# Spinors in diverse dimensions

- Imtak Jeon (HRI) -
21st APCTP Winterschool
on Fundamental Physics

- Clifford algebra
- Irreducible spinors
- Exercises : 
  - Useful identity
  - Fierz identities.

ref)
① Jeong-Hyuck Park's personal lecture note
← rigorous analysis, available at the school webpage
② Van Proeyen arXiv : hep-th/9910030
← I mostly follow this notation,
③ Kugo and P.K Townsend, Nucl. Phys. B 221 (1983) 357
← very classic note on this subject.

## Clifford algebra

$$\{\gamma_a , \gamma_b\} = 2\eta_{ab}\mathbb{1}$$

where spinor indices are omitted

$$\gamma_a{}^\alpha{}_\beta \quad , \quad \mathbb{1}^\alpha{}_\beta$$

and the $\eta_{ab}$ is Lorentz metric in arbitrary dimensions & Signature

$$\eta_{ab} = \mathrm{diag}(\underbrace{- - \cdots -}_{t}\underbrace{+ + \cdots +}_{s})$$

$$d = t + s$$

## Theorem

(a) In even dimensions, there is only one irreducible representation of the gamma matrices, up to conjugacy.

i.e. if $\gamma_a'$ and $\gamma_a$ are two irreducible representations satisfying the clifford algebra, then

$$\gamma_a' = S\gamma_a S^{-1} \quad \text{for some } S$$

and $S$ is unique up to constant.

( proof is in the reference ① )

eg) $\begin{cases} \pm\gamma_a^\dagger = A_\pm \gamma_a A_\pm^{-1} \\ \pm\gamma_a^T = C_\pm \gamma_a C_\pm^{-1} \\ \pm\gamma_a^* = B_\pm \gamma_a B_\pm^{-1} \end{cases}$

(b) Complex dimension of the irreducible rep. is $2^d$, $2^{d/2} \times 2^{d/2}$ complex matrix.

(c) We can take the chirality operator $\gamma_{(d+1)}$

s.t. $\{\gamma_{(d+1)}, \gamma_a\} = 0$ and $\gamma_{(d+1)}\gamma_{(d+1)} = 1$.

$$c\gamma_1\gamma_2\cdots\gamma_d \qquad\qquad c = \pm(-i)^{d/2+t}$$

$$\gamma_{d+1} \equiv (-i)^{d/2+t}\gamma_1\gamma_2\cdots\gamma_d$$

$\Rightarrow$ With $\gamma_{d+1}$, one can construct irreducible reps. in $(d+1)$ dimensions (odd dimensions)

$$(\gamma_1, \gamma_2, \cdots \gamma_d, \gamma_{d+1}) \qquad (\gamma_1, \gamma_2 \cdots \gamma_d, -\gamma_{d+1})$$

- size is same as d-dim.
- $\exists$ two inequivalent reps.

(☹ suppose they are equivalent

$S\gamma_a S^{-1} = \gamma_a \Rightarrow S\gamma_{d+1}S^{-1} = \gamma_{d+1}$ ⊗ )

To prove thm (b), let us take an explicit representation in $(0,d)$

$$\gamma_1 = \sigma_1 \times 1 \times 1$$
$$\gamma_2 = \sigma_2 \times 1 \times 1$$
$$\gamma_3 = \sigma_3 \times \sigma_1 \times 1$$
$$\gamma_4 = \sigma_3 \times \sigma_2 \times 1$$
$$\gamma_5 = \sigma_3 \times \sigma_3 \times \sigma_1$$
$$\gamma_6 = \sigma_3 \times \sigma_3 \times \sigma_2$$
$$\vdots$$

(f) What if we have timelike directions?

In $(t,s)$

$$\gamma_t \to i\gamma_t$$

Thus we get hermiticity property

$$\gamma_t^{\dagger} = -\gamma_t \quad , \quad \gamma_s^{\dagger} = \gamma_s$$

$$\Leftrightarrow \quad \gamma_\mu^{\dagger} = \gamma^\mu$$

size is $2^{d/2} \times 2^{d/2}$.

Is this an irreducible rep.?

i.e. Is there a smaller size rep.? No!

Note Consider a set

$$\{\gamma^M\} \equiv \{1, \gamma_a, \gamma_{a_1a_2}, \gamma_{a_1a_2a_3}, \cdots, \gamma_{12\cdots d}\}$$

where $\gamma_{a_1\cdots a_n} \equiv \gamma_{[a_1}\gamma_{a_2}\cdots\gamma_{a_n]}$

Collective coordinate which represents an element of the set.

- # of element

$$1 + d + \binom{d}{2} + \binom{d}{3} + \cdots + \binom{d}{d} = 2^d$$

- They form a group up to sign.

$\{\gamma^M\}/Z_2$ form a group. $\quad \gamma^M\gamma^N = \underset{\pm 1}{\underset{\shortparallel}{\Omega^{MN}}}\,\gamma^{L(M,N)}$

eg)
$$\gamma_{12}\gamma_3 = \gamma_{123}$$
$$\gamma_{12}\gamma_{23} = \gamma_{13}$$
$$\gamma_{12}\gamma_{21} = -1$$

- Each elements are <u>orthogonal</u> to each other

$$\mathrm{Tr}(\gamma^M \gamma^N) \propto \delta^{MN}$$

$\because$ $\mathrm{Tr}(\gamma_{a_1 \cdots a_n}) = 0$

$n =$ even

$$\mathrm{Tr}\, \gamma_{[a_1} \gamma_{a_2} \cdots \gamma_{a_n]}$$
$$= \mathrm{Tr}\, \gamma_{[a_n} \gamma_{a_1} \cdots \gamma_{a_{n-1}]}$$
$$= (-1)^{n-1} \mathrm{Tr}(\gamma_{[a_1} \cdots \gamma_{a_n]})$$

$n =$ odd

$$\mathrm{Tr}\, \gamma_{a_1 \cdots a_n} \gamma_5 \gamma_5$$
$$= \mathrm{Tr}\, \gamma_5 \gamma_{a_1 \cdots a_n} \gamma_5$$
$$= (-1)^n \mathrm{Tr}\, \gamma_{a_1 \cdots a_n}$$

$\Rightarrow$ The size should be $\geq 2^{d/2} \times 2^{d/2}$

(If size $< 2^{d/2} \times 2^{d/2} \rightarrow 2^d$ elements cannot be independent)

$\Rightarrow$ As we already saw that $\exists$ $2^{d/2} \times 2^{d/2}$ size rep.

$\therefore$ the smallest size of irreducible rep is $2^{d/2} \times 2^{d/2}$

& $\{\gamma^M\}$ forms a basis of the $2^{d/2} \times 2^{d/2}$ matrices

$\blacksquare$

(a) By uniquness

($\gamma_\mu^\dagger = \gamma^\mu$. From now on, We keep this property)

- $\gamma_a^\dagger = (-1)^t A \gamma_a A^{-1}$ , $A = \gamma_{1\cdots t}$ , $A^{-1} = A^\dagger$ —— (1)

(useful $(A\gamma_{a_1\cdots a_n})^\dagger = (-1)^{((n+t)^2+t-n)/2} A\gamma_{a_1\cdots a_n}$)

- $\gamma_a^T = -\eta C \gamma_a C^{-1}$ , $C^T = -\epsilon C$ , $C^{-1} = C^\dagger$ —— (2)

$(\gamma^{TT} = \gamma, C^{TT} = C)$

$\eta = \pm 1$, $C$: "charge conjugation matrix", $\epsilon = \pm 1$

Allowed sign for $\eta$ & $\epsilon$ are different depending on dimension d.

Which sign of $\eta$ & $\epsilon$ are allowed in dimension d ?

$$(C\gamma_{a_1\cdots a_n})^T = -\epsilon(-1)^{n(n-1)/2}(-\eta)^n C\gamma_{a_1\cdots a_n}$$
$$\equiv \underbrace{\chi_n(\epsilon,\eta)}_{=\pm 1} C\gamma_{a_1\cdots a_n}$$

Make a projection $\frac{1}{2}(1+\chi_n(\epsilon,\eta))$

and Consider the sumations,

\# of $\gamma_{a_1\cdots a_n}$ which has n-indicies.

$$\begin{cases} \sum_{n=0}^{d} \frac{1}{2}(1+\chi_n(\epsilon,\eta))\binom{d}{n} = \#\text{ of symmetric matrices} = 2^{d/2}(2^{d/2}+1)\frac{1}{2} \\ \sum_{n=0}^{d} \frac{1}{2}(1-\chi_n(\epsilon,\eta))\binom{d}{n} = \#\text{ of antisym. matrices} = 2^{d/2}(2^{d/2}-1)\frac{1}{2} \end{cases}$$

We have two variables $\epsilon$ and $\eta$ and two equations for each dimension d.

Thus we can determine the signs $\epsilon$ and $\eta$ for dimension d.

Exercise !

Result

$\eta$ for which $(C\gamma_{a_1 \cdots a_n})$ is symmetric

" " " antisymmetric

| d (mod 8) | $\epsilon$ | $\eta$ | S | A |
|---|---|---|---|---|
| 0 | $-$ | $+$ | 0, 3 | 2, 1 |
| | $-$ | $-$ | 0, 1 | 2, 3 |
| 1 | $-$ | $-$ | 0, 1 | 2, 3 |
| 2 | $-$ | $-$ | 1, 0 | 3, 2 |
| | $+$ | $+$ | 1, 2 | 3, 0 |
| 3 | $+$ | $+$ | 1, 2 | 0, 3 |
| 4 | $+$ | $+$ | 2, 1 | 0, 3 |
| | $+$ | $-$ | 2, 3 | 0, 1 |
| 5 | $+$ | $-$ | 2, 3 | 0, 1 |
| 6 | $+$ | $-$ | 3, 2 | 1, 0 |
| | $-$ | $+$ | 3, 0 | 1, 2 |
| 7 | $-$ | $+$ | 0, 3 | 1, 2 |

(Table 1)

note Odd dim ((d+1)-dimension, where d = even)

$$\gamma_{(d+1)}^T = (-1)^{\frac{d}{2}(d-1)} C\gamma_{(d+1)} C^{-1}$$

$\Rightarrow$ only one choice of $\eta$ is allowed.

(C, A are unitary)

- $\gamma_a^* = -\eta(-1)^t B\gamma_a B^{-1}$ , $B^T \equiv CA^{-1}$ , $B^\dagger = B^{-1}$ — (3)

$(\gamma_a^\dagger)^T$ combining (1) & (2)

$$B^*B = -\epsilon\eta^t(-1)^{t(t+1)/2}$$

$\rightarrow$ will be important when we define Majorana condition.

# Irreducible spinors

Dirac spinor $\lambda$ which has $2^{d/2}$ complex component can further be reduced by

- Weyl condition (in even dimension)

$$P_\pm \equiv \frac{1}{2}(1 \pm \gamma_{(d+1)})$$

"Weyl spinor" $\begin{cases} \lambda_L \equiv P_+ \lambda \\ \lambda_R \equiv P_- \lambda \end{cases}$

- Majorana condition (reality)

reality means that complex conjugate of $\lambda$ is not independent of $\lambda$.

$\lambda^* = \tilde{B}\lambda$ for some matrix $\tilde{B}$

consistency

- Lorentz

$$\left(-\frac{1}{4}\gamma_{ab}\lambda\right)^* = \tilde{B}\left(-\frac{1}{4}\gamma_{ab}\lambda\right)$$

$\parallel \leftarrow (3)$

$$-\frac{1}{4}B\gamma_{ab}B^{-1}\tilde{B}\lambda \quad \Rightarrow \quad \tilde{B} = \alpha B$$

$$((\not{\partial} - m)\psi)^* = 0$$

$$(\gamma^*\partial - m)\psi^* = 0$$

$$= -\eta(-1)^t(B\gamma B^{-1}\partial - m)B\psi$$

$$= -\eta(-1)^t B(\gamma\partial + \eta(-1)^t m)\psi$$

- $\lambda^{**} = \lambda$

$$(\tilde{B}\lambda)^* = \tilde{B}^*\tilde{B}\lambda \Rightarrow \tilde{B}^*\tilde{B} = 1 \Rightarrow |\alpha|^2 B^* B = 1$$

$$\Rightarrow |\alpha| = 1 \ \& \ B^* B = 1$$

It turns out that this condition is satisfied only for

$s - t = 0, 1, 7 \pmod 8$

$s - t = 2 \pmod 8$ with $\eta(-1)^{d/2} = +1$

$s - t = 6 \pmod 8$ with $\eta(-1)^{d/2} = -1$

eg) (1+1) dim OK

(1+3) dim OK only for $\eta = 1$

(1+9) dim OK

"Majorana spinor"

$$\lambda^* = \alpha B \lambda \iff (\bar{\lambda} = \lambda^T C) \qquad \alpha = 1 \text{ usualy.}$$

$\bar{\lambda} \equiv \lambda^\dagger A \alpha^{-1}$ "Dirac conjugate"

we can set $\alpha = 1$

cf) $\left( (\gamma^a\partial_a - m)\psi = 0 \right)^*$

$$\Rightarrow \left(-\eta(-1)^t B\gamma^a_{\partial_a} B^\dagger - m\right) B\psi = 0 \Leftrightarrow -\eta(-1)^t B\left(\gamma^a\partial_a + \eta(-1)^t m\right)\psi = 0$$

$\left( \begin{array}{ll} \eta(-1)^t = -1 & \text{Majorana} \\ \eta(-1)^t = +1 & \text{Pseudo-Majorana} \end{array} \right.$

## Symplectic Majorana condition

In case $B^*B = -1$, $\exists$ still another possibility if we have extended supersymmetry.

$$\lambda^*_i \equiv (\lambda^i)^* = B\Omega_{ij}\lambda^j$$

where $\Omega\Omega^* = -1$ ☺ $\left(\Leftarrow \lambda^{**} = \lambda \right)$

eg) (1+5) dimension

$$\lambda^{**} = B^*\Omega^* B\Omega\lambda = \underbrace{B^*B}_{-1}\,\underbrace{\Omega^*\Omega}_{-1}\,\lambda$$

## (symplectic) Majorana Weyl (even d)

If $B^*B = 1$, Try Majorana Weyl

$$\lambda^*_{L,R} = B\lambda_{L,R} = B\tfrac{1}{2}(1\pm\gamma_{(d+1)})\lambda$$
$$\lambda^*_{L,R} = \left(\tfrac{1}{2}(1\pm\gamma_{d+1})\lambda\right)^* = \tfrac{1}{2}(1\pm\gamma^*_{d+1})B\lambda$$

$\Rightarrow \gamma_{d+1}^* = B\gamma_{d+1}B^{-1}$

If $B^*B = -1$, try Symplectic MW

$$\lambda_{L,R}{}^*_i = B\Omega_{ij}\lambda_{L,R}^j = B\Omega_{ij}\tfrac{1}{2}(1\pm\gamma_{d+1})\lambda^j$$
$$\lambda_{L,R}{}^*_i = \left(\tfrac{1}{2}(1\pm\gamma_{d+1})\lambda^i\right)^* = \tfrac{1}{2}(1\pm\gamma_{d+1}^*)\Omega_{ij}B\lambda$$

$\Rightarrow \gamma_{d+1}^* = B\gamma_{d+1}B^{-1}$

In any case it should be that

$$\gamma_{d+1}^* = B\gamma_{d+1}B^{-1}.$$

Compare it with

$$\gamma_{d+1}^* = (-1)^{d/2+t} B\gamma_{d+1}B^{-1} \quad \Leftarrow \left( ☺\ \gamma_{d+1} \equiv (-i)^{\frac{d}{2}+t}\gamma_{1\ldots d} \right)$$

$\Rightarrow s+t = 0 \pmod 4$ (Symplectic) Majorana-Weyl

$B^*B=-1$ $\quad$ $B^*B=1$

## Result

only if $\eta=-1$

| $d \backslash t$ | 0 | 1 | ... |
|---|---|---|---|
| 1 | M | M | |
| 2 | $M^{-}$ | MW | |
| 3 | | M | |
| 4 | SMW | $M^{+}$ | only if $\eta=+1$ |
| 5 | | | |
| 6 | $M^{+}$ | SMW | |
| 7 | M | | |
| 8 | MW | $M^{-}$ | |
| 9 | M | M | |
| 10 | $M^{-}$ | MW | |
| 11 | | M | |
| 12 | SMW | $M^{+}$ | |

table 2

## Summary

- $\gamma_a^{\dagger} = (-1)^t A \gamma_a A^{-1}$, $\quad A = \gamma_{1\cdots t}$, $\quad A^{-1} = A^{\dagger}$
- $\gamma_a^{T} = -\eta C \gamma_a C^{-1}$, $\quad C^T = -\epsilon C$, $\quad C^{-1} = C^{\dagger}$
- $\gamma_a^{*} = -\eta (-1)^t B \gamma_a B^{-1}$, $\quad B^T = C^T A^{-1}$, $\quad B^{\dagger} = B^{-1}$

$\gamma_{d+1} \equiv (-i)^{d/2+t} \gamma_1 \gamma_2 \cdots \gamma_d$, $\quad \{\gamma_{(d+1)}, \gamma_a\} = 0$ and $\gamma_{(d+1)} \gamma_{(d+1)} = 1$

$$(A \gamma_{a_1 \cdots a_n})^{\dagger} = (-1)^{((n+t)^2 + t - n)/2} A \gamma_{a_1 \cdots a_n}$$

$$(C \gamma_{a_1 \cdots a_n})^{T} = -\epsilon (-1)^{n(n-1)/2} (-\eta)^n C \gamma_{a_1 \cdots a_n}$$

$$B^* B = -\epsilon \eta^t (-1)^{t(t+1)/2}$$

with table (1), (2).

Def Majorana : $\psi^* = \alpha B \psi \quad \Leftrightarrow \quad \underbrace{\psi^{\dagger} A \alpha^{-1}}_{\bar{\psi}} = \psi^T C$ $\quad$ set $(\alpha = 1)$

Symplectic Majorana : $(\lambda^i)^* = B \Omega_{ij} \lambda^j$, $\quad \Omega^* \Omega = -1$

## Exercise 1

Show that

$$(A\gamma_{a_1\cdots a_n})^\dagger = (-1)^{((n+t)^2+(t-n))\frac{1}{2}} A\gamma_{a_1\cdots a_n}$$

$$(C\gamma_{a_1\cdots a_n})^T = -\epsilon(-1)^{n(n-1)/2}(-\eta)^n C\gamma_{a_1\cdots a_n}$$

and determine sign factors $\alpha$, $\beta$

$$(\bar{\psi}\gamma_{a_1\cdots a_n}\lambda)^* = \alpha(\bar{\lambda}\gamma_{a_1\cdots a_n}\psi),$$

$$(\bar{\psi}\gamma_{a_1\cdots a_n}\lambda) = \beta(\bar{\lambda}\gamma_{a_1\cdots a_n}\psi)$$ for Majorana spinors $\psi$ and $\lambda$

Ans

$$\begin{aligned}
(\bar{\psi}\gamma_{a_1\cdots a_n}\lambda)^* &= (\psi^\dagger A\gamma_{a_1\cdots a_n}\lambda)^* \\
&= ((\psi^*)_\alpha (A\gamma_{a_1\cdots a_n})^\alpha{}_\beta \lambda^\beta)^* \\
&= \psi^\alpha [(A\gamma_{a_1\cdots a_n})^*{}_\alpha{}^\beta (\lambda^*)_\beta \\
&= -(\lambda^*)_\beta (A\gamma_{a_1\cdots a_n})^{\dagger\beta}{}_\alpha \psi^\alpha \\
&= -(\lambda^*)_\beta (A\gamma_{a_1\cdots a_n})^\beta{}_\alpha \psi^\alpha (-1)^{((n+t)^2+(t-n))\frac{1}{2}} \\
&= (-1)(-1)^{((n+t)^2+(t-n))\frac{1}{2}}(\bar{\lambda}\gamma_{a_1\cdots a_n}\psi)
\end{aligned}$$

$$\begin{aligned}
(\bar{\psi}\gamma_{a_1\cdots a_n}\lambda) &= (\psi^T C\gamma_{a_1\cdots a_n}\lambda) \\
&= (\psi^\alpha (C\gamma_{a_1\cdots a_n})_{\alpha\beta}\lambda^\beta) \\
&= -\lambda^\beta (C\gamma_{a_1\cdots a_n})^T{}_{\beta\alpha}\psi^\alpha \\
&= (-1)\lambda^\beta (C\gamma_{a_1\cdots a_n})_{\beta\alpha}\psi^\alpha (-\epsilon)(-1)^{n(n-1)/2}(-\eta)^n \\
&= \epsilon(-1)^{n(n-1)/2}(-\eta)^n(\bar{\lambda}\gamma_{a_1\cdots a_n}\psi)
\end{aligned}$$

# Useful relation

$$\gamma_{a_m \cdots a_1} \gamma^{b_1 \cdots b_n} = \sum_{l=0}^{\min(m,n)} l! \binom{m}{l} \binom{n}{l} \gamma_{[a_m \cdots a_{l+1}}{}^{[b_{l+1} \cdots b_n} \delta_{a_1}^{b_1} \cdots \delta_{a_l]}^{b_l]}$$

## Exercise 2

Use the above formula, show the following examples

$$\gamma_a \gamma^b = \gamma_a{}^b + \delta_a^b$$

$$\gamma_{ba} \gamma^c = \gamma_{ba}{}^c + 2\gamma_{[b} \delta_{a]}^c$$

$$\gamma_{ba} \gamma^{cd} = \gamma_{ba}{}^{cd} + 2 \cdot 2\, \gamma_{[b}{}^{[d} \delta_{a]}^{c]} + 2\, \delta_{[a}^{[c} \delta_{b]}^{d]}$$

$$\vdots$$

$$\gamma_{(d+1)} \gamma^{b_1 \cdots b_n}$$

$$= c\, \gamma_{1 \cdots d} \gamma^{b_1 \cdots b_n} \qquad c = (-i)^{\frac{d}{2}+t}$$

$$= c \frac{1}{d!} \varepsilon^{a_d \cdots a_1} \underbrace{\gamma_{a_d \cdots a_1} \gamma^{b_1 \cdots b_n}}$$

$$= c \frac{1}{d!} \varepsilon^{a_d \cdots a_1} \, n! \binom{d}{l} \gamma_{[a_d \cdots a_{n+1}} \delta_{a_1 \cdots a_n]}^{[b_1 \cdots b_n]}$$

$$= c \frac{1}{\cancel{d!}} \varepsilon^{a_d \cdots a_{n+1} b_n \cdots b_1} \, \cancel{n!} \frac{\cancel{d!}}{\cancel{n!}(d-n)!} \gamma_{a_d \cdots a_{n+1}}$$

$$= c \frac{1}{(d-n)!} \varepsilon^{a_d \cdots a_{n+1} b_n \cdots b_1} \gamma_{a_d \cdots a_{n+1}}$$

# Fierz identity

show that in even d.

$$\boxed{(\bar{\psi}\lambda)(\bar{\rho}\epsilon) = -\sum_{n=0}^{d} \frac{1}{n!}\frac{1}{2^{d/2}} (\bar{\rho}\gamma_{a_n\cdots a_1}\lambda)(\bar{\psi}\gamma^{a_1\cdots a_n}\epsilon)}$$

$$(\bar{\psi}_\alpha\lambda^\alpha)(\bar{\rho}_\beta\epsilon^\beta)$$

$$= (\bar{\psi}_\delta\delta^\alpha{}_\gamma\lambda^\gamma)(\bar{\rho}_\delta\delta^\delta{}_\beta\epsilon^\beta)$$

$$\stackrel{(-)}{=} \bar{\psi}_\delta\lambda^\gamma\bar{\rho}_\delta\epsilon^\beta \quad \delta^\alpha{}_\gamma\delta^\delta{}_\beta \leftarrow \text{matrix } M^\delta{}_\gamma{}^\alpha{}_\beta$$

Recall that $\{\gamma^M{}_{\beta}\} = \{1.\ \gamma^a.\ \gamma^{a_1a_2}, \cdots\}$ form a complet of basis for the $2^{d/2}\times 2^{d/2}$ matrix space.

Treat $\alpha\ \beta$ indices as the index of this basis matrix, and regard $\gamma\ \delta$ as the index for coefficients, And expand $\delta^\alpha{}_\gamma\delta^\delta{}_\beta$ w.r.t the complete basis

$$\delta^\alpha{}_\gamma\delta^\delta{}_\beta = \sum_{n=0}^{\infty}\frac{1}{n!}(A_{a_1\cdots a_n})^\delta{}_\gamma(\gamma^{a_1\cdots a_n})^\alpha{}_\beta$$

Using the orthogonality of the basis we can find the coefficien $(A_{a_1\cdots a_n})^\delta{}_\gamma$

multiplying $1^\beta{}_\alpha \rightarrow$ we get $A^\delta{}_\gamma = \frac{1}{2^{d/2}}\delta^\delta{}_\gamma$

" $(\gamma_b)^\beta{}_\alpha \rightarrow$ we get $(A_b)^\delta{}_\gamma = \frac{1}{2^{d/2}}(\gamma_b)^\delta{}_\gamma$

" $(\gamma_{b_1b_2})^\beta{}_\alpha \rightarrow$ we get $(A_{b_1b_2})^\delta{}_\gamma = \frac{1}{2^{d/2}}(\gamma_{b_2b_1})^\delta{}_\gamma$

$\vdots$

result $$\delta^\alpha{}_\gamma\delta^\delta{}_\beta = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{2^{d/2}}(\gamma_{a_n\cdots a_1})^\delta{}_\gamma(\gamma^{a_1\cdots a_n})^\alpha{}_\beta$$

So.

$$\stackrel{(-)}{=} -\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{2^{d/2}}(\bar{\rho}\gamma_{a_n\cdots a_1}\lambda)(\gamma^{a_1\cdots a_n})^\alpha{}_\beta$$

∎

Show that

$$(\bar\psi P_\pm \lambda)(\bar\rho P_\pm \epsilon) = -\sum_{n=\text{even}}^{d} \frac{1}{n!}\frac{1}{2^{d/2}} (\bar\rho\, \gamma_{a_n\cdots a_1} P_\pm \lambda)(\bar\psi\, \gamma^{a_1\cdots a_n} P_\pm \epsilon)$$

$$(\bar\psi P_\pm \lambda)(\bar\rho P_\mp \epsilon) = -\sum_{n\ \text{odd}} \frac{1}{n!}\frac{1}{2^{d/2}} (\bar\rho\, \gamma_{a_n\cdots a_1} P_\pm \lambda)(\bar\psi\, \gamma^{a_1\cdots a_n} P_\mp \epsilon)$$

$$(\bar\psi \underbrace{P_+\lambda})(\underbrace{\bar\rho P_+\epsilon}) = \sum_{n=0}^{d} \frac{1}{n!}\frac{1}{2^{d/2}} (\bar\rho P_+ \underbrace{\gamma_{a_n\cdots a_1} P_+}_{0} \lambda)(\bar\psi\, \gamma^{a_1\cdots a_n} P_+ \epsilon)$$

treat it as a single fermion.

$$= \sum_{n=\text{even}} \frac{1}{n!}\frac{1}{2^{d/2}} (\bar\rho\, \gamma_{a_n\cdots a_1} P_+ \lambda)(\bar\psi\, \gamma^{a_1\cdots a_n} P_+ \epsilon)$$

and use $P_+\gamma_a = \gamma_a P_-$ ■

<u>Note</u> In $(d+1)$-dimension (odd dimension)

$\{1, \gamma_a, \cdots \gamma_{a_1\cdots a_{\frac{d}{2}}}\}$ form a complet basis.