Entanglement of Identical Particle

Juyoung Lee
(SKKU)

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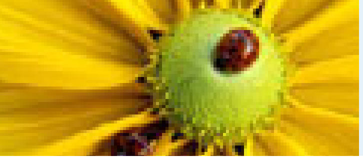


Bipartite States

- 1 qubit states in $\mathbb{C}^2$
  $|\leftrightarrow\rangle, |\updownarrow\rangle$ or these linear combination
- 2 qubits states in $\mathbb{C}^2 \times \mathbb{C}^2$
  - Natural tensor product structures
    $|\leftrightarrow\rangle \otimes |\updownarrow\rangle, \frac{|\leftrightarrow\rangle\otimes|\updownarrow\rangle+|\updownarrow\rangle\otimes|\updownarrow\rangle}{\sqrt{2}} \ldots$
  - Entangled states
    $\frac{1}{\sqrt{2}}(|\leftrightarrow\rangle \otimes |\updownarrow\rangle + |\updownarrow\rangle \otimes |\leftrightarrow\rangle)$

One Dimensional Projection

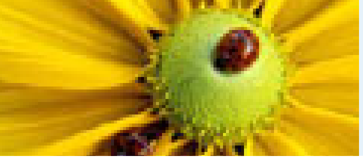
- $P_\varphi = |\varphi\rangle\langle\varphi|, \ |\varphi\rangle \in \mathbb{C}^2$

cf. $\mathrm{Tr}(\rho P_\varphi) = 1$ if and only if density matrix ρ describing the system mstate is exactly P_φ

- P in $\mathbb{C}^2 \otimes \mathbb{C}^2$
  - $P_1 = |\leftrightarrow\rangle\langle\leftrightarrow| \otimes \mathbb{I}$, $P_2 = \mathbb{I} \otimes |\updownarrow\rangle\langle\updownarrow|$, $P_1 \otimes P_2 \ldots$
  - $\mathcal{E}_P = P \otimes (\mathbb{I} - P) + (\mathbb{I} - P) \otimes P + P \otimes P$, $P = |\varphi\rangle\langle\varphi|$
  - $\langle\Psi|\mathcal{E}_{P_1}|\Psi\rangle = \langle\Psi|\mathcal{E}_{P_2}|\Psi\rangle = 1$
    - if and only if $P_1 P_2 = 0$, $P_{12}^{\mathrm{sym}} = \mathcal{E}_{P_1}\mathcal{E}_{P_2}$ and $\langle\Psi|P_{12}^{\mathrm{sym}}|\Psi\rangle = 1$

Creation & Annihilation Operators

- $\hat{a}_1^\dagger|0\rangle = |\leftrightarrow\rangle, \;\; \hat{a}_2^\dagger|0\rangle = |\updownarrow\rangle$
- CCR, $[\hat{a}_i, \hat{a}_j^\dagger] = \hat{a}_i\hat{a}_j^\dagger - \hat{a}_j^\dagger\hat{a}_i = \delta_{ij}$
- CAR, $\{\hat{a}_i, \hat{a}_j^\dagger\} = \hat{a}_i\hat{a}_j^\dagger + \hat{a}_j^\dagger\hat{a}_i = \delta_{ij}$



First Quantization

- $|\Psi\rangle = c_{00}|\phi_0\rangle \otimes |\phi_0\rangle + \sum_{j>0} c_{0j}|\phi_0\rangle \otimes |\phi_j\rangle + \sum_{i>0} c_{i0}|\phi_i\rangle \otimes |\phi_0\rangle + \sum_{i,j\neq 0} c_{ij}|\phi_i\rangle \otimes |\phi_j\rangle$
  - $\langle\Psi|\mathcal{E}_P|\Psi\rangle = 1$ & identicle $\Rightarrow$ $c_{ij} = 0$ & $c_{0j} = \pm c_{j0}$
  - Normalization : $|c_{00}|^2 + 2\sum_{j>0}|c_{0j}|^2 = 1$
- Fermions: $c_{00} = 0$, Separatable
  - $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\phi_0\rangle \otimes |\Lambda\rangle - |\Lambda\rangle \otimes |\phi_0\rangle)$ $|\Lambda\rangle = \sqrt{2}\sum_{j>0} c_{0j}|\phi_j\rangle$

First Quantization

- Bosons

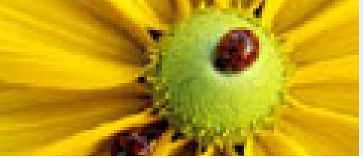
$$|\Psi\rangle = \sqrt{\tfrac{2-|c_{00}|^2}{4}}(|\phi_0\rangle \otimes |\Theta\rangle + |\Theta\rangle \otimes |\phi_0\rangle)$$

$$|\Theta\rangle = \sqrt{\tfrac{4}{2-|c_{00}|^2}}(\tfrac{c_{00}}{2}|\phi_0\rangle + \sum_{j>0} c_{j0}\phi_j)$$

(1) $c_{j0} = 0 : \ |\Psi\rangle = |\phi_0\rangle \otimes |\phi_0\rangle$

(2) $c_{00} = 0 : \ \langle\Theta|\Psi\rangle = 0$

(3) $\langle\Theta|\phi_0\rangle \neq 0 \ \& \ |\Theta\rangle \neq |\phi_0\rangle$



Second Quantization

- $\hat{a}^\dagger(\psi)|0\rangle = |\psi\rangle, \;\; \hat{a}(\psi)|\psi\rangle = |0\rangle$
- $[\hat{a}(\psi_1), \hat{a}^\dagger(\psi_2)] = \langle\psi_1|\psi_2\rangle = \{\hat{a}(\psi_1), \hat{a}^\dagger(\psi_2)\}$
- $|n_1, n_2 \cdots n_k\rangle = \frac{(\hat{a}_1^\dagger)^{n_1}(\hat{a}_2^\dagger)^{n_2}\cdots(\hat{a}_k^\dagger)^{n_k}}{\sqrt{\prod_{j=1}^k n_j!}}|0\rangle$
- $\psi(i) = \langle i|\psi\rangle,$

  $$\begin{aligned}\langle ij|\psi_1\psi_2\rangle &= \langle 0|\hat{a}_i\hat{a}_j\hat{a}^\dagger(\psi_1)\hat{a}^\dagger(\psi_2)|0\rangle \\ &= \psi_1(i)\psi_2(j) \pm \psi_1(j)\psi_2(i) \\ &\Leftrightarrow \frac{|i\rangle\otimes|j\rangle \pm |j\rangle\otimes|i\rangle}{\sqrt{2}}\end{aligned}$$

Second Quantization

- Bosons
  - $[\hat{a}(\psi_{V_1}), \hat{a}^\dagger(\psi_{V_2})] = \langle \psi_{V_1} | \psi_{V_2} \rangle$
    $= \int_{\mathbb{R}^3} \mathrm{d}^3 r \psi^*_{V_1}(r) \psi_{V_2}(r) = 0$
    $\Leftrightarrow$subalgebra $\mathcal{A}_{V_{1,2}}$ commute
- Fermions
  - $\{\hat{a}(\psi_{V_1}), \hat{a}^\dagger(\psi_{V_2})\} = \langle \psi_{V_1} | \psi_{V_2} \rangle$
    $= \int_{\mathbb{R}^3} \mathrm{d}^3 r \psi^*_{V_1}(r) \psi_{V_2}(r) = 0$
    $\Leftrightarrow$subalgebra $\mathcal{A}_{V_{1,2}}$ anti-commute
  - $[AB, C] = A\{B, C\} - \{A, C\}B$
    $\Leftrightarrow$ $\mathcal{A}_1^{\mathrm{ev}}$ and $\mathcal{A}_2^{\mathrm{ev}}$ are commute,
    as well as A_1^{ev} and A_2^{odd}, A_1^{odd} andA_2^{ev}

First and Second Quantization Approaches Compared

- Fermions
  - $$|\Psi\rangle = \hat{a}_1^\dagger \left(\sum_{j>1} c_{1j} \hat{a}_j^\dagger \right) |0\rangle$$
    $$= \mathcal{P}(\hat{a}_1^\dagger) \cdot \mathcal{Q}(\hat{a}_1^\dagger, \cdots, \hat{a}_1^\dagger, \cdots)|0\rangle$$
  - both approaches obtain separable states
- Bosons

  (1) both approaches obtain separable states
  (2) both approaches obtain separable states
  (3) second quantization approach obtain separable states but fitst way obtain entangled states