1 The matrix model approach to ABJM

The localized partition function of ABJM theory is

$$Z_k(N) \ := \ \int d^N u \, d^N w \, e^{-S_{\rm ABJM}(u,w)}$$

where

$$e^{-S_{\rm ABJM}(u,w)} \ := \ \exp\left[\frac{ik}{4\pi}\sum_{i=1}^{N}\left(u_i^2 - w_i^2\right)\right] \frac{\prod_{i<j}\sinh^2\frac{u_i-u_j}{2}\sinh^2\frac{w_i-w_j}{2}}{\prod_{i,j}\cosh^2\frac{u_i-w_j}{2}}.$$

The $\frac{1}{6}$-BPS Wilson loop is given as

$$\langle W_1 \rangle \ = \ \frac{1}{Z_k(N)}\int d^N u \, d^N w \, e^{-S_{\rm ABJM}(u,w)} \frac{1}{N}\sum_{i=1}^{N} e^{u_i},$$

$$\langle W_2 \rangle \ = \ \frac{1}{Z_k(N)}\int d^N u \, d^N w \, e^{-S_{\rm ABJM}(u,w)} \frac{1}{N}\sum_{i=1}^{N} e^{w_i}.$$

The $\frac{1}{2}$-BPS Wilson loop turns out to be

$$\langle \mathcal{W} \rangle \ = \ \frac{1}{2}\langle W_1 \rangle + \frac{1}{2}\langle W_2 \rangle.$$

QFT result:

$$\langle W_{1,2} \rangle \ = \ 1 + \frac{5}{6}\pi^2\lambda^2 + O(\lambda^4)$$

where $\lambda = \frac{N}{k}$. This can be easily reproduced from the localized partition function by the $1/k$ expansion (up to the framing factor).

To obtain the large λ behavior, which is important in AdS/CFT, we need to evaluate the integrals.

The integrals can be done by applying the matrix model techniques. To review the matrix model techniques, we start with the Gaussian matrix model

$$Z_{g_s^{-1}}(N) \;=\; \int dM\, \exp\left[-\frac{1}{2g_s}\mathrm{Tr}(M^2)\right] \qquad (1)$$

where M runs over all $N \times N$ Hermitian matrix.

M can be diagonalized as $M = U\Lambda U^\dagger$. Since $\mathrm{Tr}(M^2) = \mathrm{Tr}(\Lambda^2)$, U is a "gauge" d.o.f. To integrate U out, notice that the "metric"

$$\mathrm{Tr}(dM^2) \;=\; \sum_i d\lambda_i^2 + \sum_{i<j}(\lambda_i - \lambda_j)^2 |du_{ij}|^2,$$

where $U = e^{iu}$, induces the "volume form"

$$dM \;\propto\; \prod_i d\lambda_i \prod_{i<j} du_{ij}\sqrt{-g}, \qquad \sqrt{-g} \;:=\; \prod_{i<j}(\lambda_i - \lambda_j)^2.$$

Therefore, (1) can be written as

$$Z_{g_s^{-1}}(N) \;=\; \int d^N\lambda\, e^{-S_{\mathrm{Gauss}}(\lambda)},$$

where

$$e^{-S_{\mathrm{Gauss}}(\lambda)} \;:=\; \exp\left[-\frac{1}{2g_s}\sum_i \lambda_i^2\right]\prod_{i<j}(\lambda_i - \lambda_j)^2.$$

The observables of the Gaussian matrix model are

$$\mathrm{Tr}M^n \;=\; \sum_i \lambda_i^n.$$

The vev is given as

$$\langle \mathrm{Tr}M^n \rangle \;=\; \frac{1}{Z_{g_s^{-1}}(N)}\int d^N\lambda\, e^{-S_{\mathrm{Gauss}}(\lambda)} \sum_i \lambda_i^n. \qquad (2)$$

The evaluation of the integral can be done if N is large.

Notice that the terms in

$$S_{\text{Gauss}}(\lambda) \;=\; \frac{1}{2g_s}\sum_i \lambda_i^2 - \sum_{i<j}\log(\lambda_i-\lambda_j)^2$$

are of order $O(N^2)$ if

$$t \;:=\; g_s N \;=\; \text{fixed}.$$

In the planar limit (large N limit with t fixed), the saddle point

$$-\frac{1}{t}\lambda_i + \frac{2}{N}\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j} \;=\; 0 \qquad (3)$$

dominates the integral, and the saddle point approximation becomes exact. The vevs (2) are given as

$$\langle \text{Tr} M^n\rangle \;=\; \sum_i \lambda_i^n,$$

where λ_l are the solution of (3). It is convenient to introduce

$$\rho(\lambda) \;:=\; \frac{1}{N}\sum_i \delta(\lambda-\lambda_i)$$

by which

$$\langle \text{Tr} M^n\rangle \;=\; \int d\lambda\, \rho(\lambda)\lambda^n.$$

The "eigenvalue density" $\rho(\lambda)$ contains all the information of the planar limit.

Corrections to the saddle point approx. for large but finite N give a $1/N^2$ expansion. This is the WKB expansion if $1/N^2$ is regarded as the Planck constant.

2 Planar limit

The saddle point equations

$$-\frac{1}{t}\lambda_i + \frac{2}{N}\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j} \;=\; 0 \tag{4}$$

can be interpreted as

$$\text{(harmonic force)} + \text{(Coulomb repulsion)} = 0.$$

Therefore, the eigenvalues λ_i are distributed around $\lambda = 0$ with a width. Large $t \Leftrightarrow$ broad distribution.

As N becomes large, the eigenvalues increase, but the Coulomb repulsion becomes weak. In the planar limit,

$$\rho(\lambda) \;=\; \frac{1}{N}\sum_i \delta(\lambda-\lambda_i) \quad \longrightarrow \quad \text{a continuous function } \rho(\lambda)$$

(in the sense of the distribution) with

$$\mathrm{supp}\{\rho\} \;=\; [a,b].$$

The vevs are now given as

$$\langle \mathrm{Tr} M^n\rangle \;=\; \int_a^b d\lambda\,\rho(\lambda)\lambda^n.$$

The task is to determine $\rho(\lambda)$.

The standard tool is the resolvent defined as

$$v(z) \;:=\; \int_a^b d\lambda\,\frac{\rho(\lambda)}{z-\lambda}. \qquad (z\in\mathbb{C}\backslash[a,b])$$

This is the large N limit of

$$\frac{1}{N}\sum_i \frac{1}{z-\lambda_i}.$$

Recall

$$\frac{1}{x \pm i\epsilon} = \mathrm{P}\frac{1}{x} \mp \pi i \delta(x).$$

Using this identity, the resolvent can be written as

$$v(x \pm i\epsilon) = \mathrm{P}\int_a^b d\lambda\, \frac{\rho(\lambda)}{x-\lambda} \mp \pi i \rho(x). \qquad (x \in [a,b])$$

This implies:

- $\rho(x)$ can be recovered from $v(z)$ as

$$v(x+i\epsilon) - v(x-i\epsilon) = -2\pi i \rho(x).$$

- Since the principal integral is

$$\frac{1}{N}\sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j} \quad \longrightarrow \quad \mathrm{P}\int_a^b d\lambda\, \frac{\rho(\lambda)}{\lambda_i - \lambda},$$

  the saddle point equation (4) can be written as

$$\frac{1}{t}x = v(x+i\epsilon) + v(x-i\epsilon). \qquad (x \in [a,b]) \tag{5}$$

By construction, $v(z)$ satisfies the following conditions:

- $v(z)$ is holomorphic on $\mathbb{C}\backslash[a,b]$.
- $v(z)$ is finite at $z=a$ and $z=b$.
  (Otherwise, $\rho(\lambda)$ is delta-function-like at $z=a,b$.)
- $v(z)$ behaves asymptotically

$$v(z) = \frac{1}{z}\int_a^b d\lambda\, \rho(\lambda) + O(z^{-2}) = \frac{1}{z} + O(z^{-2}).$$

These are enough to determine $v(z)$ uniquely.

To solve (5), introduce $\omega(z)$ such that

$$v(z) \;=\; \omega(z)\sqrt{(z-a)(z-b)}.$$

Then $\omega(z)$ satisfies

$$\frac{x}{it\sqrt{|(x-a)(x-b)|}} \;=\; \omega(x+i\epsilon)-\omega(x-i\epsilon).$$

This implies that

$$\omega(z) \;=\; \frac{1}{t}\int_a^b \frac{d\lambda}{2\pi}\frac{\lambda}{z-\lambda}\frac{1}{\sqrt{|(\lambda-a)(\lambda-b)|}}$$

is a solution. The integral can be done as follows:

$$\begin{aligned}
&\int_a^b \frac{d\lambda}{2\pi}\frac{\lambda}{z-\lambda}\frac{1}{\sqrt{|(\lambda-a)(\lambda-b)|}}\\
&= \frac{1}{2}\int_C \frac{d\lambda}{2\pi i}\frac{\lambda}{z-\lambda}\frac{1}{\sqrt{(\lambda-a)(\lambda-b)}}\\
&= \frac{1}{2}\left[-\oint_z+\int_{C_\infty}\right]\frac{d\lambda}{2\pi i}\frac{\lambda}{z-\lambda}\frac{1}{\sqrt{(\lambda-a)(\lambda-b)}}\\
&= \frac{1}{2}\frac{z}{\sqrt{(z-a)(z-b)}}-\frac{1}{2}.
\end{aligned}$$

Therefore, we obtain

$$v(z) \;=\; \frac{1}{2t}z-\frac{1}{2t}\sqrt{(z-a)(z-b)}.$$

This is the right solution. The asymptotic behavior is

$$v(z) \;=\; \frac{a+b}{4t}+\frac{(b-a)^2}{16tz}+O(z^{-2}).$$

The correct behavior is obtained iff

$$-a \;=\; b \;=\; 2\sqrt{t}.$$

The eigenvalue density is therefore

$$\rho(\lambda) \;=\; \frac{1}{2\pi t}\sqrt{4t-\lambda^2}.$$

Indeed, this satisfies

$$\int_{-2\sqrt{t}}^{+2\sqrt{t}} d\lambda\, \rho(\lambda) \;=\; 1$$

as expected from the definition.

The uniqueness of the solution:

Suppose there exists another solution $\tilde{v}(z)$. The difference $\delta v(z) := \tilde{v}(z) - v(z)$ satisfies

$$0 \;=\; \delta v(x+i\epsilon) + \delta v(x-i\epsilon).$$

Define $\delta\omega(z)$ such that

$$\delta v(z) \;=\; \delta\omega(z)\sqrt{(z-a)(z-b)}.$$

Then $\delta\omega(z)$ satisfies

$$\delta\omega(x+i\epsilon) \;=\; \delta\omega(x-i\epsilon),$$

that is, $\delta\omega(z)$ is an entire function. The asymptotic behavior of $\delta v(z)$ around infinity implies that $\delta\omega(\infty) = 0$. Liouville's theorem then implies that $\delta\omega(z) = 0$. Therefore, the solution is unique.

3 Chern-Simons-matter matrix models

Let us apply the matrix model technique to CSM. The simplest example is pure Chern-Simons theory. The partition function is

$$Z = \int d^N u \exp\left[\frac{ik}{4\pi}\sum_i (u_i)^2\right]\prod_{i<j}\left(\sinh\frac{u_i-u_j}{2}\right)^2 .$$

The saddle point equations are

$$\frac{k}{2\pi i}u_i = \sum_{j\neq i}\coth\frac{u_i-u_j}{2}.$$

Introducing $x_i := e^{u_i}$ makes the equations simplified.

$$\frac{k}{2\pi i N}\log x_i = \frac{1}{N}\sum_{j\neq i}\frac{x_i+x_j}{x_i-x_j}. \tag{6}$$

These resembles (4). The 't Hooft coupling t is defined as

$$t = \frac{2\pi i N}{k}.$$

We define the resolvent $v(z)$ in this case as

$$v(z) := \frac{t}{N}\sum_i \frac{z+x_i}{z-x_i}. \tag{7}$$

Then (6) can be written as

$$2\log x = v(x_+) + v(x_-), \qquad (x\in[a,b]) \tag{8}$$

The vev of the Wilson loop is given as

$$\langle W\rangle = \frac{1}{N}\sum_i e^{u_i} = \frac{1}{N}\sum_i x_i.$$

This appears in the expansion of $v(z)$:

$$v(z) = t + 2t\langle W\rangle z^{-1} + O(z^{-2}).$$

The equation (8) can be solved in the same way. Introduce $\omega(z)$ such that

$$v(z) \;=\; \omega(z)\sqrt{(z-a)(z-b)}.$$

Then, $\omega(z)$ satisfies

$$\frac{2\log x}{i\sqrt{|(x-a)(x-b)|}} \;=\; \omega(x_+) - \omega(x_-).$$

This implies that

$$\omega(z) \;=\; \int_a^b \frac{dx}{2\pi}\frac{2\log x}{z-x}\frac{1}{\sqrt{|(x-a)(x-b)|}}$$

is a solution. This integral can be done and the result is

$$\omega(z) \;=\; \frac{1}{\sqrt{(z-a)(z-b)}}\log\frac{\left(z+\sqrt{ab}-\sqrt{(z-a)(z-b)}\right)^2}{a+b+2\sqrt{ab}}.$$

Therefore, the resolvent $v(z)$ is

$$v(z) \;=\; 2\log\frac{z+\sqrt{ab}-\sqrt{(z-a)(z-b)}}{\sqrt{a}+\sqrt{b}}.$$

The definition (7) implies $-v(0) = v(\infty) = t$ which then implies

$$ab \;=\; 1, \qquad t \;=\; 2\log\frac{\sqrt{a}+\sqrt{b}}{2}.$$

Recall that the physical t is purely imaginary. Therefore, a must be a complex number. The large $\frac{N}{k}$ corresponds to the large phase of a.

The expansion of $v(z)$ gives

$$2t\langle W\rangle \;=\; \frac{1}{2}\left(\sqrt{b}-\sqrt{a}\right)^2 .$$

The finite N result for $\langle W\rangle$ is known:

$$\langle W\rangle \;=\; e^{i\phi}\cdot\frac{1}{N}\frac{\sin\frac{\pi N}{k}}{\sin\frac{\pi}{k}}.$$

The phase $e^{i\phi}$ is determined by the "framing." In the planar limit,

$$\langle W\rangle \;\to\; e^{i\phi}\frac{\sin\frac{\pi N}{k}}{\frac{\pi N}{k}} \;=\; 2e^{i\phi}\frac{\sinh\frac{2}{t}}{t}.$$

The matrix model result is

$$\begin{aligned}\langle W\rangle \;&=\; \frac{1}{4t}\left(\sqrt{b}-\sqrt{a}\right)^2\\ &=\; \frac{1}{4t}\left[\left(\sqrt{b}+\sqrt{a}\right)^2-4\right]\\ &=\; \frac{e^t-1}{t} \;=\; e^{\frac{t}{2}}\frac{\sinh\frac{t}{2}}{t}.\end{aligned}$$

This is the correct result with $\phi=\frac{\pi N}{k}$.

Note that $|\langle W\rangle|$ is bounded from above for large $\frac{N}{k}$.

The calculation can be extended to ABJM theory. There are two eigenvalues u_i, w_i. Their distributions are encoded into two resolvents $v_1(z), v_2(z)$, satisfying

$$\begin{aligned} 2\log x &= v_1(x_+) + v_1(x_-) - 2v_2(x), \qquad (x \in [a_1, b_1]) \\ -2\log(-y) &= v_2(y_+) + v_2(y_-) - 2v_1(y). \qquad (y \in [a_2, b_2]) \end{aligned}$$

This can be simplified by introducing

$$\omega(z) := v_1(z) - v_2(z).$$

Then the equations become

$$\begin{aligned} 2\log x &= \omega(x_+) + \omega(x_-), \\ 2\log(-y) &= \omega(y_+) + \omega(y_-). \end{aligned}$$

Therefore, $\omega(z)$ is a "two-cut" solution of pure CS matrix model. Explicitly,

$$\omega(z) = \log \frac{\sqrt{(z-a_2)(z-b_2)} - \sqrt{(z-a_1)(z-b_1)}}{\sqrt{a_1+b_1-a_2-b_2}}.$$

The extension to ABJ theory is trivial, but the extension to GT theory is not. In the latter case, $\omega(z)$ satisfies

$$\begin{aligned} 2\kappa_1 \log x &= \omega(x_+) + \omega(x_-), \\ -2\kappa_2 \log(-y) &= \omega(y_+) + \omega(y_-). \end{aligned}$$

$\omega(z)$ has an integral formula

$$\omega(z) = \kappa_1 \int_{C_1} \frac{dx}{2\pi i} \frac{1}{z-x} \frac{\log x}{s(x)} - \kappa_2 \int_{C_2} \frac{dx}{2\pi i} \frac{1}{z-x} \frac{\log(-x)}{s(x)},$$

where

$$s(z) := \sqrt{(z-a_1)(z-b_1)(z-a_2)(z-b_2)},$$

but performing these integrals looks hopeless.

4 The derivative of resolvents

The calculations so far are quite tedious. For GT theory, the resulting resolvent is not explicit enough.

Observation: The derivatives of the resolvents are simple.

- Pure CS:

$$zv'(z) \;=\; 1-\frac{z-1}{\sqrt{(z-a)(z-b)}}.$$

- ABJM:

$$z\omega'(z) \;=\; 1-\frac{z^2-1}{s(z)}.$$

What happened?

The saddle point equations are also simplified.

- Pure CS:

$$2 \;=\; x_+v'(x_+)+x_-v'(x_-). \tag{9}$$

- ABJM:

$$2 \;=\; x_+\omega(x_+)+x_-\omega(x_-), \qquad 2 \;=\; y_+\omega(y_+)+y_-\omega(y_-). \tag{10}$$

$\log x$ in the LHS disappears.

The differentiation does not lose information.

$$v(z) \;=\; \sum_{n=0}^{\infty} c_n z^{-n} \qquad \Rightarrow \qquad zv'(z) \;=\; -\sum_{n=1}^{\infty} n c_n z^{-n}.$$

The constant term $c_0 = t$ can be recovered as

$$t \;=\; \frac{1}{2}\int_0^{\infty} dz\, v'(z).$$

Let us solve (9). Let

$$zv'(z) \;=\; 1+\frac{f(z)}{\sqrt{(z-a)(z-b)}}.$$

Then, $f(z)$ satisfies

$$f(x_+) \;=\; f(x_-).$$

Therefore, $f(z)$ is holomorphic. The asymptotic behavior $zv'(z) = O(z^{-1})$ implies

$$f(z) \;=\; -z+c.$$

The behavior at the origin $zv'(z) = O(z)$ implies $c = 1$.

Next, let us solve (10). Let

$$z\omega'(z) \;=\; 1+\frac{f(z)}{s(z)}.$$

Then $f(z)$ satisfies

$$f(x_+) \;=\; f(x_-), \qquad f(y_+) \;=\; f(y_-).$$

Therefore, $f(z)$ is holomorphic. The behavior at $z = 0, \infty$ implies

$$f(z) \;=\; -z^2+cz+1.$$

The leading coefficients of $z\omega'(z)$ in the expansions are proportional to $\langle W\rangle$. This implies $c = 0$.

$v'(z)$ contains the same information as $v(z)$ but the former is much simpler than the latter. In particular, $v'(z)$ is algebraic.

For GT theory, the life is not such simple, but better anyway. $z\omega'(z)$ has an integral formula

$$z\omega'(z) \;=\; \frac{1}{2}f(z) + \frac{1}{2}f(z^{-1}),$$

where

$$\begin{aligned} f(z) &= \kappa_1 \int_{C_1} \frac{d\xi}{2\pi i}\Omega(z,\xi) - \kappa_2 \int_{C_2} \frac{d\xi}{2\pi i}\Omega(z,\xi), \\ \Omega(z,\xi) &= \frac{1}{z-\xi}\frac{zs(\xi)}{\xi s(z)}. \end{aligned}$$

Note that $f(z)$ is given in terms of the elliptic integrals. Therefore, $z\omega'(z)$ is given in terms of 3 standard elliptic integrals.

$$z\omega'(z) \;=\; -\kappa_2\left[1 - \frac{z^2-1}{s(z)}\right] - \frac{\kappa_1+\kappa_2}{2}\frac{z^2-1}{s(z)}F(z), \qquad (11)$$

where $F(z)$ is given in terms of $K(k)$, $E(k)$ and $\Pi_1(z,k)$. Due to $\Pi_1(z,k)$, $z\omega'(z)$ is not algebraic.

The vev of a Wilson loop in GT theory can be obtained from (11). The expansion gives

$$z\omega'(z) \;=\; -2\left(t_1\langle W_1\rangle + t_2\langle W_2\rangle\right)z^{-1} + O(z^{-2}).$$

Note that this combination corresponds to the half-BPS Wilson loops for ABJM theory. One obtains

$$t_1\langle W_1\rangle + t_2\langle W_2\rangle \;=\; -\frac{\kappa_2}{4}(a_1+b_1+a_2+b_2) + (\text{elliptic integrals}).$$

The same technique can be applied to variants of ABJM theory in which the # of bi-fundamental matters is $n \neq 2$.

The saddle point equations are

$$\begin{aligned} 2\log x &= v_1(x_+) + v_1(x_-) - n v_2(x), \qquad (x \in [a_1, b_1]) \\ -2\log(-y) &= v_2(y_+) + v_2(y_-) - n v_1(y). \qquad (y \in [a_2, b_2]) \end{aligned}$$

It turns out that $v_1'(z)$ and $v_2'(z)$ are written in terms of the theta functions. The elliptic functions appear due to the presence of two cuts. The vevs of the Wilson loops can be obtained explicitly in terms of the theta functions.

Open issues

- Generalization to CSM matrix models with more nodes.
- Systematic study of the large 't Hooft coupling limit.
- $\mathcal{N} = 2$ CSM matrix models.
- etc.