Thermodynamic Mass and Volume in Lifshitz Spacetimes [1]

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IBS (POSTECH)

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[1] W. G. Brenna, R. B. Mann and M. Park, "Mass and Thermodynamic Volume in Lifshitz Spacetimes," Phys. Rev. D **92**, no. 4, 044015 (2015)

Contents

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  - the energy properties of matter are represented by $T_{\mu\nu}$. The local energy density of matter as measured by a given observer remain well defined.
  - A gravitational field energy should make a contribution to total energy, but is not included in $T_{\mu\nu}$. However, there is no known meaningful notion of the energy density of gravitational field in GR.

How to define a Global charge

- ADM Method in 1959
- Komar Method in 1963
- AD(T) Method in 1982, (2002)
- BY Method in 1993
- Wald's Method (Noether Formula) in 1993

How to define a Global charge

- ADM Method in 1959

$$H = -\frac{1}{8\pi}\oint_{\text{st}}\left[N(k-k_0) - N_a(K^{ab} - Kh^{ab})r_b\right]\sqrt{\sigma}d^2\theta, \quad (2)$$

$$(N=1,\ \ N_a=0) \quad M = -\frac{1}{8\pi}\lim_{\text{st}\to\infty}\oint_{\text{st}}(k-k_0)\sqrt{\sigma}d^2\theta, \quad (3)$$

$$(N=0,\ N_a=\phi^a) \quad J = -\frac{1}{8\pi}\lim_{\text{st}\to\infty}\oint_{\text{st}}(K_{ab} - Kh_{ab})\phi^a r^b\sqrt{\sigma}d^2\theta \quad (4)$$

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$$\nabla^a \nabla_a \xi_b = -R_{ab} \xi^a$$

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$$M = -\frac{1}{8\pi}\lim_{\text{st}\to\infty}\oint_{\text{st}} \nabla^\alpha \xi^\beta_{(t)} dS_{\alpha\beta} = 2\int_V (T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta}) n^\alpha \xi^\beta_{(t)} \sqrt{h} d^3 y,$$

$$J = \frac{1}{16\pi}\lim_{\text{st}\to\infty}\oint_{\text{st}} \nabla^\alpha \xi^\beta_{(\phi)} dS_{\alpha\beta} = -\int_V (T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta}) n^\alpha \xi^\beta_{(\phi)} \sqrt{h} d^3 y$$

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$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad R_L^{\mu\nu} - \frac{1}{2}\bar{g}^{\mu\nu} R_L - \Lambda h^{\mu\nu} = -\sqrt{-\bar{g}}^{-1/2} T^{\mu\nu}$$

$$(-\bar{g})^{1/2} T_{\mu\nu} = \bar{D}_\alpha \bar{D}_\beta K^{\mu\alpha\nu\beta} + X^{\mu\nu},$$

$$K^{\mu\alpha\nu\beta} = \frac{1}{2}[\bar{g}^{\mu\beta} H^{\nu\alpha} + \bar{g}^{\nu\alpha} H^{\mu\beta} - \bar{g}^{\mu\nu} H^{\alpha\beta} - \bar{g}^{\alpha\beta} H^{\mu\nu}],$$

$$H^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\bar{g}^{\mu\nu} h^a{}_a, \qquad X^{\mu\nu} = \frac{1}{2}\bar{R}^\nu{}_{\lambda\alpha\beta} K^{\mu\lambda\alpha\beta},$$

$$\bar{D}_\mu T^{\mu\nu} = 0 \;\Rightarrow\; \bar{D}_\mu (T^{\mu\nu}\bar{\xi}_\nu) = \partial_\mu (T^{\mu\nu}\bar{\xi}_\nu) = 0 \;\Rightarrow\; E(\bar{\xi}) = \frac{1}{8\pi}\int d^3x T^{0\nu}\bar{\xi}_\nu$$

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$$\tau^{ij} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma_{ij}} = \frac{2}{\sqrt{-\gamma}} (\pi^{ij} - \pi_0^{ij}), \tag{2}$$

$$Q_\xi(B) = \int_B d^2x \sqrt{\sigma} (u_i \tau^{ij} \xi_j) \tag{3}$$

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$$\mathcal{L}_\xi L(\phi) = \frac{\partial L}{\partial \phi} \mathcal{L}_\xi \phi$$

$$\omega(\phi, \delta\phi, \xi\phi) = \delta J - d(\xi \cdot \Theta), \qquad \Omega(\phi, \delta_1\phi, \delta_2\phi) = \int_C \omega(\phi, \delta_1\phi, \delta_2\phi)$$

$$\delta H = \Omega(\phi, \delta\phi, \mathcal{L}_\xi \phi) = \delta \int_C J - \int_C d(\xi \cdot \Theta) = \delta \int_C J - \int_\infty \xi \cdot \Theta$$

$$H = \int_C J - \int_\infty \xi \cdot \Theta = \int_\infty (Q - \xi \cdot B) \quad \text{where} \quad \delta \int_\infty \xi \cdot B = \int_\infty \xi \cdot \Theta$$

Black Hole Thermodynamics

In 1972, Hawking proved that the area of black hole horizon can never decreas

In 1973, this work resulted in "four laws of black hole mechanics" by Bardeen, Carter, and Hawking

0. The surface gravity κ is constant over the event horizon.

1. For two stationary black holes differing only by small variations in the parameters M, J, and Q,

$$\delta M = \frac{\kappa}{8\pi G}\delta A_{\text{hor}} + \Omega_H \delta J + \Phi_H \delta Q, \tag{2.1}$$

   where Ω_H is the angular velocity and Φ_H is the electric potential at the horizon.

2. The area of the event horizon of a black hole never decreases,

$$\delta A_{\text{hor}} \geq 0. \tag{2.2}$$

3. It is impossible by any procedure to reduce the surface gravity κ to zero in a finite number of steps.

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$$ds_E^2 \approx (\kappa x)^2 d\tau^2 + dx^2 = dx^2 + x^2 d(\kappa\tau)^2, \qquad (2)$$

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$$\begin{aligned} G_\beta(x, 0; x', t) &= \text{Tr}\left(e^{-\beta H}\varphi(x,0)\varphi(x',t)\right) = \text{Tr}\left(e^{-\beta H}\varphi(x,0)e^{-itH}\varphi(x',0)e^{itH}\right) \\ &= \text{Tr}\left(\varphi(x,0)e^{-\beta H}e^{-i(t+i\beta)H}\varphi(x',0)e^{i(t+i\beta)H}\right) \\ &= \text{Tr}\left(\varphi(x,0)e^{-\beta H}\varphi(x',t+i\beta)\right) = G_\beta(x', t+i\beta; x, 0) \end{aligned} \quad (3)$$

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- Bekenstein-Hawking Entropy in 1972, 1973

$$S = \frac{A}{4G} \tag{4}$$

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- Entropy from Noether theorem in 1993

$$S = 2\pi \int_{\Sigma} Q^{cd} \epsilon_{cd} \tag{5}$$

Komar Formula for Asymptotically Flat : Smarr Relaltion

$$M = -\frac{1}{8\pi}\oint_{\partial V} dS_{\mu\nu} D^{\mu} k^{\nu}, \quad T = \frac{\kappa}{2\pi}, \quad S = \frac{A}{4G},$$
$$\Phi_H = A_t(r_H), \quad Q = \frac{1}{4\pi}\int *F \tag{6}$$

The first law of thermodynamics of Black Holes is satisfied

$$dM = TdS + \Phi_H dQ \tag{7}$$

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Thermodynamic Mass

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and so the thermodynamic mass can be independently calculated by

$$M = \int dr_h \frac{\partial M}{\partial r_h} = \int dr_h T \frac{\partial S}{\partial r_h} \tag{10}$$

Generalised Komar Formula for AAdS

$$R_{ab} = 0$$

- Anne Magnon in 1985
- David Kastor and et al. in 2009

$$\frac{D-2}{8\pi G}\int_{\partial\Sigma} dS_{ab}\left(\nabla^a \xi^b\right) = 0$$

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  - The cosmological constant can be thought of as a perfect fluid stress-energy with pressure $P = -\Lambda/8\pi G$.

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$$dM = TdS + \Phi_H dQ + VdP \;\Rightarrow\; \frac{\partial M}{\partial l} = V\frac{\partial P}{\partial l} \;\Rightarrow\; V = \frac{\partial M}{\partial l}\Big/\frac{\partial P}{\partial l} \tag{11}$$

where V is effective volume inside the horizon.

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- We can also naively calculate a geometrical volume

$$V' = \int_{r_0}^{r_+} dr \int d\Omega \sqrt{-g} = \frac{r_+ V}{D-1} \quad (12)$$

over interior of the black hole, where the radial coordinate ranges from the singularity at $r = r_0$ to the outer horizon at $r = r_+$, and A is the area of the outer horizon.

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- V coincide with V' for simple cases such as Sch-AdS and RN-AdS black hole

Generating Smarr Relation with Eulerian Scaling

Euler theorem states that if a function $f(x, y, z)$ obeys the scaling relation

$$f(\alpha^p x, \alpha^q y, \alpha^k z) = \alpha^r f(x, y, z), \tag{13}$$

then it satisfies

$$r f(x, y, z) = p \left(\frac{\partial f}{\partial x}\right) x + q \left(\frac{\partial f}{\partial y}\right) y + k \left(\frac{\partial f}{\partial z}\right) z. \tag{14}$$

In a black hole spacetime with a charge and a cosmological constant, physical variables scale as

$$M \propto l^{D-3}, \quad A \propto l^{D-2}, \quad \Lambda \propto l^{-2}, \quad Q \propto l^{D-3}, \tag{15}$$

then the Euler's theorem yields

$$(D-3)M = (D-2)\left(\frac{\partial M}{\partial A}\right) A - 2\left(\frac{\partial M}{\partial \Lambda}\right)\Lambda + (D-3)\left(\frac{\partial M}{\partial Q}\right) Q. \tag{16}$$

2

[2] D. Kastor, S. Ray and J. Traschen, "Enthalpy and the Mechanics of AdS Black Holes," Class. Quant. Grav. **26**, 195011 (2009)

Generating Smarr Relation with Eulerian Scaling

Euler theorem states that if a function $f(x, y, z)$ obeys the scaling relation

$$f(\alpha^p x, \alpha^q y, \alpha^k z) = \alpha^r f(x, y, z), \tag{13}$$

then it satisfies

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Smarr Relation for RN-AdS in 4D

$$ds^2 = -\frac{r^2}{l^2}f(r)dt^2 + \frac{l^2}{r^2}\frac{dr^2}{f(r)} + r^2 d\Omega_k^2, \qquad f(r) = 1 + k\frac{l^2}{r^2} - \frac{2ml^2}{r^3} + \frac{q^2 l^2}{r^4}, \tag{17}$$

$$M = \frac{m}{4\pi}\omega_{k,2}, \quad T = \frac{1}{4\pi}\left(\frac{k}{r_h} + \frac{3r_h}{l^2} - \frac{q^2}{r_h}\right), \quad S = \frac{\omega_{k,2} r_h^2}{4},$$

$$\Phi_H = \frac{q}{r_H}, \quad Q = \frac{\omega_{k,2}}{4\pi}q, \quad P = \frac{3}{8\pi l^2}, \quad V = \frac{\omega_{k,2} r_h^3}{3}$$

- The first law of thermodynamics :

$$dM = TdS + \Phi_H dQ + VdP \tag{18}$$

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Spacetime with a negative cosmological constant

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where z is called dynamical critical exponent.
This anisotropic symmetry can be geometrically configured by

$$ds^2 = l^2 \left(-\frac{dt^2}{r^{2z}} + \frac{dr^2}{r^2} + \frac{dx_1^2 + dx_2^2}{r^2} \right)$$

where $z \neq 1$. When $z = 1$ it restores the AdS spacetime.

Lifshitz Spacetime Actions I

- A massive vector field with a negative cosmological constant

$$S = \int d^{n+1}x\sqrt{-g}\left(\frac{1}{2\kappa_{n+1}^2}[R+2\Lambda] - \frac{1}{{g_v}^2}\left[\frac{1}{4}H^2 + \frac{\gamma}{2}B^2\right]\right)$$

where

$$B = \frac{g_v l}{\kappa_{n+1}}\frac{q}{r^z}dt, \qquad H = dB$$

1. Miranda C.N. Cheng, Sean A. Hartnoll, Cynthia A. "Deformations of Lifshitz holography", JHEP 1003 (2010) 062
2. Miok Park, Robert B. Mann, "Deformations of Lifshitz Holography in (n+1) -dimensions", JHEP 1207 (2012) 173
3. M.H. Dehghani, Robert B. Mann, "Lovelock-Lifshitz Black Holes", JHEP 1007 (2010) 019

Lifshitz Spacetime Actions II

- higher curvature terms with a negative cosmological constant

$$S = \int d^{n+1}x\sqrt{-g}\left(\frac{1}{2\kappa_{n+1}^2}[R + 2\Lambda + f(R)]\right)$$

where f(R) is some function of the Ricci curvature

$$\mathcal{L}_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$$

1. Rong-Gen Cai, Yan Liu, Ya-Wen Sun, “A Lifshitz Black Hole in Four Dimensional R**2 Gravity”, JHEP 0910 (2009) 080
2. Yun Soo Myung, Taeyoon , “Quasinormal frequencies and thermodynamic quantities for the Lifshitz black holes”, Phys.Rev. D86 (2012) 024006
3. Eloy Ayon-Beato, Alan Garbarz, Gaston Giribet, Mokhtar Hassaine, “Analytic Lifshitz black holes in higher dimensions”, JHEP 1004 (2010) 030

Lifshitz Spacetime Actions III

- Abelian gauges fields with a negative cosmological constant

$$S = \int d^4x\sqrt{-g}\left(R + 2\Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{12}H_{\mu\nu\tau}H^{\mu\nu\tau} - \frac{C}{\sqrt{-g}}\epsilon^{\mu\nu\alpha\beta}B_{\mu\nu}F_{\alpha\beta}\right)$$

where $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$ and $H_{\mu\nu\tau} = \partial_{[\mu}B_{\nu\tau]}$ are Abelian gauge fields that are topologically coupled with coupling constant C.

1. Ulf H. Danielsson, Larus Thorlacius, "Black holes in asymptotically Lifshitz spacetime", JHEP 0903 (2009) 070
2. Robert B. Mann, "Lifshitz Topological Black Holes", JHEP 0906 (2009) 075
3. M.H. Dehghani, R.B. Mann, R. Pourhasan, "Charged Lifshitz Black Holes", Phys.Rev. D84 (2011) 046002

Lifshitz Spacetime Actions IV

- Einstein-Maxwell-dilaton with a negative cosmological constant

$$S=\int d^4x\sqrt{-g}\left(R+2\Lambda-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{4}e^{\lambda\phi}F_{\mu\nu}F^{\mu\nu}\right)$$

1. Chanyong Park, "Notes on the holographic Lifshitz theory", arXiv:1305.6690
2. Javier Tarrio, Stefan Vandoren, "Black holes and black branes in Lifshitz spacetimes", JHEP 1109 (2011) 017

Smarr Relation formulated by T.V.A.

Let us consider Sch-AdS black hole spacetime

$$f(r) = g(r) = 1 + k\frac{l^2}{r^2} - 2\frac{ml^2}{r^{D-1}}$$

This yields

$$T = \frac{1}{4\pi}\left[(D-1)\frac{r_h}{l^2} + (D-3)k\frac{1}{r_h}\right], \tag{20}$$

$$M = (D-2)\frac{\omega_{D-2}m}{16\pi} = (D-2)\frac{\omega_{D-2}}{16\pi}\left(\frac{r_h^{D-1}}{l^2} + k\omega_{D-2}r_h^{D-3}\right), \tag{21}$$

$$S = \omega_{k,D-2}\frac{r_h^{D-2}}{4}, \qquad P = -\frac{\Lambda}{8\pi G} = \frac{(D-1)(D-2)}{16\pi l^2} \tag{22}$$

and geometric arguments imply that the thermodynamic volume coincides with the geometric volume

$$V = \omega_{k,D-2}\frac{r_h^{D-1}}{(D-1)}$$

where $\omega_{k,D-2}$ is the surface area of the space orthogonal to fixed (t,r) surfaces.

Smarr Relation formulated by T.V.A.

Let us assume a generalized Smarr Relation for a uncharged case

$$M = aTS + bPV \tag{23}$$

where a, b are undetermined coefficients. Plugging all variables, we then have

$$[(D-2) - a(D-1) - b(D-2)]\frac{r_h^{D-1}}{l^2} + k\left[(D-2) - a(D-3)\right] r_h^{D-3} = 0 \tag{24}$$

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- regardless a value of k

$$a = \frac{(D-2)}{(D-3)}, \qquad b = -\frac{2}{(D-3)} \quad \Rightarrow \quad (D-3)M = (D-2)TS - 2PV \tag{25}$$

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  When $c = 0$, we could have Smarr Relation without PV even if presence of Λ

Smarr Formulas for $k = 0$ case

- From Eulerian Scaling

$$(D-3)M = (D-2)TS - 2PV \tag{28}$$

- From thermodynamic Ansatz

$$(D-1)M = (D-2)TS \tag{29}$$

Smarr Relation for $k = 0$ case

Let us consider the following metric which is Lifshitz for $z \neq 1$ and AdS for $z = 1$

$$ds^2 = -\left(\frac{r}{l}\right)^{2z} f(r)dt^2 + \frac{l^2}{r^2}\frac{dr^2}{g(r)} + r^2 d\Omega_k^2 \tag{30}$$

Employing the ansatz $M = M_0 r_h^{\beta} l^{\alpha}$ where M_0 is dimensionless and $\alpha + \beta = D - 3$,

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Employing the ansatz $M = M_0 r_h^{\beta} l^{\alpha}$ where M_0 is dimensionless and $\alpha + \beta = D - 3$,

- Thermodynamic mass relation

$$\frac{\partial M}{\partial r_h} = T\frac{\partial S}{\partial r_h} \qquad \Rightarrow \qquad M_0 = \frac{V}{\alpha l^{\alpha-1} \cdot r_h^{\beta}}\frac{\partial P}{\partial l} \tag{31}$$

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- Thermodynamic volume relation

$$\frac{\partial M}{\partial l} = V\frac{\partial P}{\partial l} \quad \Rightarrow \quad V = \frac{T\cdot\alpha\cdot r_h}{\beta\cdot l}\cdot\frac{\partial S}{\partial r_h}\left(\frac{\partial P}{\partial l}\right)^{-1} = -\frac{\alpha(D-2)}{\beta}\frac{TS}{2P} \qquad (32)$$

where it is used that Since $\Lambda \propto 1/l^2$ and $S \propto r_h^{D-2}$,

$$r_h\frac{\partial S}{\partial r_h} = (D-2)S, \qquad \frac{\partial P}{\partial l} = -\frac{2P}{l}, \qquad (33)$$

Smarr Relation for $k = 0$ case

The Smarr Relation from Eulerian scaling becomes

$$M = \frac{D-2}{D-3}TS - \frac{2}{D-3}PV = \frac{D-2}{D-3}TS\left[1 + \frac{\alpha}{\beta}\right] = \frac{D-2}{\beta}TS. \qquad (34)$$

From temperature

$$T = \left(\frac{r}{l}\right)^{z+1} \frac{1}{4\pi}\sqrt{f'(r)g'(r)}\Big|_{r=r_h}, \qquad (35)$$

we read off

$$[T] = r_h^z/l^{z+1}, \qquad [M] = [T][S] = r_h^{z+D-2}/l^{z+1} = r_h^{z+D-2}/l^{z+1} \qquad (36)$$

Since $M = M_0 r_h^{\beta} l^{\alpha}$, it turns out $\beta = D + z - 2$.
Thus the Smarr Relation without PV yields

$$M = \frac{D-2}{D+z-2}TS \qquad (37)$$

Smarr Relation without PV for charged case

Employing a mass ansatz $M = M_0 r_h^\beta l^\alpha + q_0 Q^2 r_h^\gamma l^\delta$,

$$\frac{\partial M}{\partial r_h} = T\frac{\partial S}{\partial r_h}, \qquad \frac{\partial M}{\partial l} = V\frac{\partial P}{\partial l}, \qquad \frac{\partial M}{\partial Q} = \Phi_H, \tag{38}$$

$$\begin{aligned}
\frac{\partial M}{\partial Q} =& 2q_0 Q r_h^\gamma l^\delta = \Phi_H, \qquad (39)\\
\frac{\partial M}{\partial r_h} =& M_0\beta r_h^{\beta-1} l^\alpha + q_0 Q^2 \gamma r_h^{\gamma-1} l^\delta = M_0\beta r_h^{\beta-1} l^\alpha + \frac{Q}{2}\frac{\Phi_H}{r_h}\gamma = T\frac{\partial S}{\partial r_h},\\
\frac{\partial M}{\partial l} =& M_0\alpha r_h^\beta l^{\alpha-1} + q_0 Q^2 \delta r_h^\gamma l^{\delta-1} = M_0\alpha r_h^\beta l^{\alpha-1} + \frac{Q}{2}\frac{\Phi_H}{l}\delta = V\frac{\partial P}{\partial l} \qquad (40)
\end{aligned}$$

By rearranging the last two equations, they are expressed as

$$\begin{aligned}
M_0 =& \frac{1}{\beta r_h^{\beta-1} l^\alpha}\left(T\frac{\partial S}{\partial r_h} - \frac{Q}{2}\frac{\Phi_H}{r_h}\gamma\right) = \frac{1}{\alpha r_h^\beta l^{\alpha-1}}\left(V\frac{\partial P}{\partial l} - \frac{Q}{2}\frac{\Phi_H}{l}\delta\right), \qquad (41)\\
PV =& -\frac{(D-2)}{2}\frac{\alpha}{\beta}TS - \frac{1}{4}Q\Phi_H\left(-\frac{\alpha}{\beta}\gamma + \delta\right) \qquad (42)
\end{aligned}$$

Smarr Relation without PV for charged case

By putting PV term, we arrive

$$
\begin{aligned}
M =& \frac{(D-2)}{(D-3)} TS + \Phi_H Q - \frac{2}{(D-3)} PV \\
=& \frac{(D-2)}{(D-3)} \left(\frac{\alpha+\beta}{\beta} \right) TS + \left(\frac{1}{2} - \frac{(\alpha+\beta)}{2(D-3)\beta} \gamma \right) \Phi_H Q \quad (43) \\
=& \frac{(D-2)}{\beta} TS + \left(\frac{1}{2} - \frac{\gamma}{2\beta} \right) \Phi_H Q \quad (44)
\end{aligned}
$$

Again, using $\beta = D + z - 2$, Smarr Relation without PV for charged case is obtained as

$$
M = \frac{(D-2)}{(D+z-2)} TS + \frac{(D+z-2-\gamma)}{2(D+z-2)} \Phi_H Q_* \quad (45)
$$

where Q_* again is the Maxwell charge participating in the first thermodynamic law

$$
dM = TdS + \Phi_H dQ_* \quad (46)
$$

E1: Lifshitz Blackhole with higher curvature term

Here is $z = 2$ and $D = 5$ case [3]

$$S = \int d^D x \sqrt{-g} \left(\frac{1}{\kappa} [R - 2\Lambda] + aR^2 + bR^{\mu\nu} R_{\mu\nu} + c \left[R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2 \right] \right)$$

$$a = -\frac{16l^2}{725}, \quad b = \frac{1584l^2}{13775}, \quad c = \frac{2211l^2}{11020}$$

[3] Y. Gim, W. Kim and S. H. Yi, "The first law of thermodynamics in Lifshitz black holes revisited," JHEP **1407**, 002 (2014)

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$$M = \frac{297}{1102} \cdot \frac{r_h^5 \omega_{0,3}}{l^3}, \quad S = \frac{396 r_h^3 \pi \omega_{0,3}}{551}, \quad T = \frac{5r_h^2}{8\pi l^3}$$

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$$M = \frac{297}{1102} \cdot \frac{r_h^5 \omega_{0,3}}{l^3}, \quad S = \frac{396 r_h^3 \pi \omega_{0,3}}{551}, \quad T = \frac{5r_h^2}{8\pi l^3} \quad \Rightarrow \quad M = \frac{3}{5} TS$$

$$P = \frac{2197}{4408\pi l^2}, \quad 2M = 3TS - 2PV$$

[3] Y. Gim, W. Kim and S. H. Yi, "The first law of thermodynamics in Lifshitz black holes revisited," JHEP **1407**, 002 (2014)

E1: Lifshitz Blackhole with higher curvature term

Here is $z = 2$ and $D = 5$ case [3]

$$S = \int d^D x \sqrt{-g} \left(\frac{1}{\kappa} [R - 2\Lambda] + aR^2 + bR^{\mu\nu} R_{\mu\nu} + c \left[R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2 \right] \right)$$

$$a = -\frac{16l^2}{725}, \quad b = \frac{1584l^2}{13775}, \quad c = \frac{2211l^2}{11020}$$

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[3] Y. Gim, W. Kim and S. H. Yi, "The first law of thermodynamics in Lifshitz black holes revisited," JHEP **1407**, 002 (2014)

E2: Einstein-Dilaton-Maxwell($NU(1)$)

$$S = \frac{1}{16\pi G_D}\int d^D x\sqrt{-g}\left[R - 2\Lambda - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}\sum_{i=1}^{N} e^{\lambda_i\phi}F_i^2\right],$$

$$ds^2 = -\left(\frac{r}{l}\right)^{2z} f(r)dt^2 + \frac{l^2}{r^2}\frac{1}{f(r)}dr^2 + r^2 d\Omega^2_{k,D-2}, \quad \Lambda = -\frac{(D+z-2)(D+z-3)}{2l^2}$$

$$f(r) = k\left(\frac{D-3}{D+z-4}\right)^2 \frac{l^2}{r^2} + 1 - mr^{2-D-z} + \sum_{n=2}^{N-k}\frac{q_n^2\mu^{-\sqrt{\frac{2(z-1)}{(D-2)}}} l^{2z}}{2(D-2)(D+z-4)} r^{-2(D+z-3)},$$

$$A'_{t,1} = l^{-z}\sqrt{2(D+z-2)(z-1)}\mu^{\sqrt{\frac{D-2}{2(z-1)}}} r^{D+z-3},$$

$$A'_{t,n} = q_n\mu^{-\sqrt{\frac{2(z-1)}{(D-2)}}} r^{3-D-z},$$

$$A'_{t,N} = l^{1-z}\frac{\sqrt{2k(D-2)(D-3)(z-1)}}{\sqrt{D+z-4}}\mu^{\frac{(D-3)}{\sqrt{2(D-2)(z-1)}}} r^{D+z-5},$$

$$e^{\phi} = \mu r^{\sqrt{2(D-2)(z-1)}}$$

[4]

[4] J. Tarrio and S. Vandoren, "Black holes and black branes in Lifshitz spacetimes," JHEP **1109**, 017 (2011)

- $k = 0$

- $k = 1$

- $k = -1$

- $k = 0$
  - At least one ($N = 1$) $U(1)$ gauge field is required. As the gauge and the dilaton fields diverge in order to support the Lifshitz asymptotics when $r \to \infty$, the metric does not possesses a charge and yields uncharged Lifshitz black holesolutions.

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  there is an imaginary charge density for $z \neq 1$ and so the hyperbolic case is only admitted for $z = 1$.

- Thermodynamic variable can be directly red off

$$T = \frac{r^z}{4\pi l^{z+1}} \left((D+z-2) + k\frac{l^2(D-3)^2}{r^2(D+z-4)} - \sum_{n=2}^{N-k} \frac{q_n^2 \mu^{-\sqrt{\frac{2(z-1)}{(D-2)}}} l^{2z}}{2(D-2)} r_h^{-2(D+z-3)} \right)$$

$$S = \frac{\omega_{k,D-2}}{4G_D} r_h^{D-2}, \quad P = \frac{(D+z-2)(D+z-3)}{16\pi G_D l^2},$$

$$\Phi_n = -\frac{q_n \ \mu^{-\sqrt{\frac{2(z-1)}{(D-2)}}}}{(D+z-4)} r_h^{4-D-z}, \quad Q_i = \frac{1}{16\pi G_D} \int e^{\lambda_i \phi} * F = \frac{q_i \omega_{k,D-2} l^{z-1}}{16\pi G_D}$$

- The mass is obtained by the modified Korma formula

$$M = \frac{(D-2)\omega_{k,D-2}}{16\pi G_D} \Bigg[\left(1 + k\frac{(D-3)^2 l^2}{(D+z-4)^2 r_h^2}\right) l^{z-1} r_h^{D+z-2} + \sum_{n=2}^{N-k} \frac{q_n^2 \mu^{-\sqrt{\frac{2(z-1}{(D-2)}}}}{2(D-2)(D+z}$$

- Thermodynamic variable can be directly red off

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- Smarr Relation without PV for $k=0$ is satisfied

$$M = \frac{D-2}{D+z-2} TS + \sum_{n=2}^{N}\frac{D+z-3}{D+z-2}\Phi_n Q_n$$

- Demanding Smarr Relation for $k = 1, 0$

$$(D-3)M = (D-2)TS - 2PV + (D-3)\Phi_H Q,$$

- Thermodynamic volume is

$$V = \frac{(D-2)\omega_{k,D-2}}{(D+z-3)(D+z-2)} \Bigg[\left(\frac{(z+1)}{2} + k \frac{(D-3)^2(z-1)l^2}{2(D+z-4)^2 r_h^2} \right) l^{1-z} r_h^{D+z-2} - \sum_{n=2}^{N-k} \frac{(z-1)q_n^2 \mu^{-\sqrt{\frac{2(z-1)}{(D-2)}}}}{4(D-2)(D+z-4)} l^{z+1} r_h^{4-D-z} \Bigg],$$

Summary

When existence of $\Lambda < 0$, it will be satisfied with

- General Smarr Relation in D dim. for $k = 1, 0,$ and -1

$$(D-3)M = (D-2)TS - 2PV + (D-3)\Phi_H Q,$$

- General Smarr Relation without PV in D dim Especially, $k = 0$

$$M = \frac{(D-2)}{(D+z-2)}TS + \frac{(D+z-2-\gamma)}{2(D+z-2)}\Phi_H Q$$

Acknowledgements

Thank you!