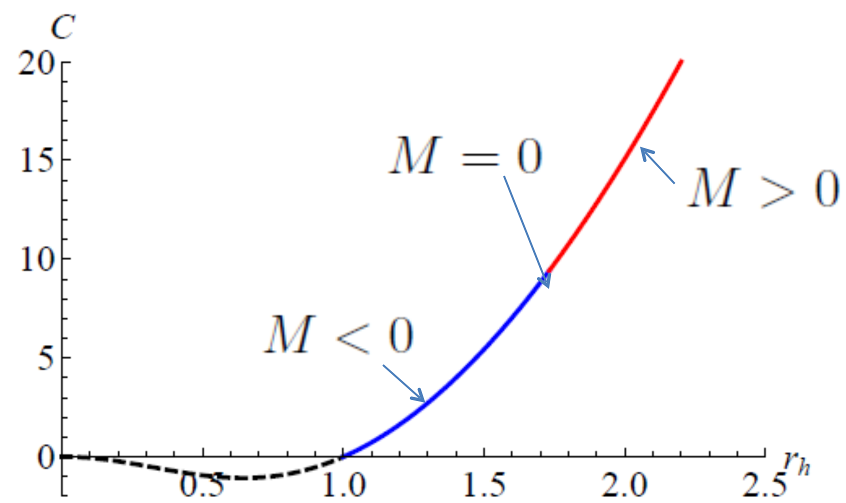
Specific heat

$$C = \frac{\partial M}{\partial T_H}$$

Where $M(r_h)$ **the mass of a black. If** $\delta(r)$ **is vanishing,** $C = \frac{2\pi}{G}\frac{r_h^2(k+\Lambda r_h^2)}{(\Lambda r_h^2 - k)}$



k=1



k=-1

3.2 Hawking-Page phase transition

We consider the Hawking-Page phase transition between a black hole in AdS space and the thermal AdS space.

In Einstein theory

Total action of a black hole

$$I_E^{\rm BE}({\rm total}) = x(\frac{\beta M}{4} - \frac{\Lambda\beta}{12G} r_h^3)$$

The first two terms give

$$I_E^{\rm TE}({\rm bulk}) + I_E^{\rm TE}({\rm YGH}) = x\beta_o\left[-\frac{\Lambda}{6G}r_{co}^3 - \frac{kr_{co}}{2G} + \frac{1-k}{2}\left(-\frac{\Lambda}{12G}r_{\rm crit}^3 + \frac{3M_{\rm crit}}{4}\right)\right]$$

where the periodicity $\beta_o\sqrt{k + \frac{\Lambda}{3}r^2 - \frac{(1-k)GM_{\rm crit}}{r}} = \beta_H\sqrt{k - \frac{2GM}{r} + \frac{\Lambda}{3}r^2}.$

Total action with the counter term

$$I_E^{\rm TE}({\rm total}) = x\frac{1-k}{2}\beta_o\left(-\frac{\Lambda}{12G}r_{\rm crit}^3 + \frac{M_{\rm crit}}{4}\right) \equiv \frac{x(1-k)}{4}\beta_o M_{\rm crit}$$

The difference of the free energy between a black hole in AdS and the thermal AdS

$$F^E = x\left(\frac{r_h(k - \frac{\Lambda}{3}r_h^2)}{8G} + \frac{(1-k)}{12G\sqrt{\Lambda}}\right)$$

We will use the relation $r_h = \frac{4\pi T \pm \sqrt{16\pi^2 T^2 - 4k\Lambda}}{2\Lambda}$

In EGB theory

For the thermal AdS, the action of the Gauss-Bonnet term

$$\int_{\mathcal{M}} \sqrt{g_E} d^4x_E \left[-\alpha(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) \right]$$
$$= x\alpha\beta_o \left[-\frac{16\pi\Lambda^2 r_{co}^3}{9} + \frac{1-k}{2} \left(-\frac{8k\pi(k+\Lambda r_h^2)}{r_h} + \frac{16\pi\Lambda G M_{\text{crit}}}{3} \right) \right]$$

The boundary term for GB term

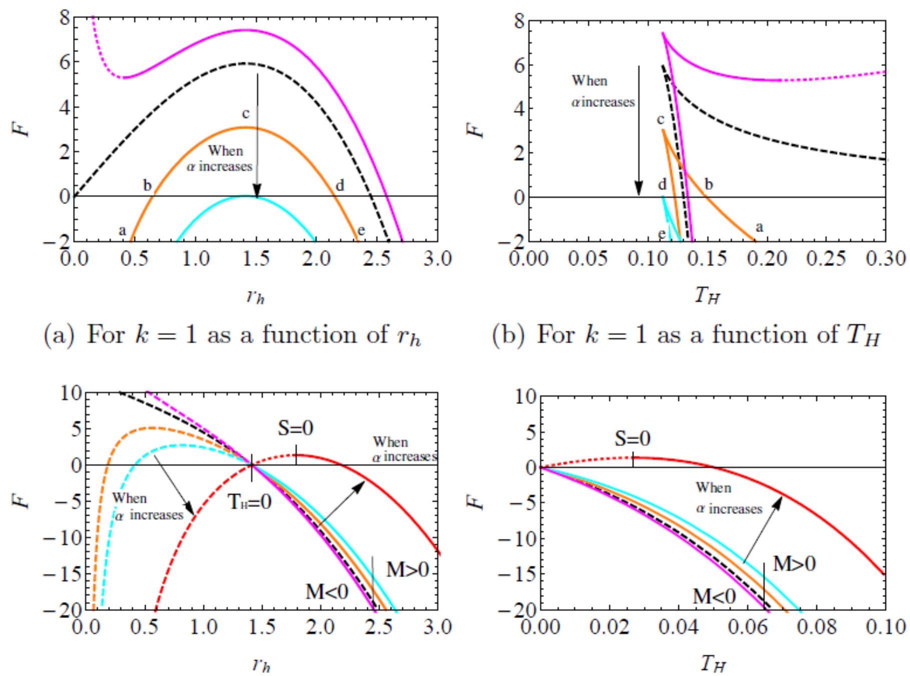
$$I_E^{\text{TE}}(\text{GBb}) = x\alpha\beta_o 16\pi \left[\frac{\Lambda^2 r_{co}^3}{9} - \frac{1-k}{2} \left(\frac{2G^2 M_{\text{crit}}^2}{r_{co}^3} + \frac{\Lambda G M_{\text{crit}}}{3} \right) \right]$$

$$I_E^{\text{TEGB}}(\text{total}) = 0 \quad k = 1 \qquad \text{constant} \quad k = -1$$

However, the additional contribution of the free energy goes to zero for the extremal BH.

The free energy

$$F^{EGB} = x\left(\frac{r_h(k-\frac{\Lambda}{3}r_h^2)}{8G} - \frac{k\alpha\kappa(k+\Lambda r_h^2)}{Gr_h} + \frac{(1-k)}{12G\sqrt{\Lambda}}\right)$$



(a) For $k = 1$ as a function of r_h (b) For $k = 1$ as a function of T_H

(c) For $k = -1$ as a function of r_h (d) For $k = -1$ as a function of T_H

Figure 7: (color online). Free energy difference in Einstein gravity and EGB theory both for $k = 1$ and for $k = -1$ with $\Lambda = 1/2$, and $\kappa = 1$. $\alpha = 0$ for the dashed line in Einstein theory. $\alpha = 0.040$ for the orange line, $\alpha = 0.083$ for the cyan line, $\alpha = 0.800$ for the green line, and $\alpha = -0.021$ for the magenta line in EGB theory.

4. Hairy black holes in DEGB theory

4.1 Action and black hole solutions

We consider the action with the Gauss-Bonnet term

$$I = \int_{\mathcal{M}} \sqrt{-g} d^4x \left[\frac{R + 2\Lambda e^{\lambda\Phi}}{2\kappa} - \frac{1}{2}\nabla^\mu \Phi \nabla_\mu \Phi + \alpha e^{-\gamma\Phi} R^2_{\text{GB}} \right] - \oint_{\partial\mathcal{M}} \sqrt{-h} d^3x \frac{K}{\kappa} + I_{\text{GBb}} + I_{\text{ct}} \,, \quad (1)$$

where $g = \det g_{\mu\nu}$ **and** $\kappa \equiv 8\pi G$

The higher curvature GB term is given $R^2_{\text{GB}} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$

The Einstein equations and the scalar field equation are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda e^{\lambda\Phi} g_{\mu\nu} = \kappa \left(\partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} \partial_\rho \Phi \partial^\rho \Phi + T^{GB}_{\mu\nu} \right) , \quad (2)$$

$$\frac{1}{\sqrt{-g}} \partial_\mu [\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi] - \alpha\gamma e^{-\gamma\Phi} R^2_{GB} + \frac{\Lambda\lambda e^{\lambda\Phi}}{\kappa} = 0 \,, \quad (3)$$

where the GB term contribute to the energy-momentum tensor

$$T^{GB}_{\mu\nu} = -8\alpha(R_{\mu\rho\nu\sigma}\nabla^\rho\nabla^\sigma e^{-\gamma\Phi} - R_{\mu\nu}\Box e^{-\gamma\Phi} + 2\nabla_\rho\nabla_{(\mu} e^{-\gamma\Phi} R^\rho{}_{\nu)} - \frac{1}{2}R\nabla_\mu\nabla_\nu e^{-\gamma\Phi}) \\ +4\alpha(2R^{\rho\sigma}\nabla_\rho\nabla_\sigma e^{-\gamma\Phi} - R\Box e^{-\gamma\Phi})g_{\mu\nu}\,, \tag{4}$$

and $\Box \equiv \nabla_\mu\nabla^\mu$ is the d'Alembertian.

We consider the metric

$$ds^2 = -B(r)e^{-2\delta(r)}dt^2 + B(r)^{-1}dr^2 + r^2 h_{ij}dx^i dx^j, \tag{5}$$

where $h_{ij}dx^i dx^j$ represents

$$(d\theta^2 + \sin^2\theta d\phi^2), \qquad \text{for } k = 1$$
$$(dx^2 + dy^2), \qquad \text{for } k = 0$$
$$(d\psi^2 + \sinh^2\psi d\phi^2), \qquad \text{for } k = -1$$

Then the dilaton field equation turns out to be

$$\Phi'' + \left(\frac{B'}{B} - \delta' + \frac{2}{r}\right)\Phi' + \frac{4\alpha\gamma e^{-\gamma\Phi}B}{r^2}\left(2\delta''B - 2\delta'^2 B - B'' + k\frac{B''}{B} + 5\delta' B' \right.$$
$$\left. - \frac{B'^2}{B} - 3k\delta'\frac{B'}{B} - 2k\delta'' + 2k\delta'^2 + \frac{\lambda r^2 \Lambda e^{\Phi(\lambda+\gamma)}}{4\alpha\gamma\kappa B}\right) = 0\,, \qquad (6)$$

And there are three Einstein equations after some rearrangement as follows:

$$2B'[4\alpha\kappa\gamma\Phi'(3B-k) + re^{\gamma\Phi}] - \kappa B\Phi'^2[16\alpha\gamma^2(B-k) - r^2 e^{\gamma\Phi}]$$
$$+16\alpha\kappa\gamma B(B-k)\Phi'' + 2e^{\gamma\Phi}(B - k - \Lambda r^2 e^{\lambda\Phi}) = 0\,, \qquad (7)$$

$$\delta'\left[4\alpha\gamma\kappa(3B-k)e^{-\gamma\Phi}\Phi' + r\right] + \frac{1}{2}\kappa r^2\Phi'^2 - 4\alpha\gamma\kappa(k-B)e^{-\gamma\Phi}\left(\Phi'' - \gamma\Phi'^2\right) = 0\,. \qquad (8)$$

$$B\kappa r e^{-\gamma\Phi-\delta}\left[32\alpha^2\gamma^2(B-k)B' - 64B\alpha^2\gamma^2\kappa(B-k)\delta' + r^2 e^{\gamma\Phi}(4B\alpha\gamma\kappa\Phi' + re^{\gamma\Phi})\right]\Phi''$$
$$-e^{-\gamma\Phi-\delta}\left[-B'\kappa\left\{8\alpha\gamma(B-k)e^{\gamma\Phi} - 8B\alpha\gamma\kappa\Phi'^2\left(4\alpha\gamma^2(B-k) - r^2e^{\gamma\Phi}\right)\right.\right.$$
$$\left. +8B\alpha\gamma\delta'\left(8\alpha\gamma\kappa k\Phi' + re^{\gamma\Phi}\right) + r^3 e^{2\gamma\Phi}\Phi'\right\} + 4\alpha\gamma\kappa B'^2\left(8\alpha\gamma\kappa k\Phi' + re^{\gamma\Phi}\right)$$
$$+re^{\gamma\Phi}\left\{4B\alpha\gamma\kappa^2(k-5B)\Phi'^2 - \Lambda e^{\lambda\Phi}\left(-8B\alpha\gamma\kappa + 8\alpha\gamma\kappa k + \lambda r^2 e^{\gamma\Phi}\right)\right.$$
$$\left. -2B\kappa r\Phi'\left(e^{\gamma\Phi} + 4\alpha\gamma\lambda\Lambda e^{\lambda\Phi}\right)\right\}$$
$$\left. +B\kappa\delta'\left\{8\alpha\gamma(B-k)e^{\gamma\Phi} - 8B\alpha\gamma\kappa\Phi'^2\left(8\alpha\gamma^2(B-k) - r^2e^{\gamma\Phi}\right) + r^3e^{2\gamma\Phi}\Phi'\right\}\right] = 0\,. \qquad (9)$$

In the present work, we choose three Eqs. (7) - (9) as dynamical equations and the remaining one Eq. (6) as the constraint equation.

We impose the boundary condition at the event horizon to examine the existence of a black hole. $g^{rr}(r_h) = 0$ or $g_{rr}(r_h) = \infty$

At the horizon, $B(r_h) = 0$, $\delta(r_h)$ **and** $\Phi(r)$ **are finite.**

From Eq. (7) we obtain the value of B′(r) at the horizon

$$B'(r_h) = \frac{e^{\gamma\Phi_h}(\Lambda r^2 e^{\lambda\Phi_h} + k)}{r e^{\gamma\Phi_h} - 4\alpha\gamma\kappa k \Phi_h'} \,. \qquad (10)$$

By plugging Eq. (10) into (9), we obtain a quadratic equation with respect to $\Phi'(r_h)$

$$\Phi'(r_h) = \frac{-D \pm \sqrt{D^2 - 4AC}}{2A}\,, \tag{11}$$

where

$$\begin{aligned} A &= 4\alpha\gamma\kappa[kr_h^2 e^{2\gamma\Phi} - 32\alpha^2\gamma^2\kappa k^2 \Lambda e^{\lambda\Phi} - 4\alpha\gamma k\lambda\Lambda r_h^2 e^{\Phi(\gamma+\lambda)} + \Lambda r_h^4 e^{\Phi(2\gamma+\lambda)}]\,, \\ D &= -r_h^3 e^{3\gamma\Phi} + 96\alpha^2\gamma^2\kappa k\Lambda r_h e^{\Phi(\gamma+\lambda)} + 32\alpha^2\gamma^2\kappa\Lambda^2 r_h^3 e^{\Phi(\gamma+2\lambda)} \\ &\quad +8\alpha\gamma\lambda\Lambda r_h^3 e^{\Phi(2\gamma+\lambda)} - \Lambda r_h^5 e^{\Phi(3\gamma+\lambda)}/k\,, \\ C &= \frac{e^{2\gamma\Phi}}{2\kappa}\left[24\alpha\gamma\kappa k + 16\alpha\gamma\kappa\Lambda r_h^2 e^{\lambda\Phi} - 2\Lambda r_h^4\left(-4\alpha\gamma\kappa\Lambda e^{2\lambda\Phi} + \lambda e^{\Phi(\gamma+\lambda)}\right)/k\right]\,, \end{aligned}$$

The term inside square root of $\Phi'(r_h)$ **ould be positive, thus only certain region of a** Φ_h **ca** r_h **make** $\Phi'(r_h)$ **The cases for k = ±1 have allowed and forbidden regions which suggest that there are lower or upper bound of the black hole size. We choose the − sign in Eq. (11).**

We now impose appropriate boundary conditions for the behavior at the asymptotic region. We expect the metric function and the field to be

$$B(r) \simeq k - \frac{2GM}{r} + \frac{\Lambda e^{\lambda\Phi}}{3} r^2, \quad \delta \simeq \delta_\infty + \frac{\delta_1}{r^\beta}, \quad \text{and} \quad \Phi \simeq \Phi_\infty + \frac{Q}{r^\sigma}, \tag{14}$$

where M is the black hole mass, Q the scalar charge, δ_∞ and Φ_∞ the asymptotic values of the lapse function and the scalar field, respectively.

The mass of a hairy black hole is represented as follows

$$M(r) = M(r_h) + M_{\text{hair}} .$$

where $M(r_h) = \frac{1}{2} r_h$ is the mass of a black hole subtracting the contribution from the scalar hair. The second term in the right-hand side represents the contribution from the scalar hair, so we will use this term as a amount of hair. The M(r) increases up to some constant as the distance from the horizon increases, if Φ' rapidly decreases to be zero.

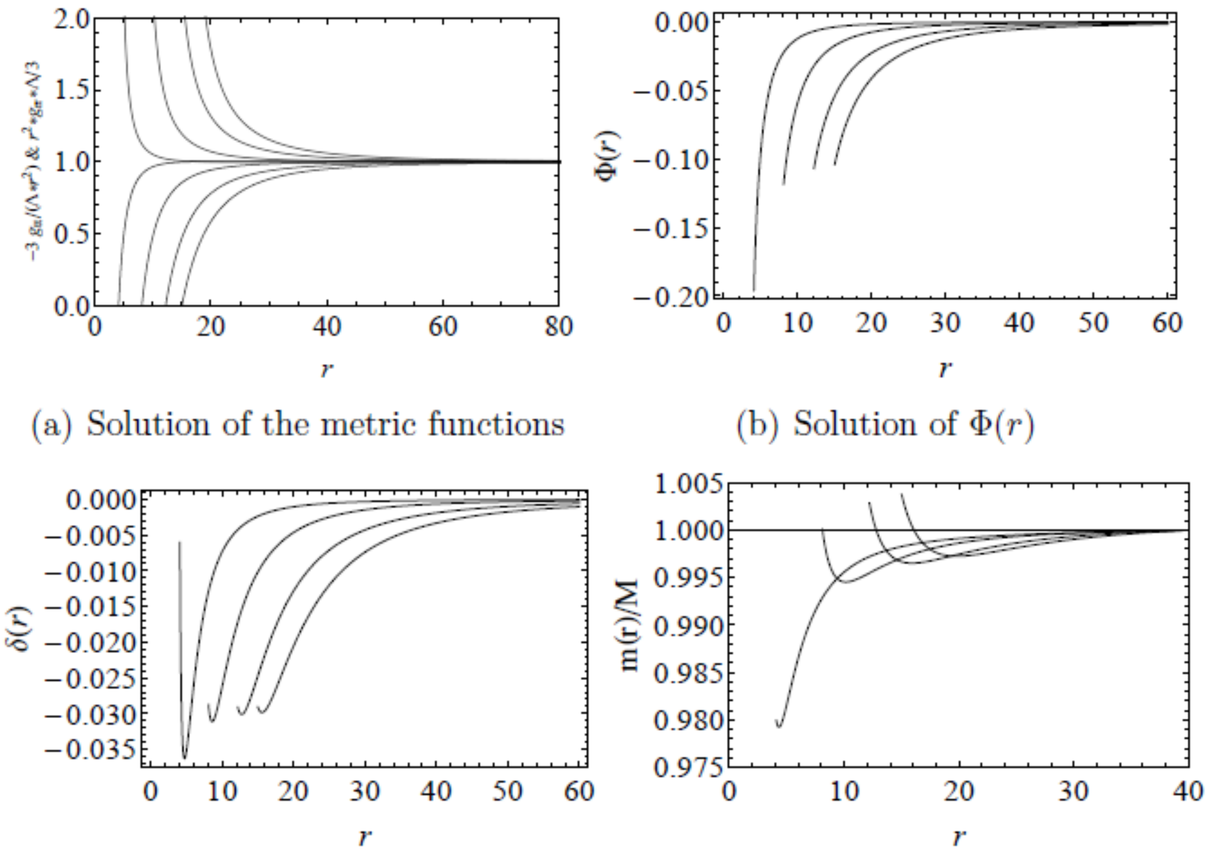
4.2 Numerical Black Hole solutions

We employ the rescaling. $\tilde{\Phi} = \Phi - \Phi_\infty \quad \tilde{\Lambda} = \Lambda e^{(\lambda-\gamma)\Phi_\infty},$

$$\tilde{r} = re^{\gamma\Phi_\infty/2},\ \tilde{t} = te^{\gamma\Phi_\infty/2},\ \tilde{M} = Me^{\gamma\Phi_\infty/2},\ \tilde{Q} = Qe^{3\gamma\Phi_\infty/2}.$$

We follow the procedure according to Refs : Z. K. Guo, N. Ohta and T. Torii, PTP 121, 253 (2009), N. Ohta and T. Torii, PTP, 121, 959 (2009).

4.2.1 Spherically symmetric solutions with k = 1



(a) Solution of the metric functions

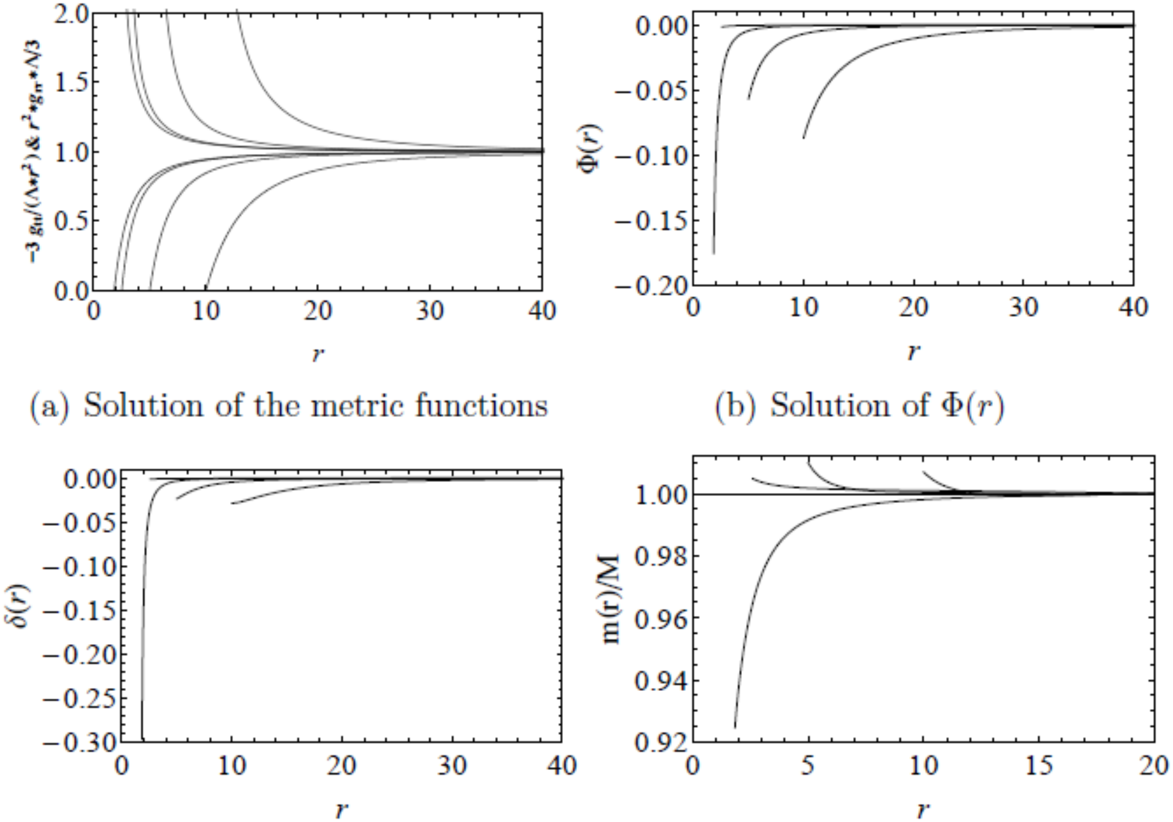
(b) Solution of $\Phi(r)$

(c) Solution of $\delta(r)$

(d) Mass function $m(r)$

Figure 1: (color online). Numerical solutions for $k = 1$ with $\alpha = 1$, $\gamma = 1/2$, $\lambda = 1/2$, $\Lambda = 1/2$, and $\kappa = 1$. We choose four solutions with the initial r_h as 4.09, 8.1, 12.2, and 15, respectively.

4.2.2 Hyperbolic solutions with k = −1



(a) Solution of the metric functions

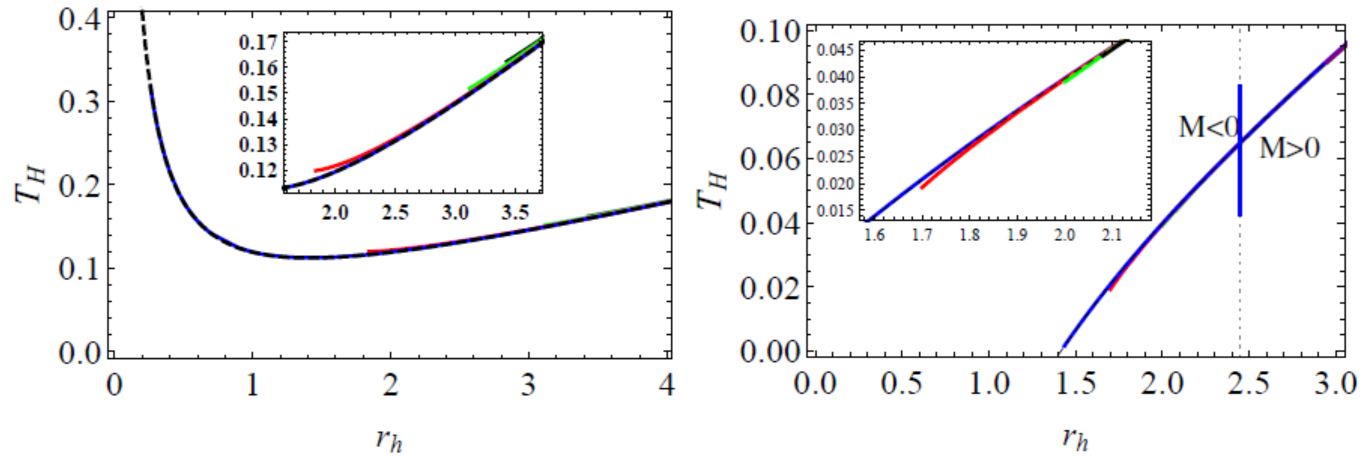
(b) Solution of $\Phi(r)$

(c) Solution of $\delta(r)$

(d) Mass function $m(r)$

Figure 2: (color online). Numerical solutions for $k = -1$ with $\alpha = 1$, $\gamma = 1/2$, $\lambda = 1/2$, $\Lambda = 1/2$, and $\kappa = 1$. We choose four solutions with the initial r_h as 1.83, 2.60, 5.00, and 10.00, respectively.

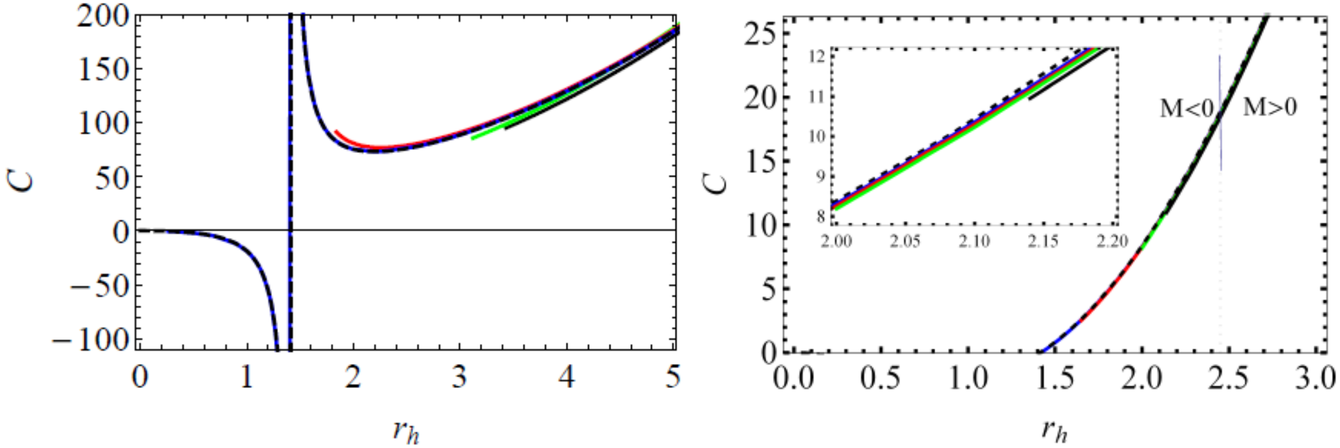
In DEGB theory



(a) Temperature for $k = 1$ (b) Temperature for $k = -1$

Figure 3: (color online). The Hawking temperature as a function of the horizon radius r_h with $\gamma = 1/2$, $\Lambda = 1/2$, and $\kappa = 1$. $\alpha = 0$ for the dashed line both in Einstein theory and EGB theory. $\alpha = 0.005$ for the blue line, $\alpha = 0.400$ for the red line, $\alpha = 0.800$ for the green line, and $\alpha = 1.000$ for the black line in DEGB theory.

In DEGB theory



(a) Specific heat for $k = 1$ (b) Specific heat for $k = -1$

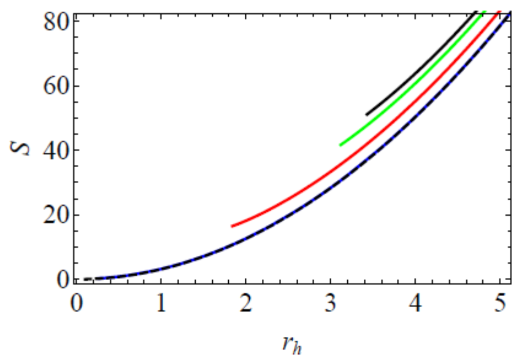
Figure 4: (color online). The specific heat as a function of the horizon radius r_h with $\gamma = 1/2$, $\Lambda = 1/2$, and $\kappa = 1$. $\alpha = 0$ for the dashed line both in Einstein theory and EGB theory. $\alpha = 0.005$ for the blue line, $\alpha = 0.400$ for the red line, $\alpha = 0.800$ for the green line, and $\alpha = 1.000$ for the black line in DEGB theory.

In DEGB theory

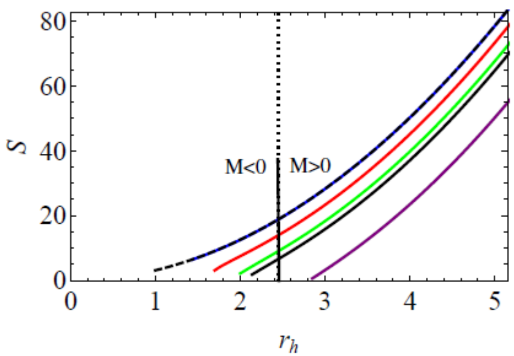
We now consider the black hole in AdS space in DEGB theory. The solutions are obtained numerically. Thus we cannot evaluate the Euclidean action and the counter term analytically.

by Torii and Maeda, PRD 58, 084004 (1998)

$$S_{DGB} = \frac{A}{4G}\left(1 + \frac{8k\alpha\kappa}{r_h^2}e^{-\gamma\Phi_h}\right) . \tag{22}$$



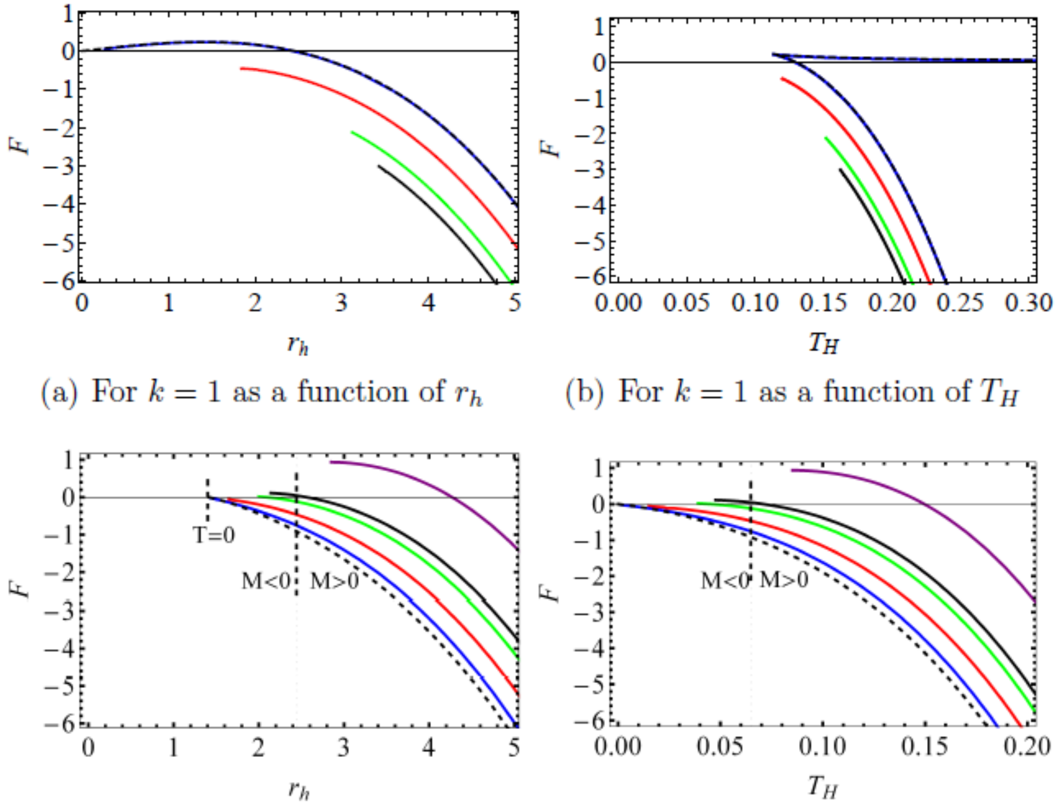
(a) Entropy for $k = 1$



(b) Entropy for $k = -1$

Figure 6: (color online). The entropy as a function of the horizon radius r_h with $\gamma = 1/2$, $\Lambda = 1/2$, and $\kappa = 1$. $\alpha = 0.000$ for the dashed line in Einstein theory. $\alpha = 0.005$ for the blue line, $\alpha = 0.400$ for the red line, $\alpha = 0.800$ for the green line, $\alpha = 1.000$ for the black line, and $\alpha = 2.000$ for the purple line in DEGB theory.

In DEGB theory



(a) For $k = 1$ as a function of r_h

(b) For $k = 1$ as a function of T_H

(c) For $k = -1$ as a function of r_h

(d) For $k = -1$ as a function of T_H

Figure 8: (color online). The free energy difference both for $k = 1$ and $k = -1$ with $\gamma = 1/2$, $\Lambda = 1/2$, and $\kappa = 1$. $\alpha = 0.000$ for the dashed line in Einstein theory. $\alpha = 0.005$ for the blue line, $\alpha = 0.400$ for the red line, $\alpha = 0.800$ for the green line, $\alpha = 1.000$ for the black line, and $\alpha = 2.000$ for the purple line in DEGB theory.

5. Summary and Discussion

We study the black hole thermodynamics and the Hawking-Page phase transition in asymptotically anti-de Sitter spacetime in both Einstein-Gauss-Bonnet and the dilatonic Einstein-Gauss-Bonnet theory.

We think that the topological information of the spacetime manifold could be additionally stored in the thermodynamic quantities for a black hole.

In four dimensions, the existence of the GB term does not influence both the equations of motion and the solutions. The observer cannot distinguish the difference between the Einstein theory and EGB theory. However, the observer can distinguish the differences of thermodynamic properties and phase transition between two theories. For this reason, the observation of the HP phase transition can determine which theory of gravitation describes our universe.