Mass and angular momentum of black holes in three dimensional gravity theories with first order formalism

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1. Introduction

3 dimensional gravity theory

graviton modes, conserved charges of black holes ......

AdS/CFT correspondence

bulk vs. boundary clash

How to get mass and angular momentum of black hole in 3-dimensions

Brown-York

Wald formalism

Abbott-Deser-Tekin (ADT)

Yi - covariant quasi local charge

*Fierz-Pauli Lagrangian : $\mathcal{L}_{FP} = -A\Big\{k^{\mu\nu}\mathcal{G}_{\mu\nu}(k) + \frac{1}{2}\mathcal{M}^2(k^{\mu\nu}k_{\mu\nu} - k^2)\Big\}$

$$L_{FP} = -AM(k_+ \cdot \bar{D}k_+ + M\bar{e}\cdot k_+ \times k_+) + AM(k_- \cdot \bar{D}k_- - M\bar{e}\cdot k_- \times k_-)$$

*Topologically massive gravity : $L^{(2)}_{TMG} \sim \frac{\mu\ell^2}{2(\mu^2\ell^2-1)}\left[p\cdot\bar{D}p + \sigma\mu\bar{e}\cdot p\times p\right]$

$\mu\ell \geq 1$ (no tachyon condition) $A = -\frac{\sigma\ell^2}{2(\mu^2\ell^2-1)} \geq 0$ (no ghost condition)

single massive state : $M = \sigma\mu$ central charges : $c_{L,R} = \frac{3\ell}{2G}\left(\sigma \pm \frac{1}{\mu\ell}\right)$

*New massive gravity :

$$\mathcal{L}^{(2)}_{NMG} \sim -\bar{\sigma}k^{\mu\nu}\mathcal{G}_{\mu\nu}(k) + \frac{1}{m^4\bar{\sigma}}\left(p^{\mu\nu}\mathcal{G}_{\mu\nu}(p) + \frac{1}{2}\mathcal{M}^2(p^{\mu\nu}p_{\mu\nu} - p^2)\right)$$

massive state : $\mathcal{M}^2 = -m^2\left(\sigma + \frac{1}{2m^2\ell^2}\right) = -\bar{\sigma}m^2 > 0$ (no tachyon condition)

$-m^4\bar{\sigma} > 0$ (no ghost condition)

central charges : $c_{L,R} = \frac{3\ell}{2G}\left(\sigma + \frac{1}{2m^2\ell^2}\right)$

*Minimal massive gravity :

field equation : $$\frac{1}{\mu}C_{\mu\nu} + \bar{\sigma}G_{\mu\nu} + \bar{\Lambda}_0 g_{\mu\nu} = -\frac{\gamma}{\mu^2}J_{\mu\nu}$$

$$J_{\mu\nu} = \frac{1}{2\det g}\epsilon_\mu{}^{\rho\sigma}\epsilon_\nu{}^{\tau\eta}S_{\rho\tau}S_{\sigma\eta}$$

$$\sqrt{-\det g}\nabla_\mu J^{\mu\nu} = -\epsilon^{\nu\rho\sigma}S_\rho{}^\tau C_{\sigma\tau} = \frac{\gamma}{\mu}\epsilon^{\nu\rho\sigma}S_\rho{}^\tau J_{\tau\sigma} \equiv 0$$

$$J_{\mu\nu} = S_\mu{}^\rho S_{\rho\nu} - SS_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(S^{\rho\sigma}S_{\rho\sigma} - S^2)$$

MMG action : $$L_{MMG} = L_{TMG} + \frac{\alpha}{2}e\cdot h\times h$$

$$\bar{\sigma} = \sigma + \alpha\Big[1 + \frac{\alpha\Lambda_0/\mu^2}{2(1+\sigma\alpha)^2}\Big] \qquad \bar{\Lambda}_0 = \Lambda_0\Big[1 + \sigma\alpha - \frac{\alpha^3\Lambda_0/\mu^2}{4(1+\sigma\alpha)^2}\Big] \qquad \gamma = -\frac{\alpha}{(1+\sigma\alpha)^2}$$

massive mode : $M = [\sigma(1+\sigma\alpha) - \alpha C]\mu = \pm\sigma\mu(1+\sigma\alpha)\sqrt{1 + \frac{\alpha\Lambda_0}{\mu^2(1+\sigma\alpha)^2}}$

$\ell M > 1$ (no tachyon condition) $\longrightarrow$ $1 - 2C = \frac{\ell^2 M^2 - 1}{(1+\sigma\alpha)^2\mu^2\ell^2} > 0$

MMG Lagrangian :

$$L^{(2)}_{MMG} \sim \frac{1}{2\mu(1-2C)}\left[p \cdot \bar{D}p + M\bar{e} \cdot p \times p\right]$$

$M\mu(1-2C) < 0$ (no ghost considtion) $\longrightarrow$ $\frac{M}{\mu} < 0$ $\longleftrightarrow$ $\pm\sigma(1+\sigma\alpha) < 0$

central charge : $c_\pm = \frac{3\ell}{2G}\left(\sigma \pm \frac{1}{\mu\ell} + \alpha C\right)$ $C = \frac{(1 - \alpha\ell^2\Lambda_0)}{2(1+\sigma\alpha)^2\mu^2\ell^2}$

2. Review of Wald formalism

variation of Lagrangian : $\delta L = E_\phi \delta\phi + d\Theta(\phi, \delta\phi)$

symplectic form : $\Omega(\phi, \delta_1\phi, \delta_2\phi) = \int_\Sigma \omega(\phi, \delta_1\phi, \delta_2\phi)$

$$\omega(\phi, \delta\phi_1, \delta\phi_2) = \delta_1\Theta(\phi, \delta_2\phi) - \delta_2\Theta(\phi, \delta_1\phi)$$

variation induced by a diffeomorphism : $\delta_\xi\phi = \pounds_\xi\phi$

$$\delta_\xi L = \pounds_\xi L = d i_\xi L = E_\phi \pounds_\xi\phi + d\Theta(\phi, \pounds_\xi\phi)$$

Noether current : $J_\xi = \Theta(\phi, \pounds_\xi\phi) - i_\xi L$

$$dJ_\xi = d\Theta(\phi, \pounds_\xi\phi) - d i_\xi L = -E_\phi \pounds_\xi\phi \quad \longrightarrow \quad J_\xi = dQ_\xi$$

arbitrary variation of the dynamical field : $\delta J_\xi = \delta\Theta(\phi, \pounds_\xi\phi) - i_\xi \delta L$

$$i_\xi \delta L = i_\xi(E_\phi \delta\phi + d\Theta(\phi, \delta\phi)) = \pounds_\xi\Theta(\phi, \delta\phi) - di_\xi\Theta(\phi, \delta\phi)$$

$$\delta J_\xi = \delta\Theta(\phi, \pounds_\xi\phi) - \pounds_\xi\Theta(\phi, \delta\phi) + di_\xi\Theta(\phi, \delta\phi)$$

variation of Hamiltonian : $\delta H_\xi = \int_\Sigma \omega(\phi, \delta\phi, \pounds_\xi\phi)$

symplectic current :

$$\omega(\phi, \delta\phi, \pounds_\xi\phi) = \delta\Theta(\phi, \pounds_\xi\phi) - \pounds_\xi\Theta(\phi, \delta\phi) = \delta J_\xi - di_\xi\Theta(\phi, \delta\phi) = \delta dQ_\xi - di_\xi\Theta$$

$$\delta H_\xi = \int_\Sigma \delta dQ_\xi - di_\xi\Theta = \oint_{\partial\Sigma} \delta Q_\xi - i_\xi\Theta$$

$$0 = \oint_{\partial\Sigma} \delta Q_\xi - i_\xi\Theta \quad \Rightarrow \quad \int_{\mathcal{H}} \delta Q_\xi = \int_\infty \delta Q_\xi - i_\xi\Theta$$

Comparison with the first law of black hole thermodynamics :

$$\int_{\mathcal{H}} \delta Q_\xi = \int_\infty \delta Q_\xi - i_\xi \Theta \quad \Longleftrightarrow \quad T_{\mathcal{H}} \delta \mathcal{S} = \delta \mathcal{E} - \Omega_{\mathcal{H}} \delta \mathcal{J}$$

asymptotic symmetries :

$$\xi = \frac{\partial}{\partial t} + \Omega_{\mathcal{H}} \frac{\partial}{\partial \phi}$$

charge variation :

$$\delta \chi_\xi = \delta Q_\xi - i_\xi \Theta$$

black hole entropy :

$$\mathcal{S}_{\text{ent}} = \frac{2\pi}{\kappa} \int_{\mathcal{H}} Q_\xi$$

Mass of black hole :

$$\delta \mathcal{M} = -\frac{1}{8\pi G} \int_\infty \delta \chi_\xi \left[\frac{\partial}{\partial t}\right]$$

Angular momentum of black hole :

$$\delta \mathcal{J} = \frac{1}{8\pi G} \int_\infty \delta \chi_\xi \left[\frac{\partial}{\partial \phi}\right]$$

3. Topologically Massive Gravity

independent variables : $e^a \quad \omega^a \quad g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu \qquad \omega_a = \frac{1}{2}\epsilon_{abc}\omega^{bc}$

Lagrangian of topologically massive gravity :

$$L = -\sigma e \cdot R + \frac{\Lambda_0}{6} e \cdot e \times e + \frac{1}{2\mu}\left(\omega \cdot d\omega + \frac{1}{3}\omega \cdot \omega \times \omega\right) + h \cdot T(\omega)$$

torsion and curvature 2-form :

$$T(\omega) = De = de + \omega \times e\,, \quad R(\omega) = d\omega + \frac{1}{2}\omega \times \omega$$

equations of motion :

$$-\sigma R + \frac{\Lambda_0}{2} e \times e + Dh = 0\,, \quad -\sigma De + \frac{1}{\mu} R + e \times h = 0\,, \quad T(\omega) = De = 0$$

symplectic potential : $$\Theta = -\sigma\delta\omega\cdot e + \frac{1}{2\mu}\delta\omega\cdot\omega + \delta e\cdot h$$

$$j_\xi = dQ_\xi = \Theta(\phi, \pounds_\xi\phi) - i_\xi L = d\big(-\sigma i_\xi\omega\cdot e + \frac{1}{2\mu}i_\xi\omega\cdot\omega + i_\xi e\cdot h\big)$$

Noether charge : $$Q_\xi = -\sigma i_\xi\omega\cdot e + \frac{1}{2\mu}i_\xi\omega\cdot\omega + i_\xi e\cdot h$$

charge variation form :

$$\delta\chi_\xi = -\sigma(i_\xi\omega\cdot\delta e + \delta\omega\cdot i_\xi e) + \frac{1}{\mu}i_\xi\omega\cdot\delta\omega + i_\xi e\cdot\delta h + \delta e\cdot i_\xi h$$

general metric form for BTZ black hole :

$$ds^2 = -f(r)^2dt^2 + \frac{dr^2}{f(r)^2} + r^2(d\phi + N(r)dt)^2$$

$$e^0 = f(r)dt\,, \quad e^1 = \frac{dr}{f(r)}\,, \quad e^2 = r(d\phi + N(r)dt)$$

connection 1-forms :

$$\omega^0 = \frac{1}{2} r N' e^0 + \frac{f}{r} e^2 , \quad \omega^1 = \frac{1}{2} r N' e^1 , \quad \omega^2 = -\frac{1}{2} r N' e^2 + f' e^0$$

curvature 2-form :

$$R^0 = \left\{ \left(\frac{1}{2} r N' \right)^2 + \frac{f f'}{r} \right\} e^1 \wedge e^2 + \left\{ \frac{3}{2} f N' + \frac{1}{2} r f N'' \right\} e^1 \wedge e^0$$

$$R^1 = -\left\{ \left(\frac{1}{2} r N' \right)^2 + \frac{f f'}{r} \right\} e^2 \wedge e^0$$

$$R^2 = \left\{ 3 \left(\frac{1}{2} r N' \right)^2 - (f f')' \right\} e^0 \wedge e^1 - \left\{ \frac{3}{2} f N' + \frac{1}{2} r f N'' \right\} e^1 \wedge e^2$$

auxiliary fields :

$$h^0 = \frac{1}{\mu} \left\{ -\frac{3}{2} \left(\frac{1}{2} r N' \right)^2 + \frac{(f f')'}{2} \right\} e^0 - \frac{1}{\mu} \left\{ \frac{3}{2} f N' + \frac{1}{2} r f N'' \right\} e^2$$

$$h^1 = \frac{1}{\mu} \left\{ -\frac{3}{2} \left(\frac{1}{2} r N' \right)^2 + \frac{(f f')'}{2} \right\} e^1$$

$$h^2 = \frac{1}{\mu} \left\{ \frac{5}{2} \left(\frac{1}{2} r N' \right)^2 + \frac{f f'}{r} - \frac{(f f')'}{2} \right\} e^2 + \frac{1}{\mu} \left\{ \frac{3}{2} f N' + \frac{1}{2} r f N'' \right\} e^0$$

functions for BTZ black hole : $f(r) = \frac{\sqrt{(r^2 - r_+^2)(r^2 - r_-^2)}}{\ell r}$, $N(r) = -\frac{r_+ r_-}{\ell r^2}$

auxiliary fields for BTZ black hole : $h^a = \frac{1}{2\mu\ell^2} e^a$

charge variation form :

$$\delta\chi_\xi = -\sigma i_\xi \omega \cdot \delta e - \sigma \delta\omega \cdot i_\xi e + \frac{1}{\mu} i_\xi \omega \cdot \delta\omega + \frac{1}{\mu\ell^2} i_\xi e \cdot \delta e$$

charge variation for mass of BTZ inTMG :

$$\delta\chi_\xi\left[\frac{\partial}{\partial t}\right] = \sigma\left(f\delta f + \frac{1}{2}r^3 N \delta N'\right)d\phi - \frac{1}{\mu}\left\{f\left(N + \frac{1}{2}rN'\right)\delta f + \frac{1}{2}r^2\left(ff' - \frac{1}{2}r^2 N N'\right)\delta N'\right\}d\phi$$

$$\delta f = -\frac{1}{\ell^2 r^2 f}\left\{(r^2 - r_-^2) r_+ \delta r_+ + (r^2 - r_+^2) r_- \delta r_-\right\} \quad \delta N' = \frac{2}{\ell r^3}(r_- \delta r_+ + r_+ \delta r_-)$$

$$\delta\mathcal{M} = -\frac{1}{8\pi G}\int_\infty \delta\chi_\xi\left[\frac{\partial}{\partial t}\right] = \frac{1}{4G}\left\{\frac{\sigma}{\ell^2}(r_+\delta r_+ + r_-\delta r_-) + \frac{1}{\mu\ell^3}(r_+\delta r_- + r_-\delta r_+)\right\}$$

$$\mathcal{M} = \sigma\frac{r_+^2 + r_-^2}{8G\ell^2} + \frac{r_+ r_-}{4G\mu\ell^3}$$

charge variation for angular momentum of BTZ in TMG :

$$\delta\chi_\xi\Big[\frac{\partial}{\partial\phi}\Big] = \Big\{\sigma\frac{1}{2}r^3\delta N' + \frac{1}{\mu}\Big(-f\delta f + \frac{1}{4}r^4 N'\delta N'\Big)\Big\}d\phi$$

$$\delta\mathcal{J} = \frac{1}{8\pi G}\int_\infty \delta\chi_\xi\Big[\frac{\partial}{\partial\phi}\Big] = \frac{1}{4G}\Big\{\frac{\sigma}{\ell}(r_+\delta r_- + r_-\delta r_+) + \frac{1}{\mu\ell^2}(r_+\delta r_+ + r_-\delta r_-)\Big\}$$

$$\mathcal{J} = \sigma\frac{r_+ r_-}{4G\ell} + \frac{r_+^2 + r_-^2}{8G\mu\ell^2}$$

$$\chi_\xi\Big[\frac{\partial}{\partial\phi}\Big] = \Big\{\sigma\frac{1}{2}r^3 N' + \frac{1}{2\mu}\Big(-f^2 + \frac{1}{4}r^4 N'^2\Big)\Big\}d\phi$$

metric for space-like warped AdS black hole :

$$ds^2 = -N(r)^2dt^2 + \frac{\ell^4 dr^2}{4R(r)^2N(r)^2} + \ell^2R(r)^2(d\theta + N^\theta(r)dt)^2$$

$$R(r)^2 = \frac{r}{4}\Big(3(\nu^2-1)r + (\nu^2+3)(r_+ + r_-) - 4\nu\sqrt{r_+r_-(\nu^2+3)}\Big)$$

$$N(r)^2 = \frac{\ell^2(\nu^2+3)(r-r_+)(r-r_-)}{4R(r)^2} \qquad N^\theta(r) = \frac{2\nu r - \sqrt{r_+r_-(\nu^2+3)}}{2R(r)^2}$$

dreibein :

$$e^0 = N(r)dt\,, \quad e^1 = \frac{\ell^2 dr}{2R(r)N(r)}\,, \quad e^2 = \ell R(r)(d\theta + N^\theta(r)dt)$$

connection 1-forms :

$$\omega^0 = \frac{R^2 N^{\theta\prime}}{\ell} e^0 + \frac{2NR'}{\ell^2} e^2\,, \quad \omega^1 = \frac{R^2 N^{\theta\prime}}{\ell} e^1\,, \quad \omega^2 = -\frac{R^2 N^{\theta\prime}}{\ell} e^2 + \frac{2RN'}{\ell^2} e^0$$

curvature 2-forms for warped AdS black hole :

$$R^0 = \Big(\frac{\nu^2}{\ell^2} + F(r)\Big) e^1 \wedge e^2 + G(r) e^1 \wedge e^0$$

$$R^1 = -\frac{\nu^2}{\ell^2}\, e^2 \wedge e^0$$

$$R^2 = \Big(-\frac{\nu^2}{\ell^2} + \frac{3(\nu^2-1)}{\ell^2} + F(r)\Big) e^0 \wedge e^1 - G(r) e^1 \wedge e^2$$

$$F(r) = \frac{3(\nu^2-1)(\nu^2+3)}{4\ell^2} \cdot \frac{1}{R^2}(r-r_+)(r-r_-)$$

$$G(r) = -\frac{3(\nu^2-1)\sqrt{(\nu^2+3)}}{4\ell^2} \cdot \frac{1}{R^2}\big(2\nu r - \sqrt{r_+ r_-(\nu^2+3)}\big)\sqrt{(r-r_+)(r-r_-)}$$

auxiliary fields for warped AdS black hole :

$$h^0 = \frac{1}{2\mu}\Big(\frac{\nu^2}{\ell^2} - \frac{3(\nu^2-1)}{\ell^2} - 2F\Big)e^0 - \frac{1}{\mu}Ge^2$$

$$h^1 = \frac{1}{2\mu}\Big(\frac{\nu^2}{\ell^2} - \frac{3(\nu^2-1)}{\ell^2}\Big)e^1$$

$$h^2 = \frac{1}{2\mu}\Big(\frac{\nu^2}{\ell^2} + \frac{3(\nu^2-1)}{\ell^2} + 2F\Big)e^2 + \frac{1}{\mu}Ge^0$$

charge variation for mass of warped AdS black hole :

$$\delta\mathcal{M} = \frac{1}{8\pi G\ell}\int_\infty \frac{\ell(\nu^2+3)}{6}\Big\{\delta r_+ + \delta r_- - \frac{1}{\nu}\frac{\sqrt{r_+r_-(\nu^2+3)}}{2r_+r_-}(r_-\delta r_+ + r_+\delta r_-)\Big\}d\theta$$

$$\mathcal{M} = \frac{(\nu^2+3)}{24G}\Big(r_+ + r_- - \frac{1}{\nu}\sqrt{r_+r_-(\nu^2+3)}\Big)$$

charge variation for angular momentum of warped AdS black hole :

$$\chi_\xi\left[\frac{\partial}{\partial\theta}\right] = \left\{\sigma\ell R^4 N^{\theta\prime} - \frac{2}{\mu\ell^2}(NRR')^2 + \frac{1}{2\mu}(R^3N^{\theta\prime})^2 + \frac{\ell^2}{2\mu}\left(\frac{\nu^2}{\ell^2} + \frac{3(\nu^2-1)}{\ell^2}\right)R^2 + \frac{\ell^2}{\mu}R^2F\right\}d\theta$$

$$\mathcal{J} = -\frac{\ell\nu(\nu^2+3)}{96G}\left[\left(r_+ + r_- - \frac{1}{\nu}\sqrt{r_+r_-(\nu^2+3)}\right)^2 - \frac{(5\nu^2+3)}{4\nu^2}(r_+ - r_-)^2\right]$$

4. Minimal Massive Gravity

Lagrangian :

$$L_{\mathrm{MMG}} = L_{\mathrm{TMG}} + \frac{\alpha}{2} e \cdot h \times h = -\sigma e \cdot R + \frac{\Lambda_0}{6} e \cdot e \times e + h \cdot T(\omega) + \frac{1}{2\mu}\Big(\omega \cdot d\omega + \frac{1}{3}\omega \cdot \omega \times \omega\Big) + \frac{\alpha}{2} e \cdot h \times h$$

equations of motion :

$$T(\omega) + \alpha e \times h = 0\,, \quad R(\omega) + \mu e \times h - \sigma\mu T(\omega) = 0\,, \quad -\sigma R(\omega) + \frac{\Lambda_0}{2} e \times e + D(\omega) h + \frac{\alpha}{2} h \times h = 0$$

symplectic potential and Noether charge :

$$\Theta = -\sigma \delta\omega \cdot e + \frac{1}{2\mu}\delta\omega \cdot \omega + \delta e \cdot h\,, \quad Q_\xi = -\sigma i_\xi \omega \cdot e + \frac{1}{2\mu} i_\xi \omega \cdot \omega + i_\xi e \cdot h$$

shifting new connection : $\Omega = \omega + \alpha$

EOMs for shifting connection :

$$T(\Omega) = 0\,, \quad R(\Omega) + \frac{\alpha\Lambda_0}{2} e \times e + \mu(1+\sigma\alpha)^2 e \times h = 0\,, \quad D(\Omega) h - \frac{\alpha}{2} h \times h + \sigma\mu(1+\sigma\alpha) e \times h + \frac{\Lambda_0}{2} e \times e = 0$$

symplectic potential and Noether charge for shifting connection :

$$\Theta = -\sigma\delta\Omega\cdot e + \frac{1}{2\mu}\delta\Omega\cdot\Omega + (1+\sigma\alpha)\delta e\cdot h - \frac{\alpha}{2\mu}(\delta\Omega\cdot h + \delta h\cdot\Omega - \alpha\delta h\cdot h)$$

$$Q_\xi = -\sigma i_\xi\Omega\cdot e + \frac{1}{2\mu}i_\xi\Omega\cdot\Omega + (1+\sigma\alpha)i_\xi e\cdot h - \frac{\alpha}{2\mu}(i_\xi\Omega\cdot h + i_\xi h\cdot\Omega - \alpha i_\xi h\cdot h)$$

auxiliary fields for BTZ in MMG :

$$h^a = -\frac{1}{\mu(1+\sigma\alpha)^2}\Big[\frac{\alpha\Lambda_0}{2} - \frac{1}{2\ell^2}\Big]e^a = -\lambda e^a$$

parameter condition : $\alpha\lambda^2 + 2\sigma\mu(1+\sigma\alpha)\lambda - \Lambda_0 = 0$

symplectic potential and Noether charge :

$$\Theta = -\sigma\delta\Omega\cdot e + \frac{1}{2\mu}\delta\Omega\cdot\Omega - \lambda(1+\sigma\alpha)\delta e\cdot e + \lambda\frac{\alpha}{2\mu}(\delta\Omega\cdot e + \delta e\cdot\Omega + \alpha\lambda\,\delta e\cdot e)$$

$$Q_\xi = -\sigma i_\xi\Omega\cdot e + \frac{1}{2\mu}i_\xi\Omega\cdot\Omega - \lambda(1+\sigma\alpha)i_\xi e\cdot e + \lambda\frac{\alpha}{2\mu}(i_\xi\Omega\cdot e + i_\xi e\cdot\Omega + \alpha\lambda\,i_\xi e\cdot e)$$

charge variation form :

$$\delta\chi_\xi = -\Big(\sigma - \lambda\frac{\alpha}{\mu}\Big)(i_\xi\Omega\cdot\delta e + i_\xi e\cdot\delta\Omega) + \frac{1}{\mu}i_\xi\Omega\cdot\delta\Omega - \Big(2\lambda(1+\sigma\alpha) - \lambda^2\frac{\alpha^2}{\mu}\Big)i_\xi e\cdot\delta e$$

charge variation for mass of BTZ in MMG :

$$\delta\chi_\xi\Big[\frac{\partial}{\partial t}\Big] = \Big(\sigma - \lambda\frac{\alpha}{\mu}\Big)\Big(f\delta f + \frac{1}{2}r^3 N\delta N'\Big)d\phi - \frac{1}{\mu}\Big\{f\Big(N + \frac{1}{2}rN'\Big)\delta f + \frac{1}{2}r^2\Big(ff' - \frac{1}{2}r^3 NN'\Big)\delta N'\Big\}d\phi$$

$$\delta\mathcal{M} = -\frac{1}{8\pi G}\int_\infty \delta\chi_\xi\Big[\frac{\partial}{\partial t}\Big] = \frac{1}{4G}\Big\{\frac{1}{\ell^2}\Big(\sigma - \lambda\frac{\alpha}{\mu}\Big)(r_+\delta r_+ + r_-\delta r_-) + \frac{1}{\mu\ell^3}(r_-\delta r_+ + r_+\delta r_-)\Big\}$$

$$\mathcal{M} = \frac{r_+^2 + r_-^2}{8G\ell^2}\left(\sigma + \frac{\alpha(1-\alpha\Lambda_0\ell^2)}{2\mu^2\ell^2(1+\sigma\alpha)^2}\right) + \frac{r_+ r_-}{4G\mu\ell^3}$$

charge variation for angular momentum of BTZ in MMG :

$$\chi_\xi\Big[\frac{\partial}{\partial\phi}\Big] = \Big\{\frac{1}{2}\Big(\sigma - \lambda\frac{\alpha}{\mu}\Big)r^3 N' + \frac{1}{2\mu}\Big(-f^2 + \frac{1}{4}r^4 N'^2\Big)\Big\}$$

$$\delta\mathcal{J} = \frac{1}{8\pi G}\int_\infty \delta\chi_\xi\Big[\frac{\partial}{\partial\phi}\Big] = \frac{1}{4G}\Big\{\frac{1}{\ell}\Big(\sigma - \lambda\frac{\alpha}{\mu}\Big)(r_-\delta r_+ + r_+\delta r_-) + \frac{1}{\mu\ell^2}(r_+\delta r_+ + r_-\delta r_-)\Big\}$$

$$\mathcal{J} = \frac{r_+ r_-}{4G\ell}\left(\sigma + \frac{\alpha(1-\alpha\Lambda_0\ell^2)}{2\mu^2\ell^2(1+\sigma\alpha)^2}\right) + \frac{r_+^2 + r_-^2}{8G\mu\ell^2}$$

metric form for new type black hole in MMG :

$$ds^2 = -f(r)^2 dt^2 + \frac{dr^2}{f(r)^2} + r^2 d\phi^2$$

$$e^0 = f(r)dt\,, \quad e^1 = \frac{dr}{f(r)}\,, \quad e^2 = r d\phi$$

connection 1-forms : $\Omega^0 = \frac{f}{r} e^2\,, \quad \Omega^1 = 0\,, \quad \Omega^2 = f' e^0$

curvature 2-forms :

$$R^0 = \frac{ff'}{r} e^1 \wedge e^2\,, \quad R^1 = -\frac{ff'}{r} e^2 \wedge e^0\,, \quad R^2 = -(ff')' e^0 \wedge e^1$$

auxiliary fields for new type black hole in MMG :

$$h^0 = \frac{1}{\mu(1+\sigma\alpha)^2}\left\{\frac{(ff')'}{2} - \frac{\alpha\Lambda_0}{2}\right\} e^0\,, \qquad h^1 = \frac{1}{\mu(1+\sigma\alpha)^2}\left\{\frac{(ff')'}{2} - \frac{\alpha\Lambda_0}{2}\right\} e^1\,,$$

$$h^2 = \frac{1}{\mu(1+\sigma\alpha)^2}\left\{\frac{ff'}{r} - \frac{(ff')'}{2} - \frac{\alpha\Lambda_0}{2}\right\} e^2$$

charge variation form :

$$\delta\chi_\xi = -\sigma(i_\xi\Omega\cdot\delta e + i_\xi e\cdot\delta\Omega) + \frac{1}{\mu}i_\xi\Omega\cdot\delta\Omega + (1+\sigma\alpha)(i_\xi e\cdot\delta h + i_\xi h\cdot\delta e) - \frac{\alpha}{\mu}(i_\xi\Omega\cdot\delta h + i_\xi h\cdot\delta\Omega - \alpha i_\xi h\cdot\delta h)$$

function for new type black hole in MMG : $f(r) = \frac{\sqrt{(r-r_+)(r-r_-)}}{\ell}$

charge form for new type black hole in MMG :

$$\chi_\xi\left[\frac{\partial}{\partial t}\right] = \left\{\frac{1}{2}\left(\sigma - \lambda\frac{\alpha}{\mu}\right)f^2 - \frac{\alpha}{2\mu^2(1+\sigma\alpha)^2}(ff')^2\right\}d\phi$$

parameter conditions : $\Lambda_0 = \frac{1}{\alpha\ell^2} - \frac{2\sigma\mu^2(1+\sigma\alpha)^3}{\alpha^2}$

$$\frac{\alpha}{\mu^2(1+\sigma\alpha)^4}\left(\frac{1}{2\ell^2} - \frac{\alpha\Lambda_0}{2}\right)^2 - \frac{\sigma}{\ell^2(1+\sigma\alpha)} - \frac{\Lambda_0}{1+\sigma\alpha} = 0$$

$$\Downarrow$$

$$\sigma - \lambda\frac{\alpha}{\mu} = \sigma + \frac{\alpha(1-\alpha\Lambda_0\ell^2)}{2\mu^2\ell^2(1+\sigma\alpha)^2} = 2\sigma + \alpha = \frac{\alpha}{\mu^2\ell^2(1+\sigma\alpha)^2}$$

simplified charge form :

$$\chi_\xi\left[\frac{\partial}{\partial t}\right] = \frac{1}{2}\Big(\sigma - \lambda\frac{\alpha}{\mu}\Big)(f^2 - (ff')^2)d\phi = -\Big(\sigma + \frac{\alpha}{2}\Big)\frac{1}{4\ell^2}(r_+ - r_-)^2 d\phi$$

mass of the new type black hole in MMG :

$$\mathcal{M} = -\frac{1}{8\pi G}\int_\infty \chi_\xi\left[\frac{\partial}{\partial t}\right] = \frac{1}{16G\ell^2}\Big(\sigma + \frac{\alpha}{2}\Big)(r_+ - r_-)^2$$

5. Summary

Investigation of the charge variation form through the Wald formalism

Masses and angular momenta of asymptotically AdS black holes

Mass of new type black hole in MMG as a new result