M2-branes on orbifold and exact large N expansion

Tomoki Nosaka,*[1]

[1]: *Korea Institute for Advanced Study, Seoul 02455, Korea*

Abstract

In this talk we consider the partition function of the ABJM theory and its generalization, $\mathcal{N}=4$ $U(N)$ circular quiver superconformal Chern-Simons theories. Using the Fermi gas formalism we compute the all order $1/N$ corrections to the partition function in the eleven dimensional limit $N\to\infty$ with k kept finite. We also study the non-perturbative effects in $1/N$ by small k expansion. This is two hour talk in the "Mini Workshop on Gauge theory and Supergravity" at APCTP (7/25-29, 2016).

*nosaka(at)yukawa.kyoto-u.ac.jp

Contents

1 S^3 Partition function of ABJM theory

The partition function of the ABJM theory [1] is known to be written as the following $2N$ dimensional integration [2]

$$Z(N) = \frac{1}{(N!)^2} \int d^N\lambda d^N\widetilde{\lambda} e^{\pi i k \sum_{i=1}^N (\lambda_i^2 - \widetilde{\lambda}_i^2)} Z_{\text{1-loop}}, \tag{1.1}$$

with

$$Z_{\text{1-loop}} = \frac{\prod_{i<j}(2\sinh\pi(\lambda_i - \lambda_j))^2 \prod_{i<j}(2\sinh\pi(\widetilde{\lambda}_i - \widetilde{\lambda}_j))^2}{\prod_{i,j}(2\cosh\pi(\lambda_i - \widetilde{\lambda}_j))^2.} \tag{1.2}$$

Below we compute the partition function, or the free energy $F = -\log Z$ in the limit $N \to \infty$, through two different ways: (i) saddle point approximation [3] and (ii) Fermi gas formalism [4].

1.1 Saddle point approximation for $N \to \infty$ with finite k

First rewrite the partition function (1.2) as

$$Z(N) = \int d^N \lambda d^N \widetilde{\lambda} e^{-f(\lambda,\widetilde{\lambda})} \tag{1.3}$$

with

$$f(\lambda,\widetilde{\lambda}) = -\pi i k \sum_{i=1}^{N} (\lambda_i^2 - \widetilde{\lambda}_i^2) - \sum_{i<j} \log(2\sinh\pi(\lambda_i - \lambda_j))^2 - \sum_{i<j} \log(2\sinh\pi(\widetilde{\lambda}_i - \widetilde{\lambda}_j))^2$$
$$+ \sum_{i,j} \log(2\cosh\pi(\lambda_i - \widetilde{\lambda}_j))^2. \tag{1.4}$$

In the limit $N \to \infty$, we can approximate the partition function as

$$Z(N) \approx e^{-f(\lambda_{\text{saddle}},\widetilde{\lambda}_{\text{saddle}})}, \tag{1.5}$$

where the subscript "saddle" means that we choose the saddle point configuration, which is the solution to the following saddle point equations

$$\frac{\partial f}{\partial \lambda_i} = 0, \quad \frac{\partial f}{\partial \widetilde{\lambda}_i} = 0. \tag{1.6}$$

In the 't Hooft limit $k, N \to \infty$ with N/k kept finite, all the terms in the free energy f (1.4) scale as $\mathcal{O}(N^2)$ for generic values λ_i, $\widetilde{\lambda}_i$ of $\mathcal{O}(1)$. This implies that we can solve the saddle point equations for some $(\lambda_i, \widetilde{\lambda}_i)$ which are of $\mathcal{O}(1)$.

In the eleven dimensional limit $N \to \infty$ with k kept finite, the same choice does not guarantee the balancing of terms. Instead we have to pose the following ansatz

$$\lambda_i = \sqrt{N} x(i/N) + y(i/N), \quad \widetilde{\lambda}_i = \sqrt{N} x(i/N) - y(i/N). \tag{1.7}$$

Substiuting these ansatz to (1.4) we indeed find that the first terms and the terms coming from $Z_{\text{1-loop}}$ have the same scaling $\mathcal{O}(N^{3/2})$ [3, 5]:

$$f = 4\pi N^{3/2} H + \mathcal{O}(N),$$
$$H = \int_0^1 ds \left[-ikxy + \frac{2}{\dot{x}} \Big(\frac{1}{16} + y^2 \Big) \right], \tag{1.8}$$

where $s = i/N \in (0,1)$ and we have assumed that $\text{Re}(x)$ is monotonically increasing function in s in the derivation.

Now we can determine the saddle point solution by solving the extremization problem of H

$$\frac{\delta H}{\delta x} = 0 \longrightarrow -iky + \frac{d}{ds} \Big[\frac{2}{\dot{x}^2} \Big(\frac{1}{16} + y^2 \Big) \Big] = 0,$$

$$\frac{\partial H}{\partial x(s=0,1)} = 0 \longrightarrow -\frac{2}{\dot{x}^2}\Big(\frac{1}{16}+y^2\Big)\Big|_{s=0,1} = 0,$$
$$\frac{\delta H}{\delta y} = 0 \longrightarrow -ikx + \frac{4y}{\dot{x}} = 0, \tag{1.9}$$

instead of expanding and solving the original saddle point equations (1.6). The solution to (1.9) is

$$x(s) = as+b, \quad y(s) = \frac{ika(as+b)}{4}, \quad (a=\sqrt{2/k}, \quad b=-1/\sqrt{2k}) \tag{1.10}$$

with which the free energy (1.4) is evaluated as

$$f = \frac{\pi\sqrt{2k}}{3}N^{3/2} + \mathcal{O}(N). \tag{1.11}$$

The saddle point approximation itself is available for various kinds of matrix model obtained by the localization for general theories. However the construction of the solution is not straightforward but rather heuristic, as it requires us to pose an non-trivial ansatz like (1.7). This is in contrast to the Fermi gas formalism. Though it is applicable only to some special theories, once we obtain the Fermi gas formalism for the partition function it is straightforward to compute the free energy in the large N limit. Using the Fermi gas formalism we can also compute the $1/N$ corrections to the free energy easily.

2 Fermi gas formalism

Using the cauchy determinant formula (see appendix A for derivation)

$$\frac{\prod_{i<j}(x_i-x_j)\prod_{i<j}(y_i-y_j)}{\prod_{i,j}(x_i-y_j)} = (-1)^{\frac{N(N-1)}{2}}\det_{i,j}\Big(\frac{1}{x_i-y_j}\Big), \tag{2.1}$$

we can rewrite the partition function as

$$Z = \frac{1}{(N!)^2}\int\Big(\frac{dx}{2\pi}\Big)^N\Big(\frac{dy}{2\pi}\Big)^N\Big[\det_{i,j}\mathcal{O}_1(x_i,y_j)\Big]\Big[\det_{i,j}\mathcal{O}_0(y_i,x_j)\Big], \tag{2.2}$$

with

$$\mathcal{O}_1(x,y) = e^{\frac{i}{4\pi k}x^2}\frac{1}{2k\cosh\frac{x-y}{2k}}e^{-\frac{i}{4\pi k}y^2}, \quad \mathcal{O}_0(x,y) = \frac{1}{2k\cosh\frac{x-y}{2k}}. \tag{2.3}$$

Now we remind some quantum statistical mechanics: a state of N fermions of same kind is written, by taking the anti-symmetric property into account, as

$$|\{x_1,x_2,\cdots,x_N\}\rangle = \frac{1}{\sqrt{N!}}\sum_{\sigma\in S_N}(-1)^\sigma\bigoplus_i|\psi_i = x_{\sigma(i)}\rangle, \tag{2.4}$$

where the overall factor $1/\sqrt{N}$ is determined by the normalization:

$$\langle\{x_1,x_2,\cdots,x_N\}|\{y_1,y_2,\cdots,y_N\}\rangle=\prod_i\langle x_i|y_i\rangle. \tag{2.5}$$

Then each determinant in (2.2) can be regarded as the matrix element of $\widehat{\mathcal{O}}_s\otimes\widehat{\mathcal{O}}_s\otimes\cdots\otimes\widehat{\mathcal{O}}_s$ between $|\{x_1,x_2,\cdots,x_N\}\rangle$ and $|\{y_1,y_2,\cdots,y_N\}\rangle$, while each set of the N integrations multiplied with $1/N!$ can be regarded as the insertion of unity. Hence the partition function is in the same form as the partition function of N particle ideal Fermi gas

$$Z(N)=\frac{1}{N!}\int\Bigl(\frac{dx}{2\pi}\Bigr)^N\det_{i,j}\langle x_i|\widehat{\rho}|x_j\rangle, \tag{2.6}$$

with the one-particle Hamiltonian

$$\widehat{\rho}=e^{-\widehat{H}}=\widehat{\mathcal{O}}_1\widehat{\mathcal{O}}_0. \tag{2.7}$$

Here we have defined the one-particle operators $\widehat{\mathcal{O}}_s$ by $\langle x|\widehat{\mathcal{O}}_s|y\rangle=\mathcal{O}_s(x,y)$ with $|x\rangle,|y\rangle$ one-particle position eigenstates. We can find that $\widehat{\mathcal{O}}$ can be written explicitly as

$$\widehat{\mathcal{O}}_0=\frac{1}{2\cosh\frac{\widehat{P}}{2}},\quad \widehat{\mathcal{O}}_1=\frac{1}{2\cosh\frac{\widehat{P}+\widehat{Q}}{2}}. \tag{2.8}$$

where $\widehat{Q}$ and $\widehat{P}$ are the canonical position/momentum operators obeying $[\widehat{Q},\widehat{P}]=i\hbar$ with

$$\hbar=2\pi ik. \tag{2.9}$$

For later convenience we shall redefine $\widehat{P}+\widehat{Q}$ as $\widehat{Q}$ so that

$$\widehat{\rho}=\frac{1}{2\cosh\frac{\widehat{Q}}{2}}\frac{1}{2\cosh\frac{\widehat{P}}{2}}. \tag{2.10}$$

2.1 Free energy in large N limit

Suppose the eigenvalues of the 1-particle Hamiltonian $\widehat{H}$ are $0\le E_1\le E_2\le\cdots$. Since any two fermions cannot occupy the same state, the lowest total energy is realized by the occupying the first N states counted from the ground state, hence the partition function is

$$Z(N)\approx\exp\Bigl[-\sum_{i=1}^N E_i\Bigr]. \tag{2.11}$$

If we introduce $n(E)$, the "number of 1-particle states with $\widehat{H}\le E$" as we often do in the exercises on the quantum statistical mechanics, then

$$\sum_{i=1}^N E_i=\int_0^{E_{\max}}dE\frac{dn}{dE}E,\quad (n(E_{\max})=N). \tag{2.12}$$

We can compute the large E expansion of $n(E)$ by using the small k (semiclassical) expansion as [4]

$$n(E) = CE^2 + n_0 + \mathcal{O}(e^{-E}), \quad C = \frac{2}{\pi^2 k}, \quad n_0 = -\frac{1}{3k} + \frac{k}{24}. \tag{2.13}$$

Noticing the fact $H \sim (|Q|+|P|)/2$ for $|Q|, |P| \gg 1$, it is not difficult to understand the leading part of $n(E)$ as the phase space volume inside the polygon

$$n(E) \sim \int_{\frac{|Q|+|P}{2} \leq E} \frac{dQdP}{2\pi\hbar} = CE^2. \tag{2.14}$$

On the other hand, the sub-leading part comes from the deviation of the actual Fermi surface from the approaching polygon which is significant only around the vertices of the polygon (see figure 1).

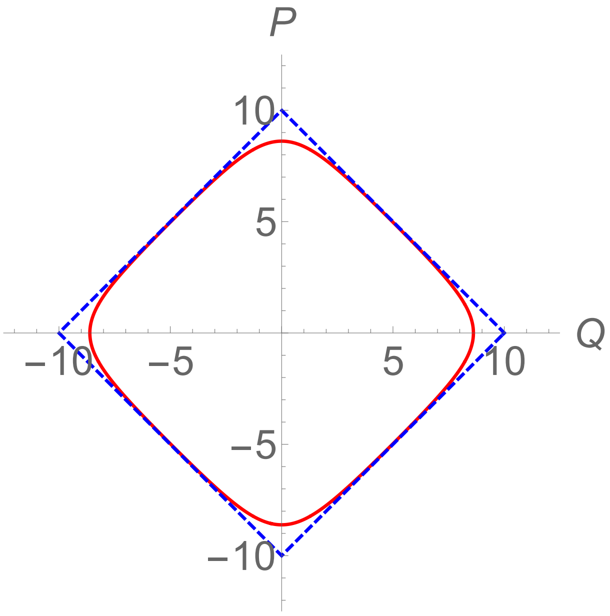

Figure 1: Red: the surface of $H = E$ with $H(Q, P) \sim \log(2 \cosh Q/2) + \log(2 \cosh P/2)$; Blue: the polygon $|Q|/E + |P|/2 = E$ to which $H = E$ approaches in the limit $E \to \infty$.

Hence the free energy in the large N limit is computed by (2.12) as

$$F = -\log Z \approx \frac{2}{3} C^{-\frac{1}{2}} N^{\frac{3}{2}}, \tag{2.15}$$

which coincide with the result obtained from the saddle point approximation (1.11).

3 Perturbative corrections in $1/N$

In this section we explain the determination of the all order perturbative corrections in $1/N$ to the large N free energy (2.15). Our goal is to show the following expression

$$F = -\log Z^{\mathrm{pert}} + (\text{non-perturbative in } 1/N), \tag{3.1}$$

where

$$Z^{\mathrm{pert}} = e^{A} C^{-\frac{1}{3}} \,\mathrm{Ai}[C^{-\frac{1}{3}}(N-B)] \tag{3.2}$$

with A some constant independent of N,

$$C = \frac{2}{\pi^2 k}, \quad B = \frac{1}{3k} + \frac{k}{24}, \tag{3.3}$$

and the Airy fuction given by the following integration

$$\mathrm{Ai}(x) = \int_{-i\infty}^{i\infty} \frac{dt}{2\pi} e^{\frac{1}{3}t^3 - tx}. \tag{3.4}$$

3.1 Grand potential $J(\mu)$ for large μ

First we introduce the auxiliary parameter μ called the chemical potential and define the grand potential $\widetilde{J}(\mu)$ by

$$e^{\widetilde{J}(\mu)} = 1 + \sum_{N \geq 1} e^{\mu N} Z(N) = e^{\widetilde{J}(\mu)}. \tag{3.5}$$

The grand potential is written as [4]

$$\widetilde{J}(\mu) = \mathrm{Tr} \log(1 + e^{\mu} \widehat{\rho}). \tag{3.6}$$

Then, from the large E expansion of the number of states $n(E)$ we can evaluate the large μ expansion of the grand potential $\widetilde{J}(\mu)$ as follows.

$$\begin{aligned} \widetilde{J}(\mu) &= \int_0^\infty dE \frac{dn}{dE} \log(1 + e^{\mu - E}) \\ &= \Big[n(E) \log(1 + e^{\mu - E}) \Big]_{E=0}^{E=\infty} + \int_0^\infty dE n(E) \frac{e^{\mu - E}}{1 + e^{\mu - E}} \\ &= \int_0^\infty dE n(E) \frac{e^{\mu - E}}{1 + e^{\mu - E}}. \end{aligned} \tag{3.7}$$

This convert each term in the large E expansion of the number of states $n(E)$ (2.13) in linear manner, as

$$\begin{aligned} n(E) = CE^2 + \cdots &\longrightarrow \widetilde{J}(\mu) = -2C\,\mathrm{Li}_s(-e^\mu) + \cdots \\ n(E) = n_0 + \cdots &\longrightarrow \widetilde{J}(\mu) = n_0 \log(1+e^\mu) + \cdots\,, \end{aligned} \tag{3.8}$$

where the polylogarithm function is defined as

$$\mathrm{Li}_s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s} \tag{3.9}$$

and we have used the following integration formula

$$\int_0^\infty dE E^s \frac{e^{\mu-E}}{1+e^{\mu-E}} = -\Gamma(s+1)\,\mathrm{Li}_{s+1}(-e^\mu). \tag{3.10}$$

If we assume $|\mathrm{Im}(\mu)| < \pi i$ the polylogarithm functions can be expanded for large μ as

$$\mathrm{Li}_3(-e^\mu) = -\frac{1}{6}\mu^3 - \frac{\pi^2}{6}\mu + \mathcal{O}(e^{-\mu}), \quad \log(1+e^\mu) = \mu + \mathcal{O}(e^{-\mu}), \tag{3.11}$$

hence we obtain

$$\widetilde{J}(\mu) = \frac{C}{3}\mu^3 + \Big(n_0 + \frac{\pi^2 C}{3}\Big)\mu + A + \mathcal{O}(e^{-\mu}). \tag{3.12}$$

Note that, besides the non-perturbative corrections $\mathcal{O}(e^{-\mu})$ we cannot determine the constant A from the parturbative expansion of $n(E)$ in $1/E$ (2.13) alone. Indeed, if we have $n(E) = \cdots + e^{-\alpha E}$, for example, then this term contribute to the large μ expansion of $\widetilde{J}(\mu)$ as

$$n(E) = e^{-\alpha E} + \cdots \longrightarrow \widetilde{J}(\mu) = \sum_{\ell=1}^{\infty} (-1)^{\ell-1} \frac{e^{\ell\mu}}{\ell+\alpha} + \cdots = \frac{1}{\alpha} + \mathcal{O}(e^{-\mu}) + \cdots\,. \tag{3.13}$$

To obtain the exact value of A we need more complicated analysis. Here we would like to just quote the final result: [6]

$$\begin{aligned} A(k) &= \frac{2\zeta(3)}{\pi^2 k}\Big(1 - \frac{k^3}{16}\Big) + \frac{k^2}{\pi^2}\int_0^\infty dx \frac{x}{e^{kx}-1}\log(1-e^{-2x}) \\ &= \frac{2\zeta(3)}{\pi^2 k} - \frac{k}{12} - \frac{\pi^2 k^3}{4320} + \frac{\pi^4 k^5}{907200} - \frac{\pi^6 k^7}{50803200} + \mathcal{O}(k^9), \end{aligned} \tag{3.14}$$

where the second line is the result we can obtain by the small k expansion of the grand potential [4], with which the first line is understood as the resummation.

3.2 All order perturbative corrections in $1/N$

Since the inverse transformation $\widetilde{J}(\mu) \to Z(N)$ to (3.5) is

$$Z(N) = \int_{-i\pi}^{i\pi} \frac{d\mu}{2\pi i} e^{\widetilde{J}(\mu)-\mu N}, \tag{3.15}$$

now we have almost reproduced the Airy function (3.2), except the discrepancy in the integration domains and the discrepancy in the integrand by the multiplication of $e^{\mathcal{O}(e^{-\mu})}$.

First consider the difference in the integration domain. This is resolved by introduce the modified grand potential $J(\mu)$

$$e^{\widetilde{J}(\mu)} = \sum_{n\in\mathbb{Z}} e^{J(\mu+2\pi in)}. \tag{3.16}$$

with which

$$Z(N) = \int_{-i\infty}^{i\infty} \frac{d\mu}{2\pi i} e^{J(\mu)-\mu N}. \tag{3.17}$$

The difference between the original grand potential $\widetilde{J}$ and the modified grand potential $J(\mu)$ can be evaluated as follows. First rewrite (3.16) as

$$\widetilde{J}(\mu) = J(\mu) + J_{\mathrm{osc}}(\mu), \tag{3.18}$$

with

$$J_{\mathrm{osc}}(\mu) = \log\Big[1 + \sum_{n\neq 0} e^{J(\mu+2\pi in)-J(\mu)}\Big]. \tag{3.19}$$

Now suppose

$$J(\mu) = \frac{C}{3}\mu^3 + B\mu + A + \mathcal{O}(e^{-\mu}), \tag{3.20}$$

then, since

$$\begin{aligned} J(\mu+2\pi in) - J(\mu) &= C\Big(2\pi i\mu^2 + (2\pi i)^2\mu + \frac{(2\pi i)^3}{3}\Big) + B(2\pi i) + \mathcal{O}(e^{-\mu}) \\ &= -\frac{8n^2\mu}{k} + i(2\pi C\mu^2 + 2\pi B) + \mathcal{O}(e^{-\mu}), \end{aligned} \tag{3.21}$$

we find that the large μ expansion of $J_{\mathrm{osc}}(\mu)$ consist only of the non-perturbative effects:

$$J_{\mathrm{osc}} = \log(1 + \sum_{n\neq 0} e^{-\frac{8n^2\mu}{k}+\cdots}) = \mathcal{O}(e^{-\mu/k}). \tag{3.22}$$

Hence the assumption (3.20) is indeed consistent, and we obtain

$$Z(N)=\int_{-i\infty}^{i\infty}\frac{d\mu}{2\pi i}e^{J_{\rm pert}(\mu)+\mathcal{O}(e^{-\mu})-\mu N} \tag{3.23}$$

with $J_{\rm pert}$ the perturbative part of the large μ expansion of the modified grand potential.

$$J_{\rm pert}(\mu)=\frac{C}{3}\mu^3+B\mu+A. \tag{3.24}$$

Second let us consider the deviation $e^{\mathcal{O}(e^{-\mu})}$ in the integrand (3.23). In the large N limit, the effect of this term can be evaluated by a "probe approximation" as follows. First let us neglect this term, then the integration (3.23) can be evaluated for large N by the saddle point approximation:

$$\int_{-i\infty}^{i\infty}\frac{d\mu}{2\pi i}e^{J_{\rm pert}(\mu)-\mu N}\approx e^{J_{\rm pert}(\mu_*)-\mu_* N} \tag{3.25}$$

with

$$\frac{d}{d\mu}(J_{\rm pert}-\mu N)=0\longrightarrow \mu_*=\sqrt{\frac{N-B}{C}}. \tag{3.26}$$

Now suppose J have an additional non-perturbative term $ae^{-\omega\mu}$. Although this insertion modifies the saddle point equation (3.26) by $(-a\omega+\partial_\mu a)e^{-\omega\mu}$, we can neglect this effect as the saddle point value μ_* is large. Hence we can evaluate the effect of $e^{\mathcal{O}(e^{-\mu})}$ by using the same saddle as

$$\int_{-i\infty}^{i\infty}e^{J_{\rm pert}(\mu)-\mu N}e^{ae^{-\omega\mu}}\approx Z_{\rm pert}(N)\cdot e^{ae^{-\omega\sqrt{(N-B)/C}}}, \tag{3.27}$$

where

$$Z_{\rm pert}=\int_{-i\infty}^{i\infty}\frac{d\mu}{2\pi i}e^{J_{\rm pert}(\mu)-\mu N}=e^AC^{-\frac{1}{3}}\,{\rm Ai}[C^{-\frac{1}{3}}(N-B)]. \tag{3.28}$$

Hence the non-perturbative effects $\mathcal{O}(e^{-\mu})$ in large μ in $J(\mu)$ correspond to the non-perturbative effects in large N in the free energy F

$$J=J_{\rm pert}+ae^{-\omega\mu}\longrightarrow F-(-\log Z_{\rm pert})\approx ae^{-\omega\sqrt{\frac{N-B}{C}}}. \tag{3.29}$$

Therefore we have finally obtained the Airy function expression for the all order perturbative corrections in $1/N$ to the free energy (3.1).

4 M2-branes on more general orbifold

The Fermi gas formalism exist for more general 3d $U(N)$ circular quiver superconformal Chern-Simons theories. An example is the $\mathcal{N}=4$ $U(N)$ circular quiver superconformal Chern-Simons theory with the levels $U(N)_k \times U(N)_0^{q-1} \times U(N)_{-k} \times U(N)_0^{p-1}$ [7] called the $(q,p)_k$ model. This theory corresponds to the $\mathrm{AdS}_4 \times S^7/\Gamma_{q,p,k}$ spacetime where

$$R_{\mathrm{AdS}} = (32\pi^2 qpkN)^{\frac{1}{6}}, \tag{4.1}$$

and $S^7/\Gamma_{q,p,k}$ is the radial section of $(\mathbb{C}^2/\mathbb{Z}_q \times \mathbb{C}^2/\mathbb{Z}_p)/\mathbb{Z}_k$, where the orbifolds act as

$$\begin{aligned}
S^7 &= \left\{ (z_1, z_2, z_3, z_4) \middle| \sum_{i=1}^{4} |z_i|^2 = R_{\mathrm{AdS}}^2 \right\}, \\
\mathbb{Z}_q &: (z_1, z_2, z_3, z_4) \to (e^{\frac{2\pi i}{q}} z_1, e^{\frac{2\pi i}{q}} z_2, z_3, z_4), \\
\mathbb{Z}_p &: (z_1, z_2, z_3, z_4) \to (z_1, z_2, e^{\frac{2\pi i}{p}} z_3, e^{\frac{2\pi i}{p}} z_4), \\
\mathbb{Z}_k &: (z_1, z_2, z_3, z_4) \to (e^{\frac{2\pi i}{qk}} z_1, e^{\frac{2\pi i}{qk}} z_2, e^{\frac{2\pi i}{pk}} z_3, e^{\frac{2\pi i}{pk}} z_4).
\end{aligned} \tag{4.2}$$

From the localization technique the partition function of the $(q,p)_k$ model is given as

$$\begin{aligned}
Z(N) &= (-1)^{\frac{(q+p)N(N-1)}{2}} \prod_{a=1}^{q+p} \Big(\frac{1}{N!} \int \prod_{i=1}^{N} d\lambda_i^{(a)} \Big) e^{\pi i k \sum_{i=1}^{N} (\lambda_i^{(1)2} - \lambda_i^{(q+1)2})} \\
&\times \prod_{a=1}^{q+p} \frac{\prod_{i<j} 2\sinh \pi(\lambda_i^{(a)} - \lambda_j^{(a)}) \prod_{i<j} 2\sinh \pi(\lambda_i^{(a+1)} - \lambda_j^{(a+1)})}{\prod_{i,j} 2\cosh \pi(\lambda_i^{(a)} - \lambda_j^{(a+1)})},
\end{aligned} \tag{4.3}$$

which can be rewritten into the Fermi gas partition function (2.6) with one-particle Hamiltonian

$$\widehat{\rho} = e^{-\widehat{H}} = \Big(\frac{1}{2\cosh \frac{\widehat{Q}}{2}} \Big)^q \Big(\frac{1}{2\cosh \frac{\widehat{P}}{2}} \Big)^p. \tag{4.4}$$

The large E expansion of the number of state $n(E)$ for this theory is [4, 8]

$$n(E) = C_{q,p} E^2 + n_{0q,p} + \mathcal{O}(e^{-E}) \tag{4.5}$$

with

$$C_{q,p} = \frac{2}{\pi^2 qpk}, \quad n_{0q,p} = -\frac{1}{6k} \Big(\frac{p}{q} + \frac{q}{p} \Big) + \frac{kqp}{24}. \tag{4.6}$$

and the large μ expansion of the grand potential is

$$J(\mu) = \frac{C_{q,p}}{3} \mu^3 + B_{q,p} \mu + A_{q,p} + \mathcal{O}(e^{-\mu}), \quad B_{q,p} = n_{0q,p} + \frac{\pi^2 C_{q,p}}{3} \tag{4.7}$$

with $A_{q,p}$ observed to be written in terms of the A in the ABJM case (3.14) as

$$A_{q,p} = \frac{p^2 A(qk) + q^2 A(pk)}{2}. \tag{4.8}$$

5 Membrane instantons in $k \to 0$ limit

Finally let us evaluate the non-perturbative effects. For this purpose we need to evaluate the grand potential $\widetilde{J}(\mu)$ more exactly. Such a computation is extremely difficult, except in the limit $k \to 0$. Here we would like to consider the $(q,p)_k$ model, not the simpest ABJM case, to keep some orbifold-dependence of the non-perturbative effect, which will help us to argue the gravitational interpretation of these effects.

First we rewrite the grand potential (3.6) in an integration of an auxiliary parameter t [9]

$$J = \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{dt}{2\pi i} \frac{\pi}{t \sin \pi t} \mathcal{Z}(t) e^{t\mu} \tag{5.1}$$

where

$$\mathcal{Z}(t) = \mathrm{Tr} \widehat{\rho}^t. \tag{5.2}$$

If we assume $\mathrm{Re}(\mu) < 0$, we can pinch the contour to surround the right half of the complex plane ($\mathrm{Re}(t) > \epsilon$) as $e^{\mu t} \to 0$ for $\mathrm{Re}(t) \to \infty$. Hence we can evaluate the integration by collecting the residues therein:

$$\mathrm{Re}(\mu) < 0 \longrightarrow J = \sum_{n=1}^{\infty} \mathrm{Res}\Big[\frac{\pi}{t \sin \pi t} \mathcal{Z}(t) e^{t\mu}, t \to n\Big] \tag{5.3}$$

where we have assumed that $\mathcal{Z}(t)$ have no poles in $\mathrm{Re}(t) > 0$. Hence the integration indeed reproduce the infinite sum coming from the expansion of $\log(1 + e^{\mu}\widehat{\rho})$ (3.6).

Now assume $\mu > 0$. Then the integration can be evaluated by pinching the contour opposite way, hence we obtain

$$\mathrm{Re}(\mu) > 0 \longrightarrow J = \sum_{t_a}^{\infty} \mathrm{Res}\Big[-\frac{\pi}{t \sin \pi t} \mathcal{Z}(t) e^{t\mu}, t \to t_a\Big], \tag{5.4}$$

where the overall minus sign comes from the orientation of the integration contour and t_a are the poles of the integrand in $\mathrm{Re}(t) \leq 0$. Therefore we can obtain all the non-perturbative effects by computing $\mathcal{Z}(n)$, continuing its expression $n \in \mathbb{N} \to n \in \mathrm{R}$ and identifying its pole structures.

In the classical limit $k \to 0$, the computation of $\mathcal{Z}(n)$ is straightforward and we obtain

$$\begin{aligned} \mathcal{Z}(n) &= \frac{1}{\hbar} \mathcal{Z}_0(n) + \mathcal{O}(\hbar), \\ \mathcal{Z}_0 &= \int \frac{dQdP}{2\pi} \Big(\frac{1}{2\cosh\frac{Q}{2}}\Big)^q \Big(\frac{1}{2\cosh\frac{P}{2}}\Big)^p = \frac{1}{2\pi} \frac{\Gamma(\frac{nq}{2})^2 \Gamma(\frac{np}{2})^2}{\Gamma(nq)\Gamma(np)}. \end{aligned} \tag{5.5}$$

This function indeed have no poles in $\mathrm{Re}(t) > 0$, hence our strategy (5.3) works.

The poles of the integrand in (5.3) in $\mathrm{Re}(t) \leq 0$ are

$$\begin{aligned} &t = 0, \\ &t = -\frac{2\ell}{q}, \quad (\ell = 1, 2, \cdots) \\ &t = -\frac{2\ell}{p}, \quad (\ell = 1, 2, \cdots) \\ &t = -\ell. \quad (\ell = 1, 2, \cdots) \end{aligned} \tag{5.6}$$

The last three lines says that there are three kinds of the non-perturbative effects $e^{-2\ell\mu/q}$, $e^{-2\ell\mu/p}$ and $e^{-\ell\mu}$.

From the gravity side, the non-perturbative effect can be interpreted as a closed membrane winding on $S^7/\Gamma_{q,p,k}$ (see figure 2). For example, according to the inverse transformation (3.29)

Figure 2: Left: closed M2-brane without winding (red line); Right: closed M2-brane winding on some 3-cycle in $S^7/\Gamma_{q,p,k}$, which looks like zero dimensional object in AdS_4 spacetime (instanton).

the non-perturbative effect of $e^{-2\mu/q}$ corresponds in the free energy to

$$F^{\text{non-pert}} = (\cdots) \cdot e^{-\frac{2}{q}\sqrt{\frac{N-B_{q,p}}{C_{q,p}}}}. \tag{5.7}$$

Using the explicit expression of $C_{q,p}$ (4.6), the exponent can be expressed in terms of R_{AdS} (4.1) as (we temporary neglect the shift B)

$$\frac{2}{q}\sqrt{\frac{N-B_{q,p}}{C_{q,p}}} \sim \frac{\pi}{q}\sqrt{2qpkN} = T_{\mathrm{M2}} \cdot R_{\mathrm{AdS}}^3 \cdot \frac{\pi^2}{q} \tag{5.8}$$

where T_{M2} is the tension of the M2-brane $T_{\mathrm{M2}} = 1/(2\pi)^2$. As the coefficient of T_{M2} in the exponent indeed completely coincides with the volume of $\mathbb{RP}^3/\mathbb{Z}_q$ ($\mathbb{RP}^3 = \{(z_1, z_2, z_3, z_4) \in S^7/\mathbb{Z}_k | z_1 = z_3^*, z_2 = z_4^*\}$). We call such effect as the "instantons". On the other hand, the gravitational interpretation for the coefficient in front of the exponent is still obscure.

From the residues of these poles we obtain

$$\begin{aligned} J(\mu) &= \frac{1}{\hbar} J_0(\mu) + \mathcal{O}(\hbar), \\ J_0 &= J_0^{\text{pert}} + J_0^{\text{np}}, \end{aligned} \tag{5.9}$$

with J_0^{pert} the perturbative part coming from the residue at $t = 0$

$$J_0^{\text{pert}} = \frac{4}{3\pi qp}\mu^3 + \frac{\pi(4 - q^2 - p^2)}{3qp}\mu + \frac{2(q^3 + p^3)\zeta(3)}{\pi qp}, \tag{5.10}$$

and J_0^{np} the contributions from the other poles

$$J_0^{\text{np}} = \sum_{\ell=1}^{\infty} a_\ell e^{-\frac{2\ell\mu}{q}} + \sum_{\ell=1}^{\infty} b_\ell e^{-\frac{2\ell\mu}{p}} + \sum_{\ell=1}^{\infty} c_\ell e^{-\ell\mu}, \tag{5.11}$$

where

$$\begin{aligned} a_\ell &= \binom{2\ell}{\ell} \frac{1}{\ell \sin\frac{2\pi\ell}{q}} \frac{\Gamma(-\frac{p\ell}{q})^2}{\Gamma(-\frac{2p\ell}{q})}, \\ b_\ell &= \binom{2\ell}{\ell} \frac{1}{\ell \sin\frac{2\pi\ell}{p}} \frac{\Gamma(-\frac{q\ell}{p})^2}{\Gamma(-\frac{2q\ell}{p})}, \\ c_\ell &= \frac{(-1)^{\ell-1}}{2\pi\ell} \frac{\Gamma(-\frac{q\ell}{2})^2\Gamma(-\frac{p\ell}{2})^2}{\Gamma(-q\ell)\Gamma(-p\ell)}. \end{aligned} \tag{5.12}$$

Interestingly, the coefficients of the instanton sometimes diverges. We can see that a divergence occurs at the instanton number ℓ such that its instanton exponent is included in either of the other two series of the instantons. For example, a_ℓ diverges at $\ell = q$ where the instanton exponent is $e^{-2\mu}$, which is included in the second series $\{e^{-2\ell\mu/p}\}_{\ell=1}^{\infty}$ $(\ell = p)$ as well as the third series $\{e^{-\ell\mu}\}_{\ell=1}^{\infty}$ $(\ell = 2)$. The divergence simultaneously occurs in these instantons with the same exponent and completely cancel together to give a finite coefficient in total. Nevertheless the remaining coefficient after the pole cancellation have non-trivial μ depenence:

$$\text{instnaton coefficient} \sim \mathcal{O}(\mu^{\#}), \tag{5.13}$$

with # the number of species contribute to the pole cancellation. As this is reflected to the coefficient of the corresponding non-perturbative effect in the free energy as well

$$F^{\text{non-pert}} = \cdots + \mathcal{O}(N^{\#/2}) \cdot e^{-T_{\text{M2}}\cdot\text{vol(M2)}}, \tag{5.14}$$

hopefully the structure of divergence and cancellation will help us to understand the instnaton coefficient in gravity side in future.

A Proof of Cauchy determinant formula

Proof goes as induction. First consider the case of $N=1$. Then $\prod_{i<j}(x_i-x_j)=\prod_{i<j}(y_i-y_j)=1$ and $\prod_{i,j}(x_i-y_j)=(x_1-y_1)$ and the formula (2.1) trivially follows. To proof the formula for $N\geq 2$, let us rewrite (2.1) as

$$\prod_{i<j}(x_i-x_j)\prod_{i<j}(y_i-y_j)=(-1)^{\frac{N(N-1)}{2}}\prod_{i,j}(x_i-y_j)\det_{i,j}\frac{1}{x_i-x_j}. \tag{A.1}$$

If we regard both sides as functions of x_1, then both of them are polynomials of degree $N-1$. Moreover, since the left-hand side vanishes at $x_1=x_i$ ($i=2,3,\cdots,N$) and so does the right-hand side (because i-th degenerate with the 1st row when $x_1=x_i$), these two polynomials are written as

$$\begin{aligned}\text{left-hand side} &= a_L\prod_{i=2}^{N}(x_1-x_i)\\ \text{right-hand side} &= a_R\prod_{i=2}^{N}(x_1-x_i).\end{aligned} \tag{A.2}$$

Hence what we have to do to prove (2.1) for rank N is to show $a_L=a_R$. From (A.1) a_L and a_R are written as

$$\begin{aligned}a_L &= \prod_{2\leq i<j}(x_i-x_j)\prod_{i<j}(y_i-y_j),\\ a_R &= \prod_{i\geq 2}\prod_{j}(x_i-y_j)\lim_{x_1\to\infty}x_1\det_{i,j}\frac{1}{x_i-y_j}.\end{aligned} \tag{A.3}$$

The limit in a_R can be computed as

$$\begin{aligned}x_1\det_{i,j}\frac{1}{x_i-y_j} &= \det\begin{pmatrix}\frac{x_1}{x_1-y_1} & \frac{x_1}{x_1-y_2} & \cdots & \frac{x_1}{x_1-y_N}\\ \frac{1}{x_2-y_1} & \frac{1}{x_2-y_2} & \cdots & \frac{1}{x_2-y_N}\\ & \vdots & & \end{pmatrix}\\ &\underset{x_1\to\infty}{\to}\det\begin{pmatrix}1 & 1 & \cdots & 1\\ \frac{1}{x_2-y_1} & \frac{1}{x_2-y_2} & \cdots & \frac{1}{x_2-y_N}\\ & \vdots & & \end{pmatrix}\\ &\quad\downarrow\ \text{subtract (1st row)}\times(x_i-y_1)^{-1}\text{ from }i\text{-th row}\\ &= \det\begin{pmatrix}1 & 1 & \cdots & 1\\ 0 & (\frac{1}{x_2-y_2}-\frac{1}{x_2-y_1}) & \cdots & (\frac{1}{x_2-y_N}-\frac{1}{x_2-y_1})\\ & \vdots & & \end{pmatrix}\end{aligned}$$

$$= \prod_{i\geq 2} \frac{1}{x_i - y_1} \prod_{j\geq 2}(y_j - y_1) \det_{\substack{2\leq i\leq N\\ 2\leq j\leq N}} \frac{1}{x_i - y_j} \tag{A.4}$$

Hence the condition $a_L = a_R$ is identical to

$$a_L = a_R \Longleftrightarrow \frac{\prod_{2\leq i<j}(x_i - x_j)\prod_{2\leq i<j}(y_i - y_j)}{\prod_{i,j\geq 2}(x_i - y_j)} = (-1)^{\frac{(N-1)(N-2)}{2}} \det_{2\leq i,j\leq 1} \frac{1}{x_i - y_j}, \tag{A.5}$$

which follows if (2.1) for rank $N-1$ is true. Since (2.1) is true for rank $N = 1$, we conclude that (2.1) is true for general $N \geq 2$.

References

[1] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, "N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals," JHEP **0810** (2008) 091 doi:10.1088/1126-6708/2008/10/091 [arXiv:0806.1218 [hep-th]].

[2] A. Kapustin, B. Willett and I. Yaakov, "Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter," JHEP **1003** (2010) 089 doi:10.1007/JHEP03(2010)089 [arXiv:0909.4559 [hep-th]].

[3] C. P. Herzog, I. R. Klebanov, S. S. Pufu and T. Tesileanu, "Multi-Matrix Models and Tri-Sasaki Einstein Spaces," Phys. Rev. D **83** (2011) 046001 doi:10.1103/PhysRevD.83.046001 [arXiv:1011.5487 [hep-th]].

[4] M. Marino and P. Putrov, "ABJM theory as a Fermi gas," J. Stat. Mech. **1203** (2012) P03001 doi:10.1088/1742-5468/2012/03/P03001 [arXiv:1110.4066 [hep-th]].

[5] T. Nosaka, K. Shimizu and S. Terashima, to appear

[6] Y. Hatsuda and K. Okuyama, "Probing non-perturbative effects in M-theory," JHEP **1410** (2014) 158 doi:10.1007/JHEP10(2014)158 [arXiv:1407.3786 [hep-th]].

[7] Y. Imamura and K. Kimura, "On the moduli space of elliptic Maxwell-Chern-Simons theories," Prog. Theor. Phys. **120** (2008) 509 doi:10.1143/PTP.120.509 [arXiv:0806.3727 [hep-th]].

[8] S. Moriyama and T. Nosaka, "Partition Functions of Superconformal Chern-Simons Theories from Fermi Gas Approach," JHEP **1411** (2014) 164 doi:10.1007/JHEP11(2014)164 [arXiv:1407.4268 [hep-th]].

[9] Y. Hatsuda, "Spectral zeta function and non-perturbative effects in ABJM Fermi-gas," JHEP **1511** (2015) 086 doi:10.1007/JHEP11(2015)086 [arXiv:1503.07883 [hep-th]].

[10] S. Moriyama and T. Nosaka, "ABJM membrane instanton from a pole cancellation mechanism," Phys. Rev. D **92** (2015) no.2, 026003 doi:10.1103/PhysRevD.92.026003 [arXiv:1410.4918 [hep-th]].