Workshop 2017 on “Discrete Approaches to the Dynamics of Fields and Space-Time”
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Geometry of Quantum Entanglement

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“Entanglement and Geometry of Quantum Systems”
H. Matsueda, (Morikita Co.,Ltd., 2016) in Japanese

HM and T.Suzuki, JPSJ86, 104001 (2017)

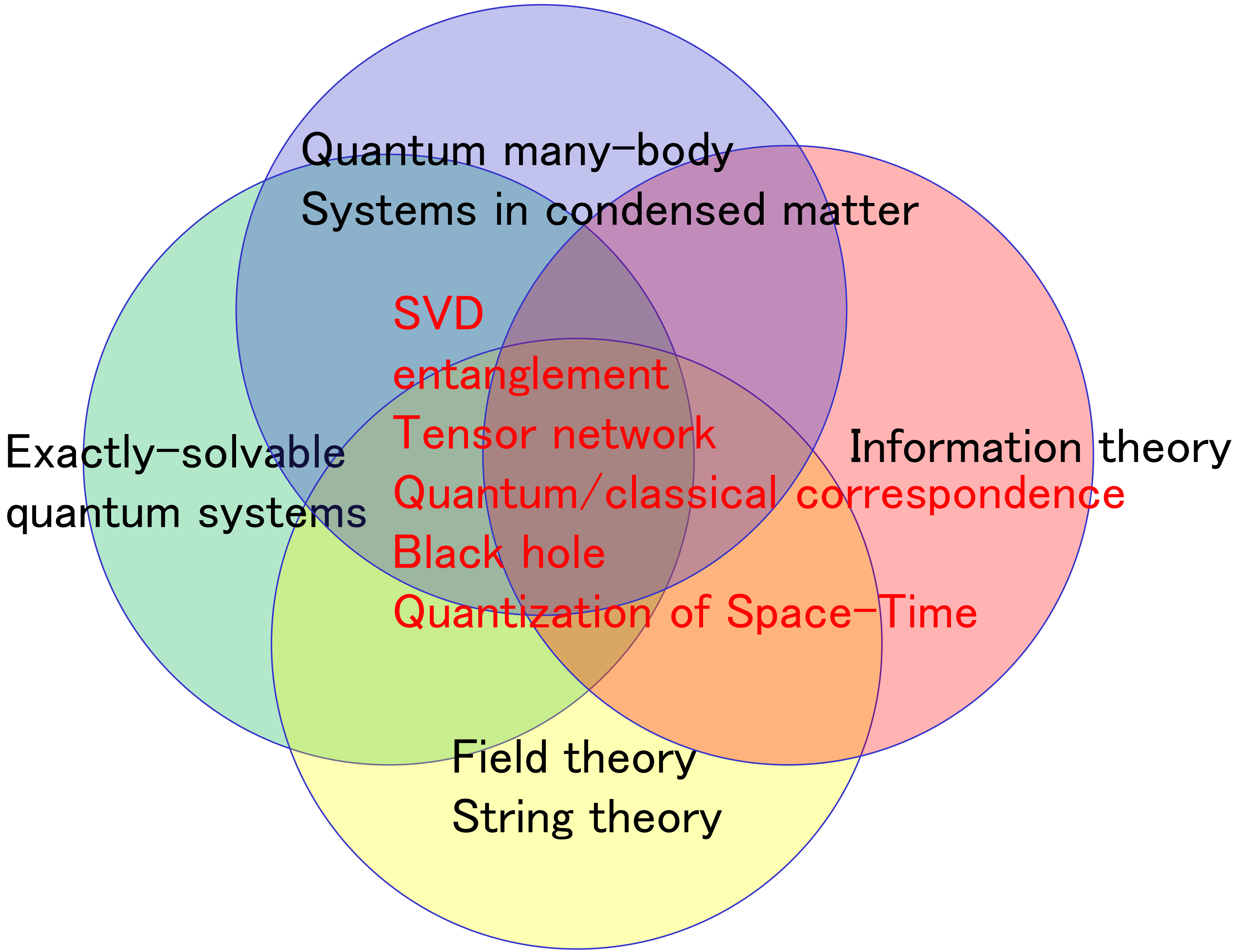
Contents of my talk

1. General introduction
   (summary of important concepts)

2. Tensor network states, extra dimensions,
   and holographic geometry

3. Information-geometrical analysis of
   the correspondence between
   BTZ black hole and finite-T CFT

Recent development of interdisciplinary physics research



Key concepts of this research field

"quantum entanglement" and "holography principle"

Entanglement entropy

- similar to the logarithm of two-point correlator
- scaling formula ⇔CFT

Tensor network states Network geometry ⇔ RG

- variational ansatz for quantum-many body system which satisfies the entropy scaling ⇔Bethe ansatz K, post-K

Holography principle (bulk/edge correspondence) ⇔wavelets

- $AdS_{d+2} \Leftrightarrow CFT_{d+1}$
- classical side behaves as a memory to efficiently storage quantum data ⇔Information geometry

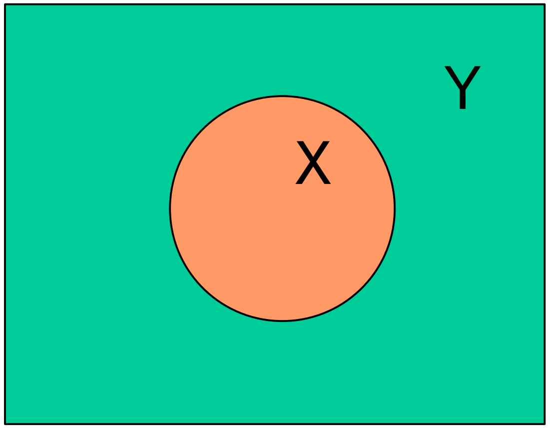
$$g_{\mu\nu}(\theta) \approx \partial_\mu \partial_\nu S(\theta)$$

Reconstruction of statistical mechanics and field theory by the information-theoretical concepts

Entanglement entropy

Total system (superblock, universe) = X+Y

$$|\psi\rangle = \sum_{x,y} \psi(x,y) |x\rangle \otimes |y\rangle \qquad x \in X \quad y \in Y$$



Reduced density matrices for X and Y

$$\rho_X = \mathrm{Tr}_Y |\psi\rangle\langle\psi|$$

$$\rho_Y = \mathrm{Tr}_X |\psi\rangle\langle\psi|$$

Entanglement entropy

$$S_X = -\mathrm{Tr}_X \left(\rho_X \log \rho_X\right)$$

$$S_Y = -\mathrm{Tr}_Y \left(\rho_Y \log \rho_Y\right)$$

Entanglement ⇔ information flow across the boundary of X and Y

Singular Value Decomposition (SVD)

SVD of matrix Ψ

$$\psi(x,y)=\sum_l U_l(x)\sqrt{\Lambda_l}V_l(y)$$

$\sqrt{\Lambda_l}$: singular value

$U_l(x), V_l(y)$: unitary matrices

$$\rho_X(x,x')=\sum_y \psi(x,y)\psi^*(x',y)=\sum_l U_l(x)\Lambda_l U_l^*(x')$$

$$\rho_Y(y,y')=\sum_x \psi(x,y)\psi^*(x,y')=\sum_l V_l(y)\Lambda_l V_l^*(y')$$

Von Neumann entropy for partial systems
＝ entanglement entropy → Area-law scaling

$$S_X=-\sum_l \lambda_l \log\lambda_l = S_Y \qquad \lambda_l=\Lambda_l/\sum_l \Lambda_l$$

Universal scaling formulae for the entanglement entropy

Scaling formula → criticality, space dimension d, linear size L

Gapped systems → Area Law

$$S = \alpha L^{d-1} + \cdots$$

Critical systems → logarithmic formula

$$S = \frac{1}{3} C L^{d-1} \log L + \cdots$$

C: central charge (d=1)

Topological Entanglement Entropy (d=2)

$$S = \alpha L - \gamma$$

Finite-entanglement scaling (d=1)

$$S_{MPS} = \frac{1}{\sqrt{12/c} + 1} \log \chi$$

Holography principle

Holography principle

String, black hole physics:
t' Hooft (1974,1993), Susskind (1995), Maldacena (1997)

AdS/CFT correspondence

(d+1)-dim. Quantum system
with conformal symmetry

⇕

(d+2)-dim. Classical
General Relativity on
Hyperbolic Space-time

Supersymmetric Yang-Mills

⇕

Type IIB string theory
on $AdS_5 \times S^5$

Universal (AdS_5)   Model-dependent (S^5)

GKP–Witten relation and Ryu–Takayanagi formula

Gubser–Klevanov–Polyakov(GKP)–Witten relation

$$\left\langle O(x_1)\cdots O(x_n)\right\rangle_{CFT} = \frac{\delta}{\delta\phi(x_1)}\cdots\frac{\delta}{\delta\phi(x_n)}\exp\left(-\frac{1}{2\kappa}I(\phi(x))\right)\Bigg|_{\phi=\phi_0}$$

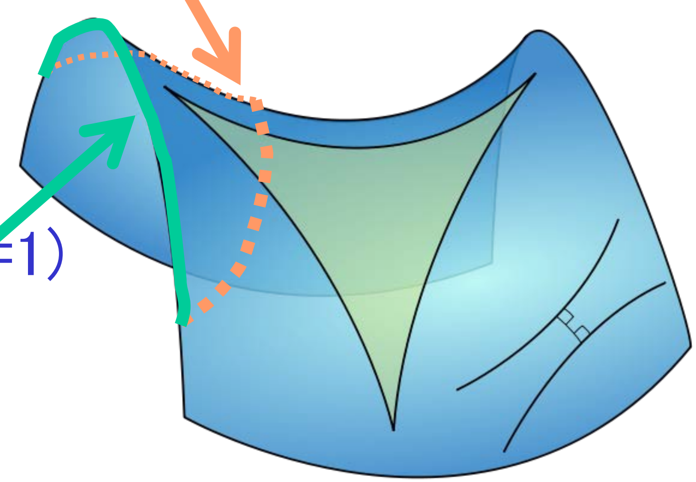
Ryu–Takayanagi formula (2006)

γ : gedesic distance (d=2)

$$S = \frac{\gamma}{4G}$$

$$\gamma = 2l\log L$$

γ : area of the minimal surface



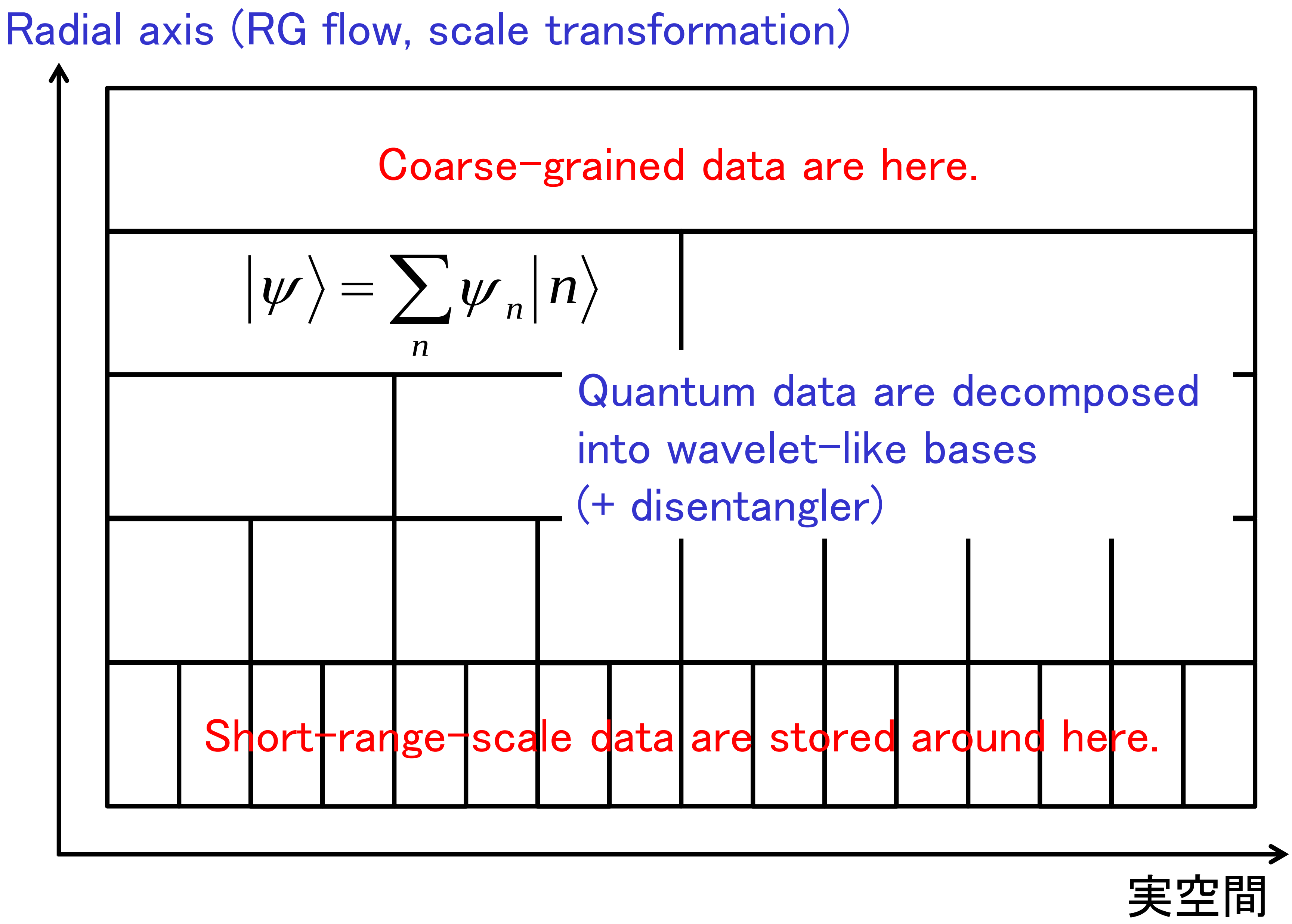
Logarithmic entropy formula (d=1)

$$S = \frac{1}{3}c\log L$$

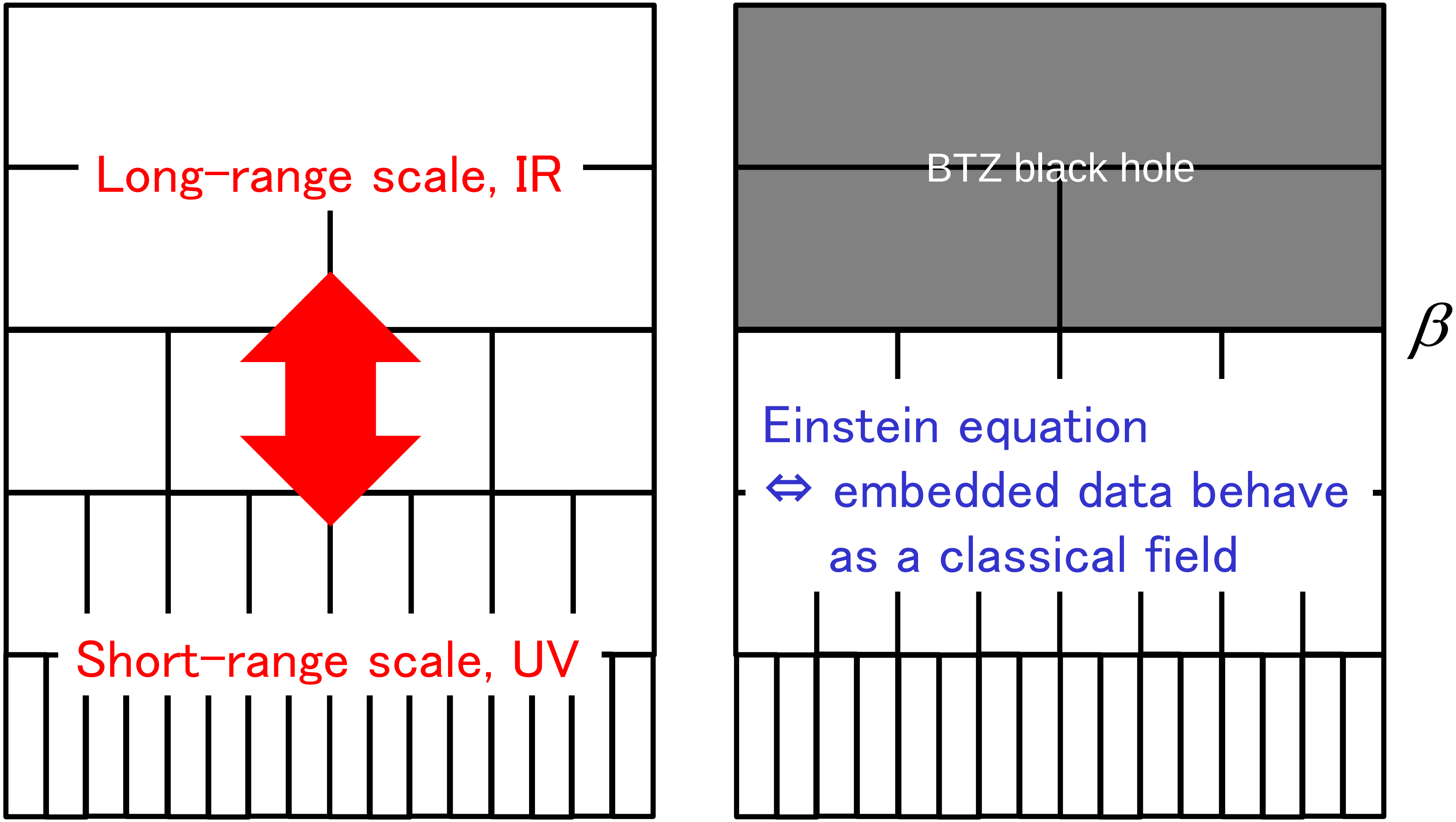
$$c = \frac{3l}{2G}$$

Brown–Henneaux central charge

Quantum-data storage to hyperbolic space



Information-theoretical interpretation of AdS/CFT



Critical ground state

Excitation, finite-T

Tensor Networks States,
Extra Dimensions,
and
Holographic Geometry

Factorization of entangled states: core algorithm for TNS

S=1/2 Heisenberg antiferromagnet (2 sites)

$$H = \vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left(S_1^+ S_2^- + S_1^- S_2^+\right) + S_1^z S_2^z$$

bases: $\left|\uparrow\uparrow\right\rangle, \left|\uparrow\downarrow\right\rangle, \left|\downarrow\uparrow\right\rangle, \left|\downarrow\downarrow\right\rangle$

$$H = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}$$

Single ground state (entangled)

$$\left|0\right\rangle = \frac{1}{\sqrt{2}}\left(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle\right)$$

$$E_0 = -\frac{3}{4}$$

We would like to exactly transform this non-local state to a kind of local representation by using extra dimension

Local approximation cannot represent singlet !

$$|\psi\rangle = \sum_{s_1=\uparrow,\downarrow} a^{s_1}|s_1\rangle \otimes \sum_{s_2=\uparrow,\downarrow} c^{s_2}|s_2\rangle$$

$$= a^{\uparrow}c^{\uparrow}|\uparrow\uparrow\rangle + a^{\uparrow}c^{\downarrow}|\uparrow\downarrow\rangle + a^{\downarrow}c^{\uparrow}|\downarrow\uparrow\rangle + a^{\downarrow}c^{\downarrow}|\downarrow\downarrow\rangle$$

$$|0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

$$a^{\uparrow}c^{\uparrow} = 0$$
$$a^{\downarrow}c^{\downarrow} = 0$$
$$a^{\uparrow}c^{\downarrow} = 1/\sqrt{2}$$
$$a^{\downarrow}c^{\uparrow} = -1/\sqrt{2}$$

No solution !

$$|\psi\rangle = |1\rangle \otimes |2\rangle$$

$$\rho_1 = Tr_2|\psi\rangle\langle\psi| = |1\rangle\langle 1| \qquad S_1 = -Tr_1\rho_1 \log\rho_1 = 0$$

Vector product state

$$|\psi\rangle = \sum_{s_1,s_2} a^{s_1} c^{s_2} |s_1 s_2\rangle \Rightarrow \sum_{s_1,s_2} A^{s_1} C^{s_2} |s_1 s_2\rangle$$

$$A^{s_1} = \left(a_1^{s_1}, a_2^{s_1}\right)$$

$$C^{s_2} = \begin{pmatrix} c_1^{s_2} \\ c_1^{s_2} \end{pmatrix}$$

Local representation, but exact

→ Introduction of extra dimension associated with entanglement

$$|\psi\rangle = \sum_{\alpha=1}^{\chi=2} \left\{ \sum_{s_1=\uparrow,\downarrow} a_\alpha^{s_1} |s_1\rangle \otimes \sum_{s_2=\uparrow,\downarrow} c_\alpha^{s_2} |s_2\rangle \right\}$$

$$= \left(a_1^\uparrow c_1^\uparrow + a_2^\uparrow c_2^\uparrow\right)|\uparrow\uparrow\rangle + \left(a_1^\uparrow c_1^\downarrow + a_2^\uparrow c_2^\downarrow\right)|\uparrow\downarrow\rangle$$

$$+ \left(a_1^\downarrow c_1^\uparrow + a_2^\downarrow c_2^\uparrow\right)|\downarrow\uparrow\rangle + \left(a_1^\downarrow c_1^\downarrow + a_2^\downarrow c_2^\downarrow\right)|\downarrow\downarrow\rangle$$

$$a_1^\uparrow = c_2^\uparrow = a_2^\downarrow = c_1^\downarrow = 0$$

$$a_2^\uparrow c_2^\downarrow = 1/\sqrt{2}$$

$$a_1^\downarrow c_1^\uparrow = -1/\sqrt{2}$$

$$|\psi\rangle = |0\rangle$$

χ =2 → exact

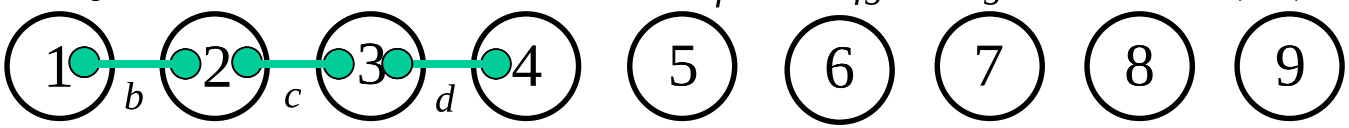
$$A^\uparrow = (x, y), A^\downarrow = (z, w), C^\uparrow = \begin{pmatrix} \frac{y}{xw-yz} \\ \frac{x}{yz-xw} \end{pmatrix}, C^\downarrow = \begin{pmatrix} \frac{w}{xw-yz} \\ \frac{z}{yz-xw} \end{pmatrix}$$

Matrix Product State (MPS)

Open boundary condition

$$\left|\psi\right\rangle = \sum_{\{s_1, s_2, \cdots, s_n\}} \left\langle s_1 \right| A_2^{s_2} A_3^{s_3} \cdots A_{n-1}^{s_{n-1}} \left| s_n \right\rangle \left| s_1 s_2 \cdots s_n \right\rangle$$

$\left\langle s_1 \right| = A_b^{s_1}$ $A_{bc}^{s_2}$ $A_{cd}^{s_3}$ $A_{de}^{s_4}$ $A_{ef}^{s_5}$ $A_{fg}^{s_6}$ $A_{gh}^{s_7}$ $A_{hi}^{s_8}$ $\left| s_9 \right\rangle = A_i^{s_9}$



$A_j^{s_j}$ $\chi \times \chi$, χ : unphysical degree of freedom

$s_j = \uparrow, \downarrow$: physical degree of freedom

Matrix = map from unphysical to physical degrees of freedom

↓

What is unphysical degree ? → entanglement !

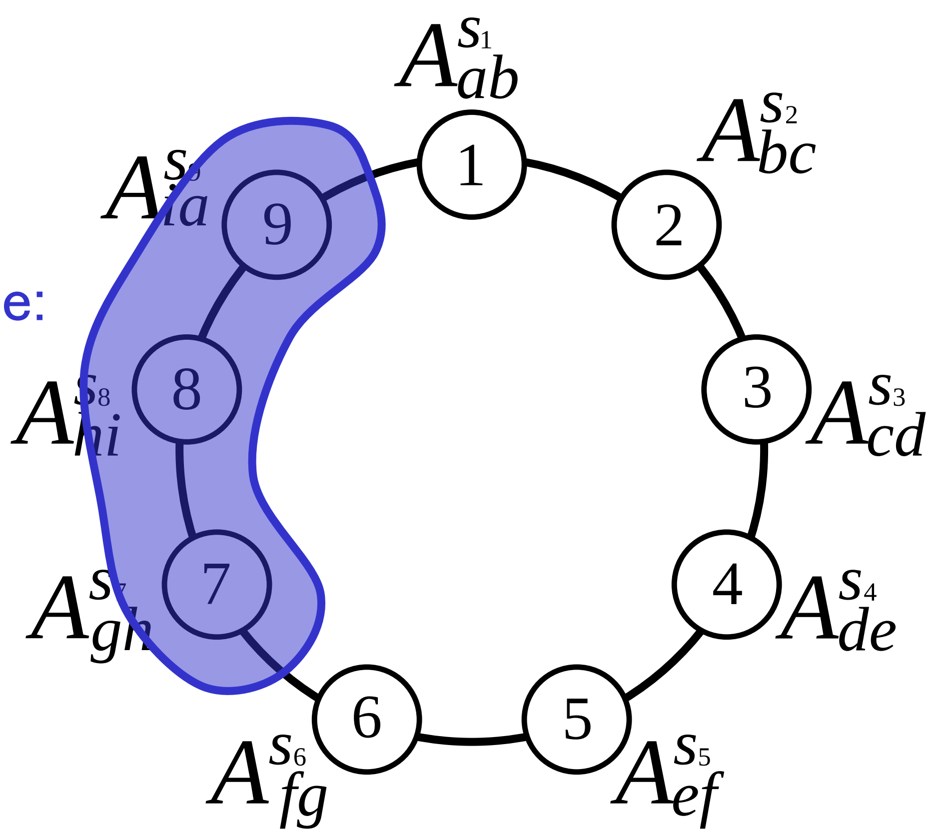
Ψ : product of local matrix ⇔ non-local correlation

Periodic boundary condition

$$|\psi\rangle = \sum_{\{s_1, s_2, \cdots, s_n\}} tr\left(A_1^{s_1} A_2^{s_2} \cdots A_n^{s_n}\right) |s_1 s_2 \cdots s_n\rangle$$

Entanglement entropy

$$S = 2\log\chi$$



Evaluation of proper χ value:

Gapped → L^{1-1} = const.

Critical → $O(L^{c/6})$

$$2\log\chi = \frac{c}{3}\log L$$

$$\Rightarrow \chi = L^{c/6}$$

Equivalence of MPS to the Bethe ansatz

- basis change of Algebraic Bethe ansatz ⇒ MPS
- matrix-product Bethe ansatz

M excitations from the highest weight state

$$\psi_{\Omega}(x_1,\dots,x_M) = Tr\left(E^{x_1-1} A E^{x_2-x_1-1} A \cdots E^{x_M-x_{M-1}-1} A E^{L-x_M} \Omega\right)$$

$$A = \sum_{j=1}^{M} A_{k_j} E \qquad E A_{k_j} = e^{ik_j} A_{k_j} E$$

$$A_{k_j} A_{k_j} = 0 \qquad A_{k_j} A_{k_l} = s(k_j, k_l) A_{k_l} A_{k_j}$$

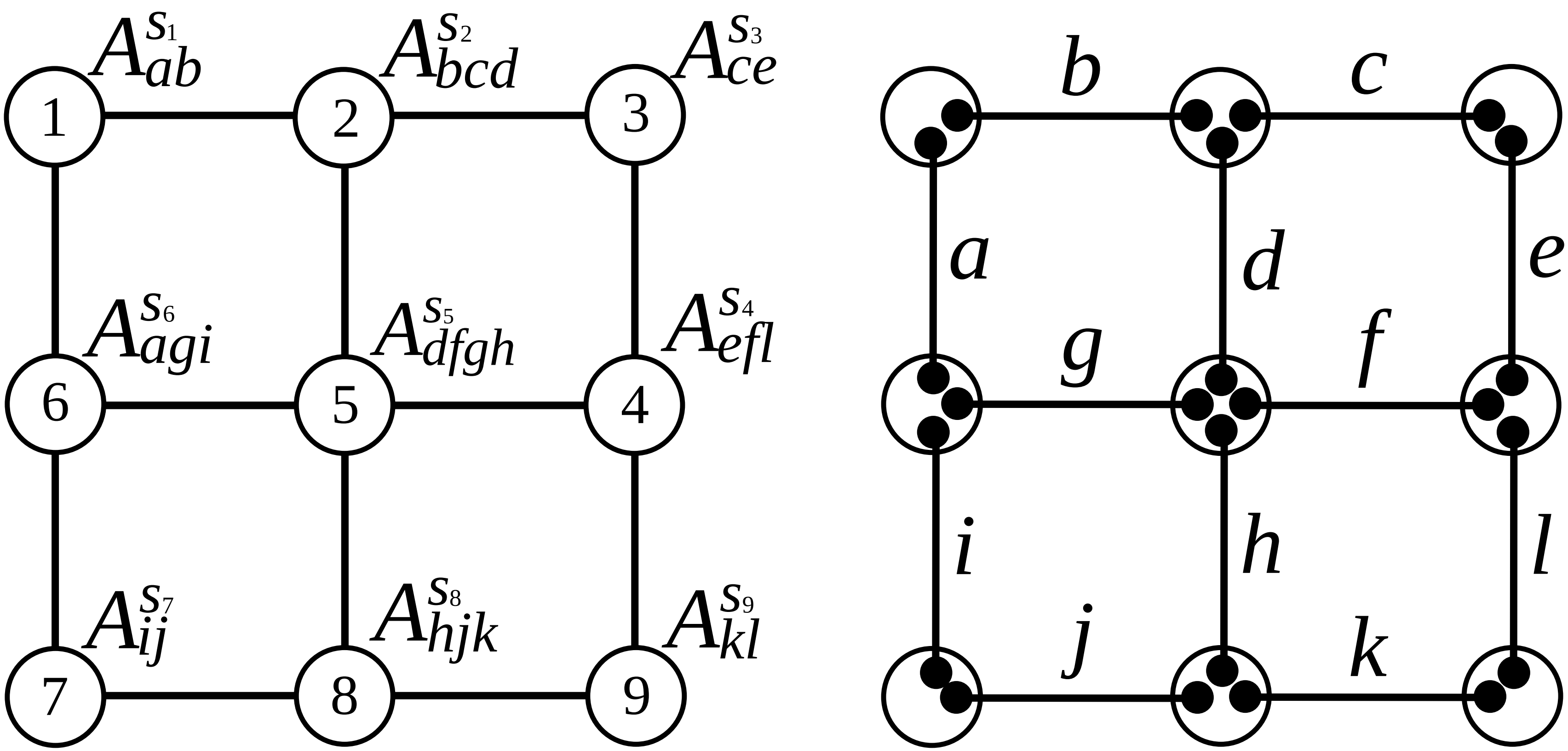
$$E\Omega = e^{-ip} \Omega E$$

- simple interpretation

$$\psi = e^{ik_1x_1+ik_2x_2} + \Theta e^{ik_1x_2+ik_2x_1} = \begin{pmatrix} e^{ik_1x_1} & e^{ik_2x_1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \Theta & 0 \end{pmatrix} \begin{pmatrix} e^{ik_1x_2} \\ e^{ik_2x_2} \end{pmatrix}$$

Tensor Product State (TPS), Tensor Network State (TNS)

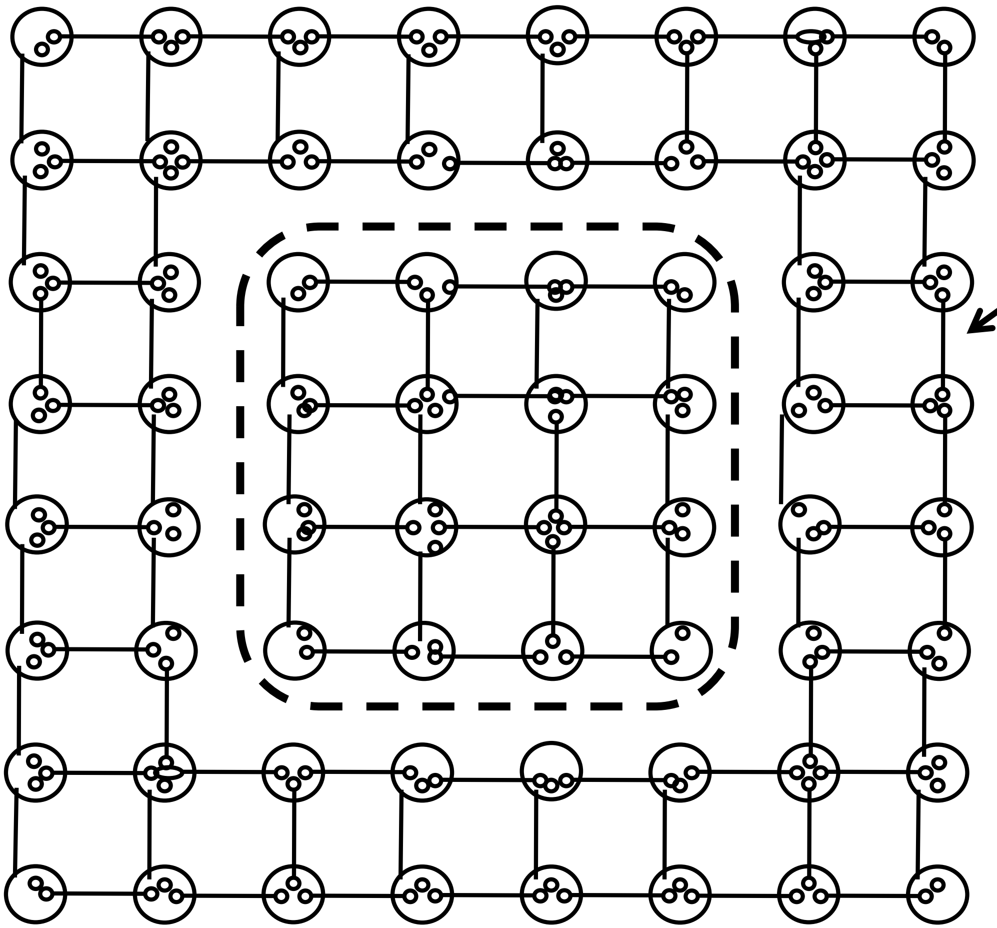
Projected Entangled Pair State (PEPS)



(a) (b)

$$|\psi\rangle = \sum_{\{s_j\}} \sum_{a,b,\ldots,l} A_{ab}^{s_1} A_{bcd}^{s_2} A_{ce}^{s_3} A_{efl}^{s_4} A_{dfgh}^{s_5} A_{agi}^{s_6} A_{ij}^{s_7} A_{hjk}^{s_8} A_{kl}^{s_9} |s_1 s_2 \cdots s_9\rangle$$

Entanglement structure of tensor network states



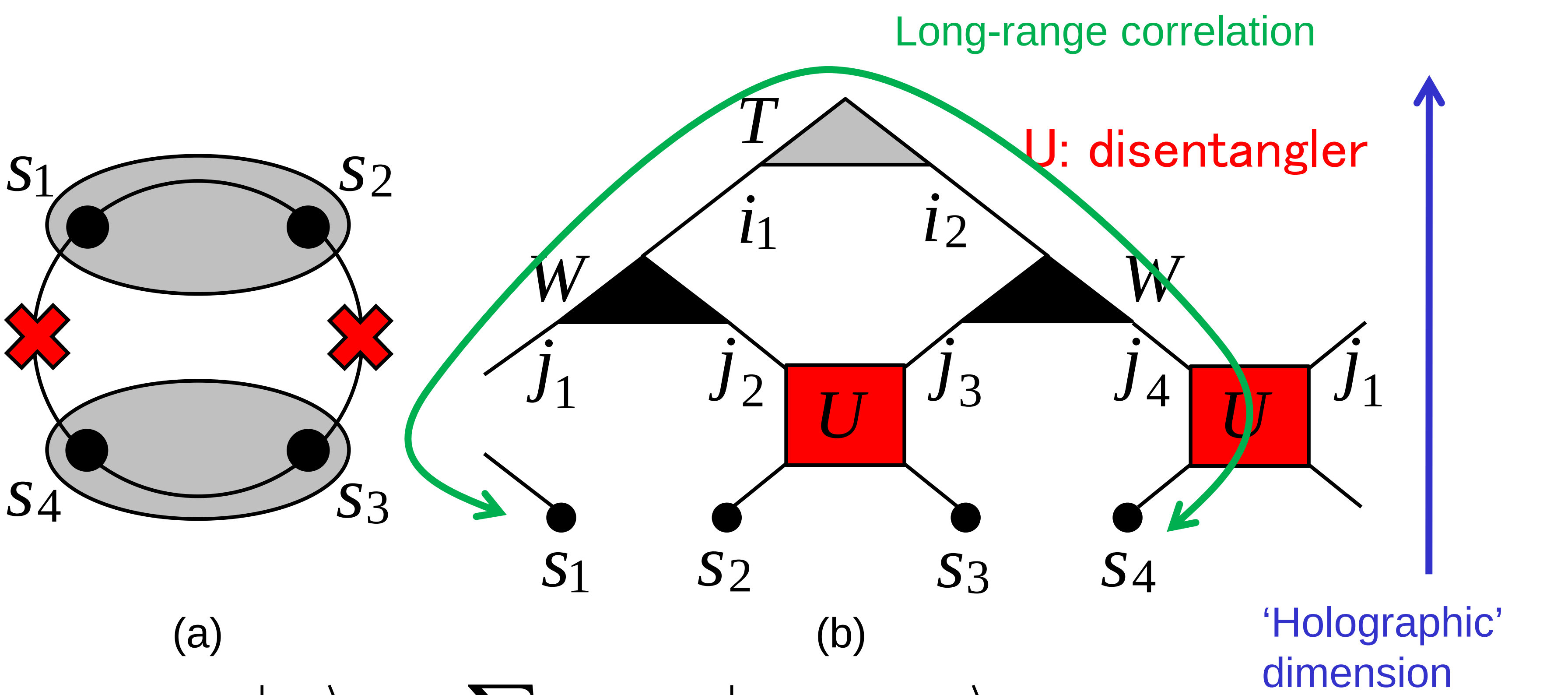
$$|\psi\rangle = \frac{1}{\sqrt{\chi}}\sum_{\alpha=1}^{\chi}|\alpha\alpha\rangle$$

$S = N_{bond} \log \chi$ ⇒automatically satisfies the area law

However, large χ of order L is necessary for critical systems

Hierarchical tensor networks, entanglement renormalization

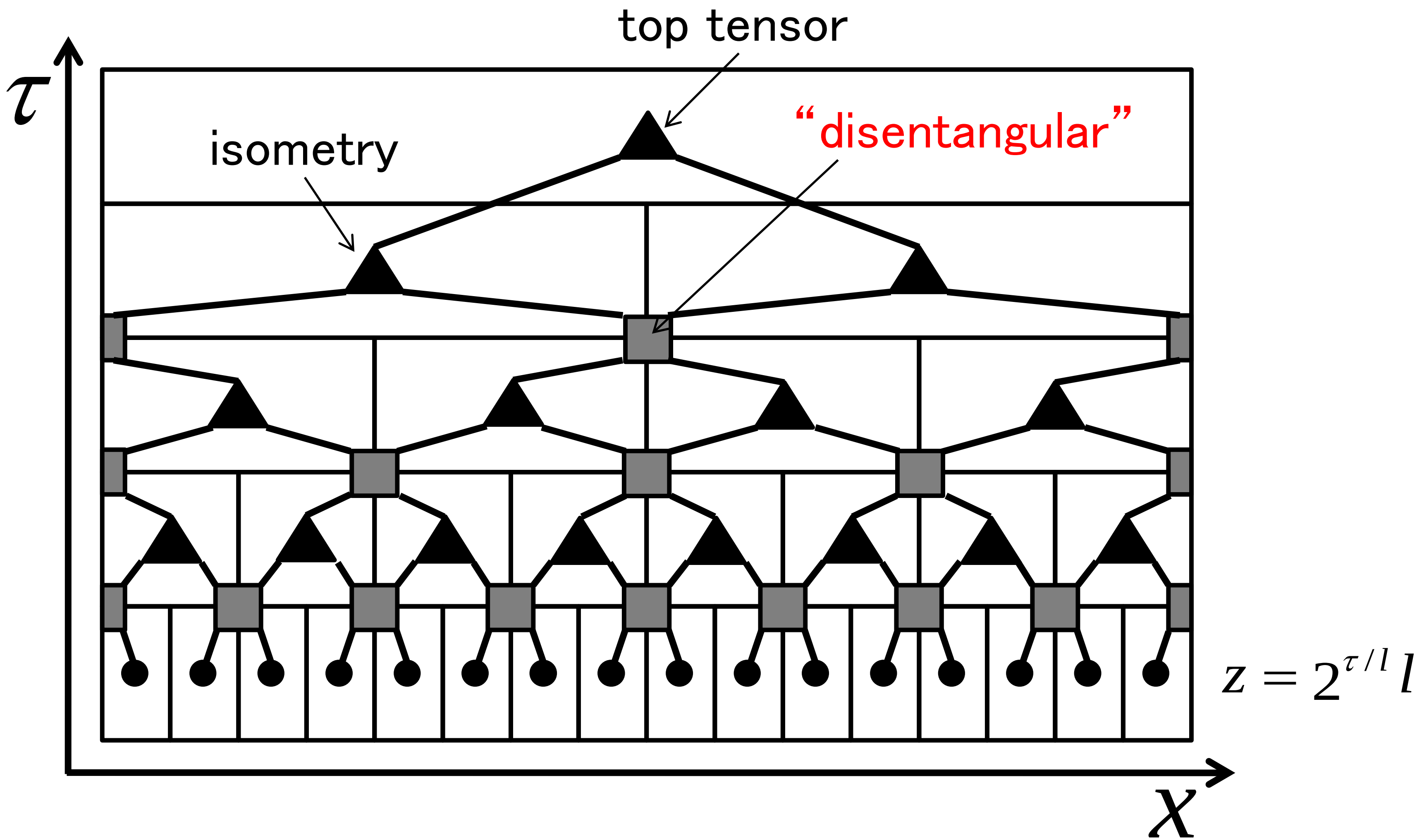
Multiscale Entanglement Renormalization Ansatz (MERA)



$$|\psi\rangle = \sum_{s_1,s_2,s_3,s_4,} T_{s_1 s_2 s_3 s_4} |s_1 s_2 s_3 s_4\rangle$$

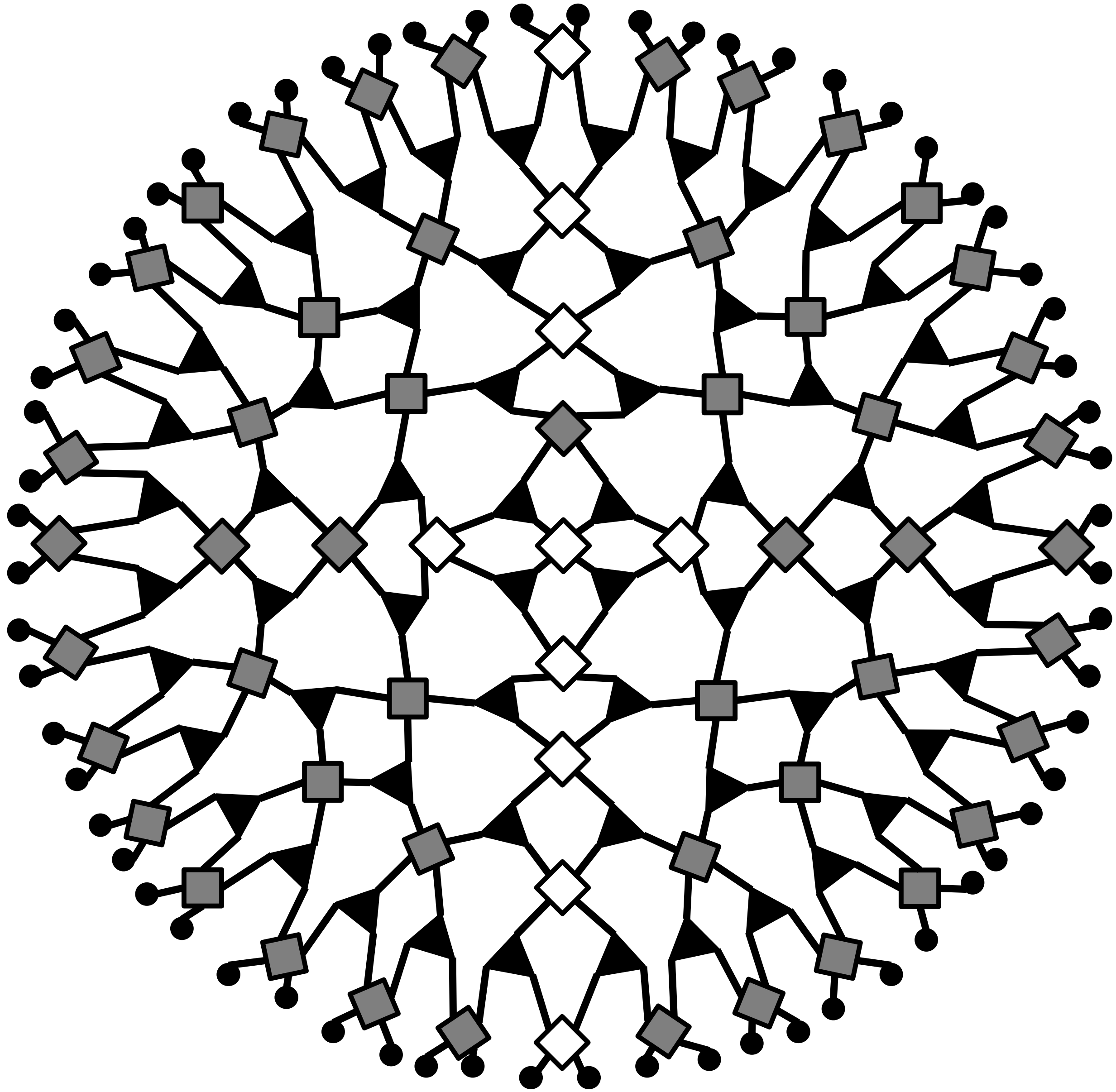
$$|\psi\rangle = \sum_{i_1,i_2} \sum_{j_1,\ldots,j_4} \sum_{s_1,\ldots,s_4} T_{i_1 i_2} W^{i_1}_{j_1 j_2} W^{i_2}_{j_3 j_4} U^{j_2 j_3}_{s_2 s_3} U^{j_4 j_1}_{s_4 s_1} |s_1 s_2 s_3 s_4\rangle$$

MERA network → discretized hyperbolic space

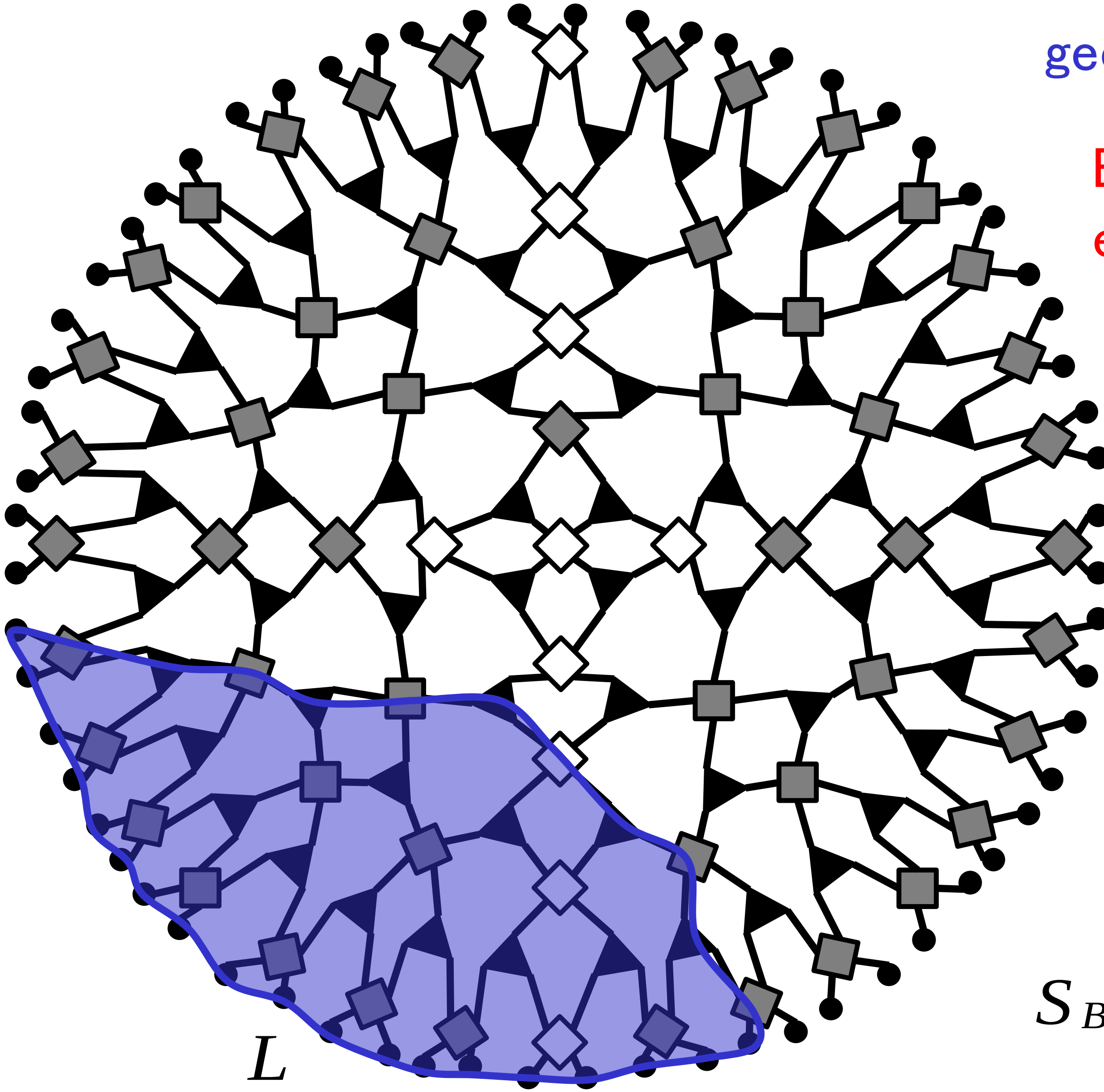


$$ds^2 = \left(d\tau \log 2\right)^2 + \left(2^{-\tau/l}\, dx\right)^2 = \frac{l^2}{z^2}\left(dz^2 + dx^2\right)$$

Poincare disk representation of MERA network



Geometric structure of MERA network (causal cone)



geodesic～ $2l \ln L$

Bekenstein–Hawking entropy

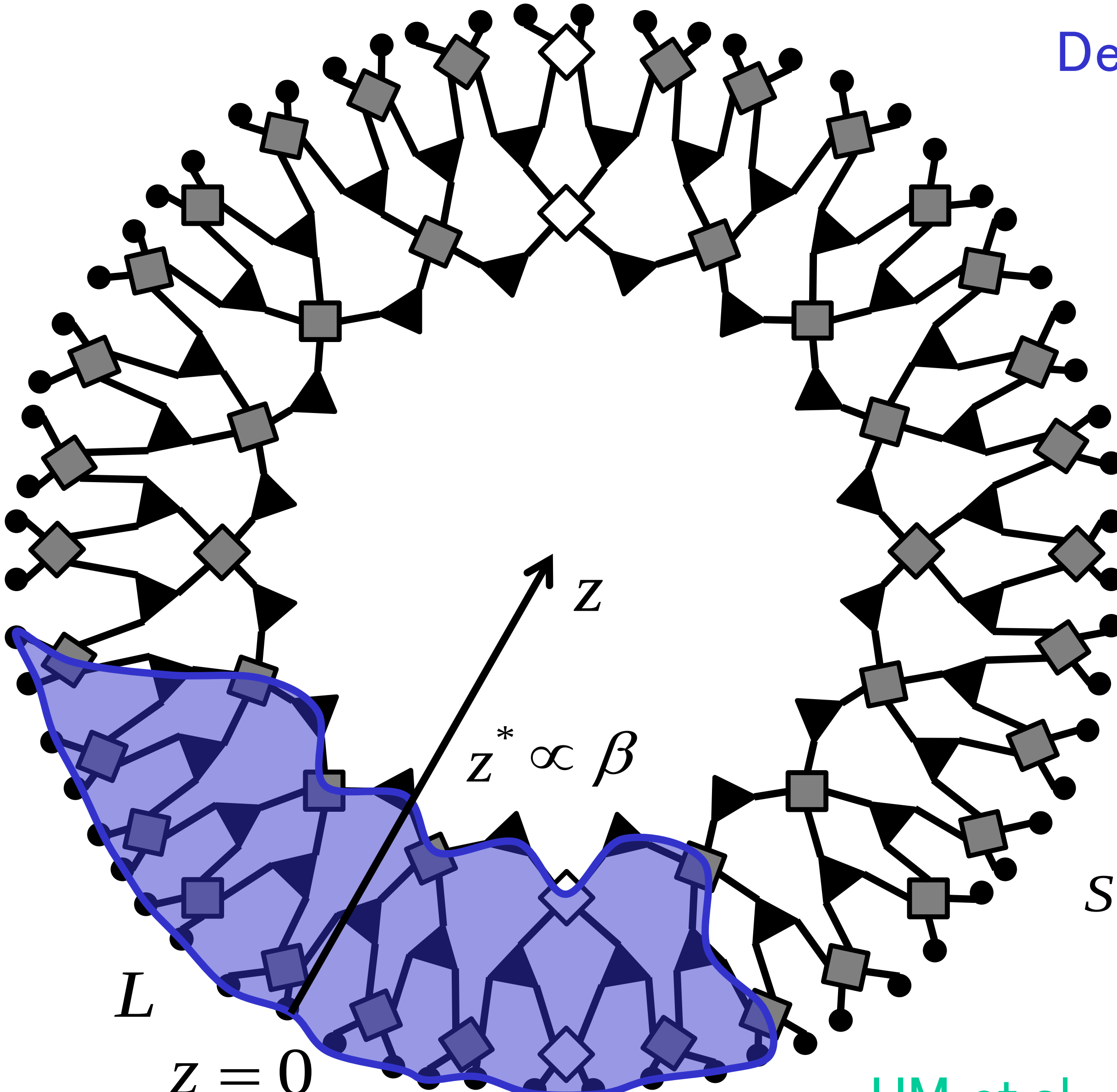
$$S_{BH} = \frac{A}{4G}$$

Brown–Henneaux charge

$$c = \frac{3l}{2G}$$

$$S_{BH} = \frac{c}{3} \log L = S_{EE}$$

Correspondence between BTZ and finite-T MERA



Deformation of geodesic

Event horizon of BTZ black hole

$$z^* \propto \beta$$

$$S = \frac{c}{3}\log\left(\frac{\beta}{\pi}\sinh\left(\frac{\pi L}{\beta}\right)\right)$$

HM et.al., PRD 87, 066002 (2015)

Recent topics motivated from MERA networks

Exact Holographic Mapping (EHM)
Xiao-Liang Qi, arXiv:1309.6282
Haal wavelet

Analytic MERA
Glen Evenbly and Stenven R White, PRL116, 140403 (2016)
Disentangler ⇔ Daubechies 4-tap wavelet

MERA and Quantum Integrability
H. Matsueda, arXiv
Variational optimization ⇔ Bethe equation

Relation with Loop Quantum Gravity and Spin Network
Muxin Han and Ling-Yan Hung, arXiv:1610.02134

Information-Geometrical Analysis of the Correspondence between BTZ Black Hole and Finite-T CFT

HM and T.Suzuki, JPSJ86, 104001 (2017)

BTZ black hole and holography

BTZ black hole: vacuum solution of the Einstein equation in d=2 with negative Λ

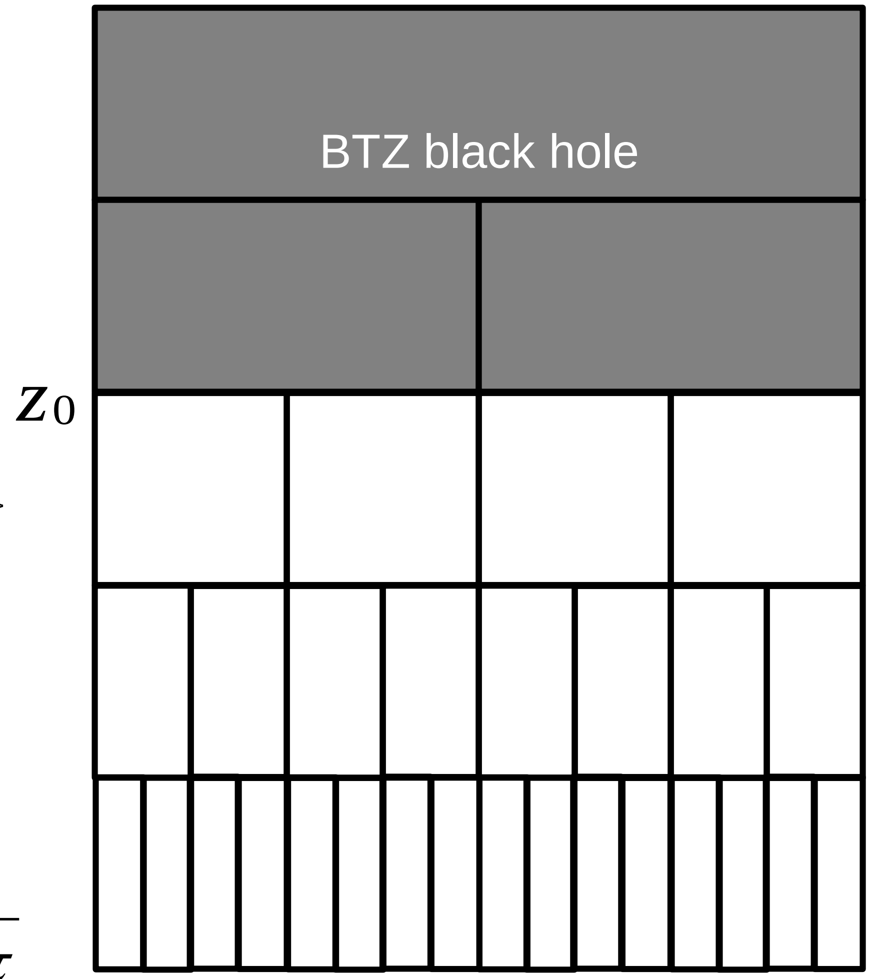
Dual QFT → finite-T CFT

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = 0$$

$$ds^2 = \frac{R^2}{z^2} \left\{ - f(z) dt^2 + \frac{dz^2}{f(z)} + dx^2 \right\}$$

$$f = 1 - a z^2 = 1 - \left(\frac{z}{z_0} \right)^2$$

$$S = \frac{c}{3} \log \left(\frac{\beta}{\pi} \sinh \left(\frac{\pi L}{\beta} \right) \right) \qquad z_0 = \frac{\beta}{2\pi}$$



Relative entanglement entropy

Schmidt decomposition of quantum pure states

(Finite-T case is also represented by the same form using TFD)

$$|\psi(\theta)\rangle = \sum_n \sqrt{\lambda_n(\theta)}\,|n\rangle_A \otimes |n\rangle_{\bar{A}} \qquad \langle\psi(\theta)|\psi(\theta)\rangle = \sum_n \lambda_n(\theta) = 1$$

Θ: canonical parameters (function of model parameters)

Relative entanglement entropy

(Entropy is a measure of difference between two quantum states)

$$D(\theta) = -\sum_n \lambda_n(\theta)\log\lambda_n(\theta) + \sum_n \lambda_n(\theta)\log\lambda_n(\theta + d\theta)$$

$$= \frac{1}{2} g_{\mu\nu}(\theta) d\theta^\mu d\theta^\nu + \cdots$$

Entanglement spectrum

$$\gamma_n(\theta) = -\log\lambda_n(\theta)$$

Fisher metric

$$g_{\mu\nu}(\theta) = \sum_n \lambda_n(\theta) \frac{\partial\log\lambda_n(\theta)}{\partial\theta^\mu}\frac{\partial\log\lambda_n(\theta)}{\partial\theta^\nu} = \langle\partial_\mu\gamma\,\partial_\nu\gamma\rangle = \langle\partial_\mu\partial_\nu\gamma\rangle$$

Purpose of this study

$$|\psi(\theta)\rangle = \sum_n \sqrt{\lambda_n(\theta)}\,|n\rangle_A \otimes |n\rangle_{\bar{A}} \qquad \langle\psi(\theta)|\psi(\theta)\rangle = \sum_n \lambda_n(\theta) = 1$$

$$\lambda_n(\theta)$$

$$\gamma_n(\theta) = -\log \lambda_n(\theta)$$

Entangled entropy

$$S(\theta) = -\sum_n \lambda_n(\theta) \log \lambda_n(\theta) = \langle \gamma \rangle$$

Fisher metric

$$g_{\mu\nu}(\theta) = \langle \partial_\mu \gamma \, \partial_\nu \gamma \rangle = \langle \partial_\mu \partial_\nu \gamma \rangle$$

Once the Schmidt coefficients of a quantum system are determined, both of the entanglement entropy and the classical geometrical quantity are calculated simultaneously !

Hessian Geometry and Entanglement Thermodynamics

Exponential family form
(environment→finite-T effect, def. of canonical parameters)

$$|\psi(\theta)\rangle = \sum_n \sqrt{\lambda_n(\theta)}\,|n\rangle_A \otimes |n\rangle_{\bar{A}}$$

$$\lambda_n(\theta) = e^{-\gamma_n(\theta)} = \exp\{\theta^\mu F_{n,\mu} - \psi(\theta)\} = \frac{1}{Z} e^{\theta^\mu F_{n,\mu}} \qquad \psi(\theta) = \log Z$$

Hessian Geometry

$$\gamma_n(\theta) = \psi(\theta) - \theta^\mu F_{n,\mu} \qquad g_{\mu\nu}(\theta) = \langle \partial_\mu \partial_\nu \gamma \rangle = \partial_\mu \partial_\nu \psi(\theta)$$

Thermodynamics law for the entanglement entropy

$$S(\theta) = \langle \gamma(\theta) \rangle = \psi(\theta) - \theta^\mu \langle F_\mu \rangle = \psi(\theta) - \theta^\mu \partial_\mu \psi(\theta)$$

⇕

$$TS = -F + E$$

Geometry of Gaussian Distribution

Wave function of harmonic oscillator

$$p(X)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(X-\mu)^2}{2\sigma^2}\right\}$$

$$=\exp\left\{-\frac{X^2}{2\sigma^2}+\frac{\mu X}{\sigma^2}-\frac{\mu^2}{2\sigma^2}-\log\left(\sqrt{2\pi}\sigma\right)\right\}$$

$$=\exp\left\{\theta^1 F_1(X)+\theta^2 F_2(X)-\psi(\theta)\right\}$$

$$\psi=\log\left(\sqrt{2\pi}\sigma\right)+\frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2$$

$$F_1=X,\ F_2=-\frac{1}{2}X^2$$

$$\theta^1=\frac{\mu}{\sigma^2},\ \theta^2=\frac{1}{\sigma^2}$$

$$\mu=\frac{\theta^1}{\theta^2},\ \sigma=\frac{1}{\sqrt{\theta^2}}$$

$$\psi=-\frac{1}{2}\log\theta^2+\frac{1}{2}\log 2\pi+\frac{1}{2}\frac{\left(\theta^1\right)^2}{\theta^2}$$

$$ds^2=g_{\mu\nu}d\theta^\mu d\theta^\nu=\left(\partial_\mu\partial_\nu\psi\right)d\theta^\mu d\theta^\nu=\frac{d\mu^2+2d\sigma^2}{\sigma^2}$$

Information-geometrical representation of BTZ black hole

$$\mu = \frac{\theta^1}{\theta^2},\ \sigma = \frac{1}{\sqrt{\theta^2}} \quad \Rightarrow \quad t = \frac{\theta^0}{\theta^2 - a},\ x = \frac{\theta^1}{\theta^2},\ z = \frac{1}{\sqrt{\theta^2}}$$

Time near the event horizon: $t = \frac{\theta^0}{\theta^2 - a} = \frac{z^2}{1 - a z^2}\theta^0 \Rightarrow \infty$

Hesse potential that exactly derives the BTZ metric

$$\psi = \frac{1}{4a}\left\{\left(\theta^2 - a\right)\log\left(\theta^2 - a\right) - \theta^2 \log\theta^2\right\} + \frac{1}{2}\frac{\left(\theta^1\right)^2}{\theta^2} - \frac{1}{2}\frac{\left(\theta^0\right)^2}{\theta^2 - a}$$

Entanglement entropy

$$\begin{aligned} S &= \psi - \theta^\alpha \partial_\alpha \psi \\ &= -\frac{1}{4a}\log\left(\theta^2 - a\right) - \frac{1}{2}a\left(\frac{\theta^0}{\theta^2 - a}\right)^2 \\ &= \frac{1}{4}\log\left(\frac{z^2}{1 - a z^2}\right) - \frac{1}{2}a t^2 \end{aligned}$$

Derivation of Ryu–Takayanagi Formula

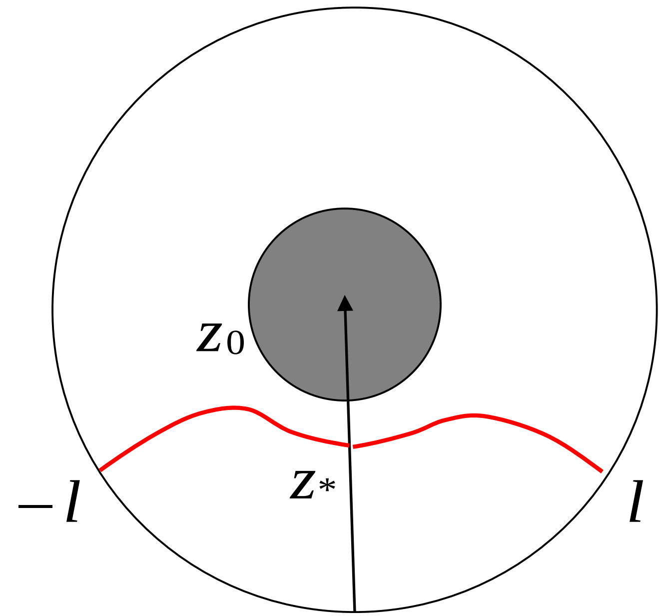
Entropy formula derived from information geometry

$$a = \left(\frac{1}{z_0}\right)^2$$

$$S = \frac{1}{4}\log\left(\frac{z^2}{1-a\,z^2}\right) - \frac{1}{2}a\,t^2$$

$$z = z_* = z_0 \tanh\left(\frac{l}{z_0}\right)$$

$$S \Rightarrow \frac{1}{2}\log\left(z_0 \sinh\left(\frac{l}{z_0}\right)\right)$$



$$\theta^2 = \frac{1}{z^2} = \frac{1}{\left\{z_0 \tanh\left(\frac{l}{z_0}\right)\right\}^2} \Rightarrow \frac{1}{l^2}$$

Duality by the Legendre Transformation and Bulk/Edge

Dual parameters: $\eta_\alpha = -\partial_\alpha \psi$

$$\theta^0 = a\frac{e^V}{1-e^V}\eta_0,\ \theta^1 = -a\frac{1}{1-e^V}\eta_1,\ \theta^2 = a\frac{1}{1-e^V}$$

$$V = -4a\left(\eta_2 + \frac{1}{2}(\eta_0)^2 - \frac{1}{2}(\eta_1)^2\right)$$

Dual potential = entanglement free energy for bosonic system

$$\varphi = -\theta^\alpha \eta_\alpha - \psi = -\frac{1}{4}\log\left(1-e^V\right) + \frac{1}{4}\log a + \frac{1}{4}V + \frac{1}{2}a(\eta_0)^2$$

Summary

Reconstruction of statistical mechanics and field theory by using information-theoretical concepts

Informaion theory ⇔ CFT, integrability, geometry, ･･･

(1) Entanglement entropy scaling

(2) extra dimension and tensor networks

(3) Network structure + RG concept → discretized geometry

(4) Information-geometrical interpretation of AdS/CFT

Anti de Sitter space and CFT

AdS metric

z: radial axis, z→0: boundary

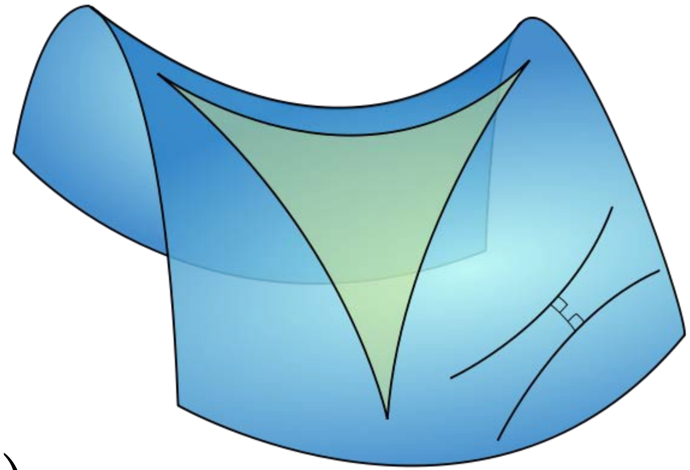
$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{l^2}{z^2}\left(dz^2 + \eta_{ij} dx^i dx^j\right)$$

$$ds \sim \frac{l}{z} dz$$

$$s \sim l \log z$$

Infinitesimal trans.

$$\bar{x}^i = x^i + \xi^i(x)$$
$$\bar{z} = z + z\zeta(x)$$



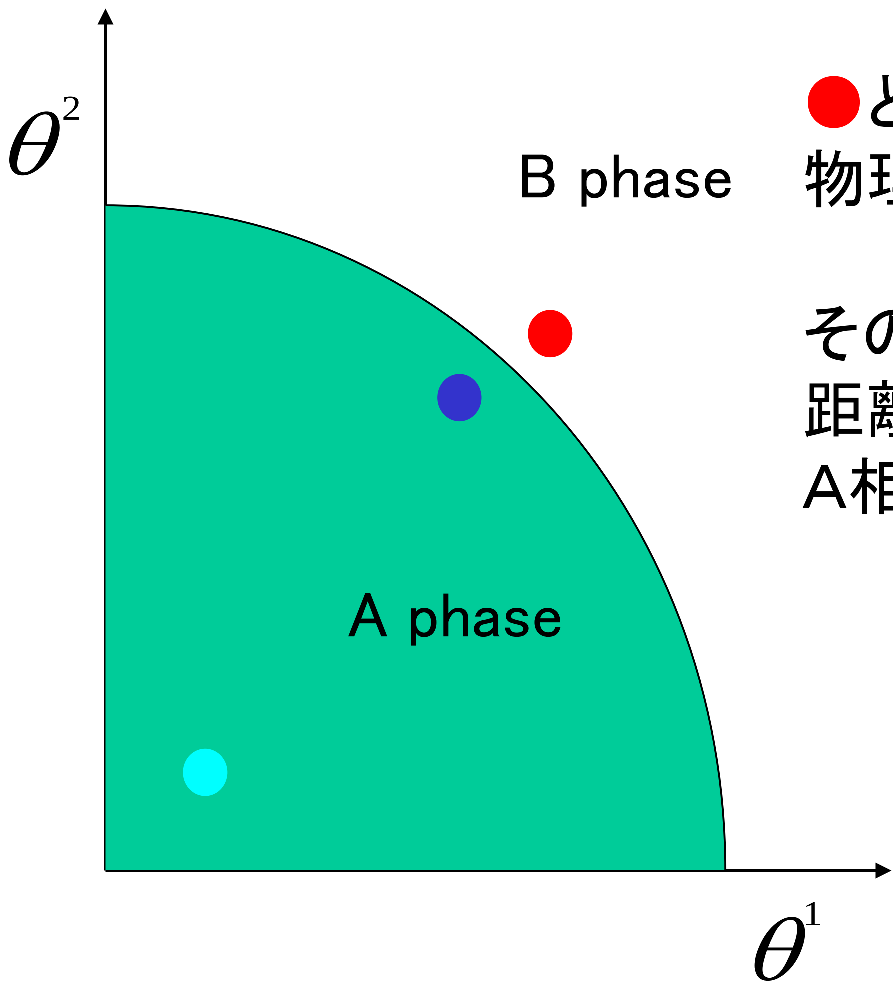
$$z \to 0$$

$$d\bar{s}^2 = \bar{g}_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu$$
$$= ds^2 + \left(\partial_i \xi_j + \partial_j \xi_i - 2\zeta \eta_{ij}\right) dx^i dx^j$$

Isometry trans. ⇒ conformal Killing equation at z→0

Boundary of AdS_{d+1} ⇒ CFT_d

相対エントロピーを計量とみなすことのイメージ



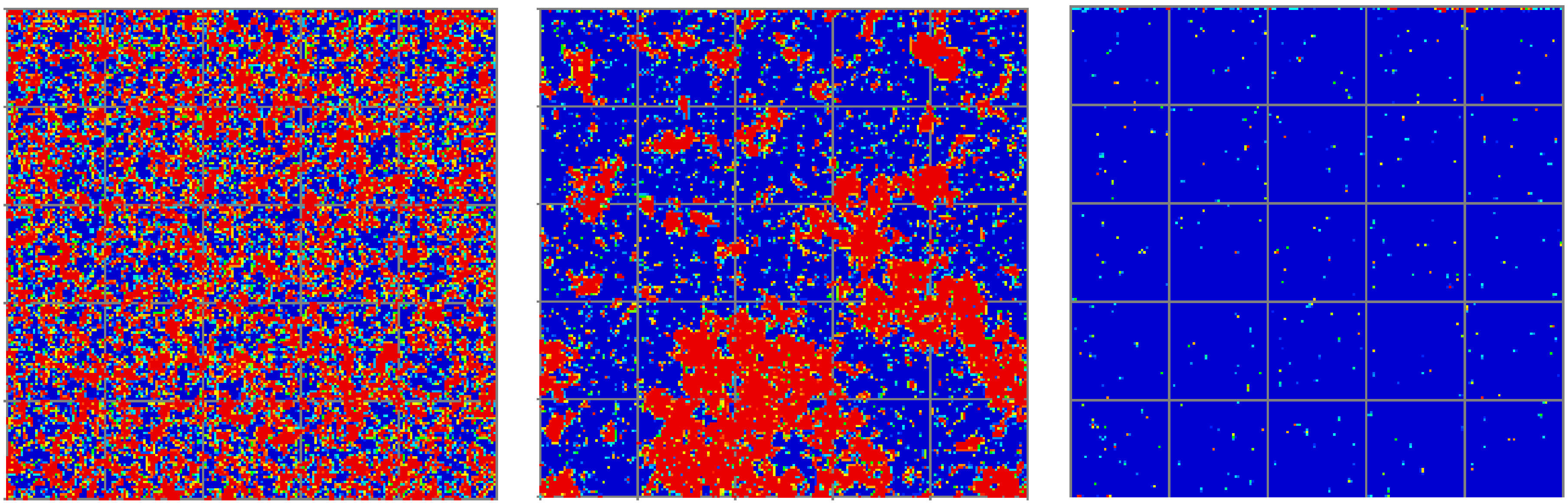
●と●は相図上では近いが，
物理的な性質は大きく異なる.

その一方で，●と●は相図上の
距離は遠くても同じ物理的な
A相に属している，

Monte Carlo Simulation of the 2D Ising Model

Classical Ising Spin Model: $H = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z \qquad \sigma_i^z = \pm 1$

Snapshots at various temperature



(a) $T = 3.02J$ (b) $T \approx T_c = 2.27J$ (c) $T = 1.52J$

$$L = 256 \qquad T_c = \frac{2J}{\log(1+\sqrt{2})} = 2.2692J$$

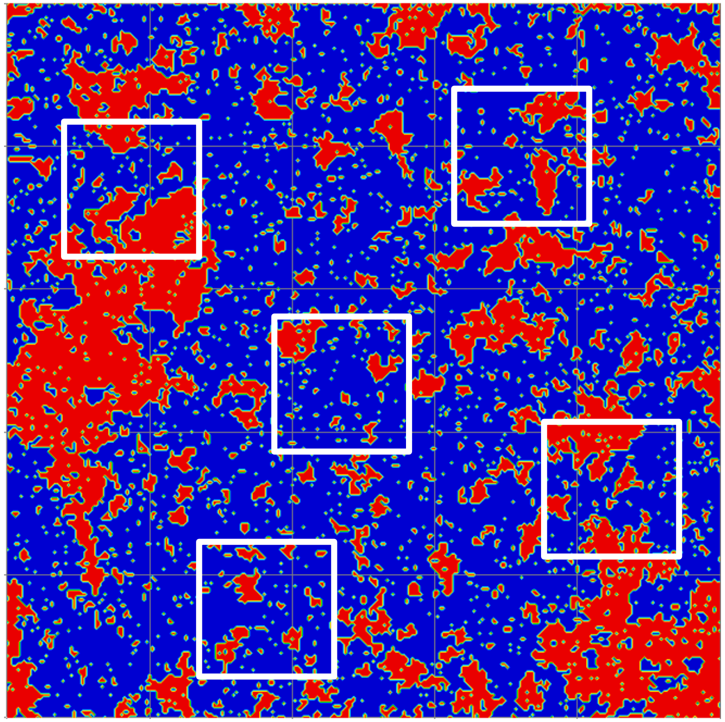
臨界性・フラクタル性と情報量

2次元イジング模型

$$H = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z$$

臨界点でのスナップショット
→ フラクタル的なスピン構造

白枠で囲まれた部分系の集合
→全ての熱揺らぎを近似的に表す

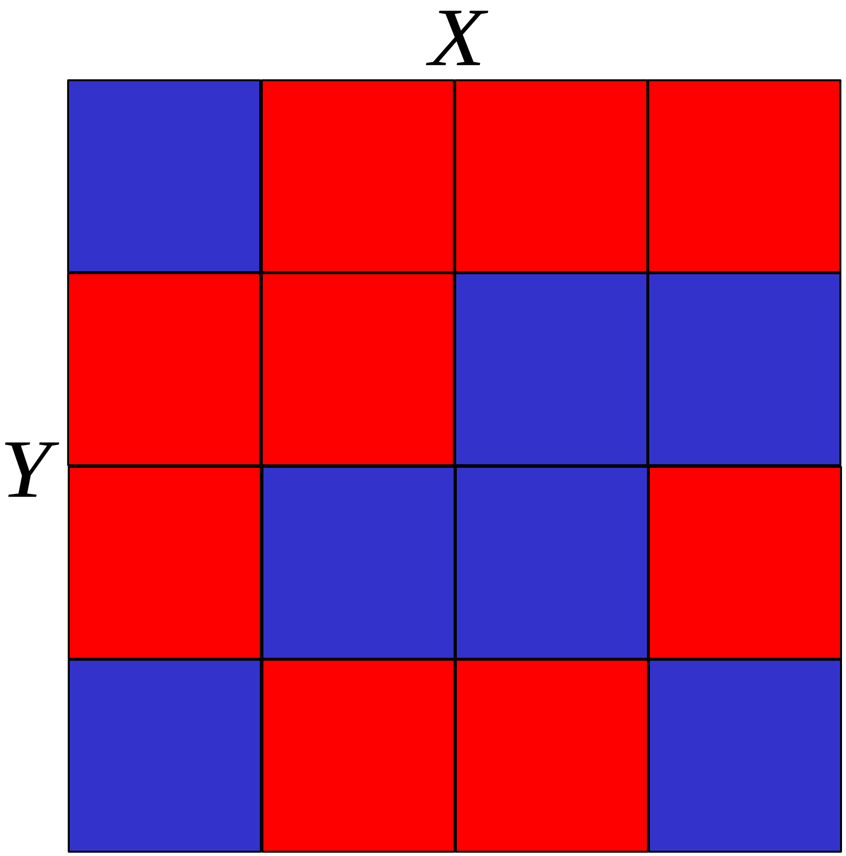


A typical snapshot of the Ising model 256x256, T=2.26J

臨界点での1枚のスナップショット⇔分配関数とほぼ同じ情報量

Density matrix of a snapshot

A snapshot determined by Monte Carlo simulation



$$\psi = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$$

$$\rho_Y = \psi \psi^* \qquad \rho_X = \psi^* \psi$$

Matrix product → trace over partial degree of freedom

Singular Value Decomposition (SVD)

Singular Value Decomposition of matrix Ψ (Snapshot Data)

$$\psi(x,y)=\sum_{l} U_l(x)\sqrt{\Lambda_l}\,V_l(y)$$

Λ_l : singular value (non-negative, uniquely determined)

$U_l(x), V_l(y)$: (unitary matrices, various choices)

$$\rho_X(x,x')=\sum_{y}\psi(x,y)\psi^*(x',y)=\sum_{l} U_l(x)\,\Lambda_l U_l^*(x')$$

$$\rho_Y(y,y')=\sum_{x}\psi(x,y)\psi^*(x,y')=\sum_{l} V_l(y)\,\Lambda_l V_l^*(y')$$

Snapshot Entropy → boundary law (not extensive)

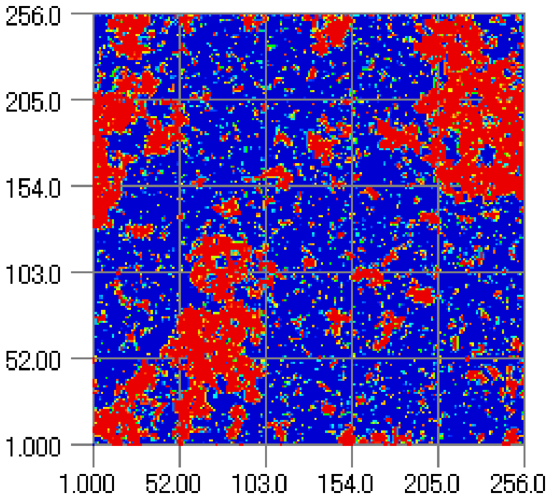
$$S_X=-\sum_{l}\lambda_l\log\lambda_l=S_Y \qquad \lambda_l=\Lambda_l/\sum_{l}\Lambda_l$$

スナップショット・SVDに隠れた双曲的空間構造

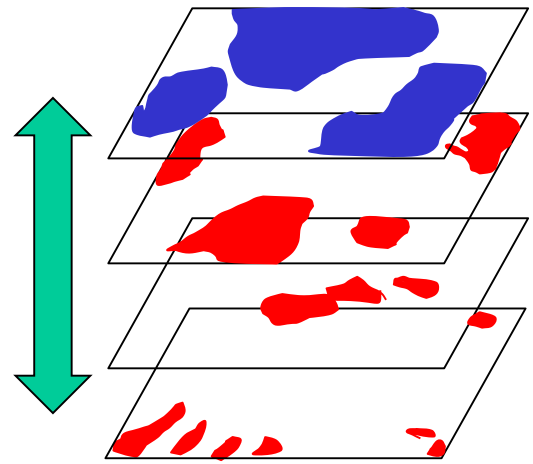
±1エンコーディング

HM, Phys. Rev. E85, 031101 (2012)

$$\psi(x,y)=\sum_{l=1}^{L}\psi^{(l)}(x,y) \qquad \psi^{(l)}(x,y)=U_l(x)\sqrt{\Lambda_l}V_l(y)$$



$$ds \propto \frac{dl}{l}$$

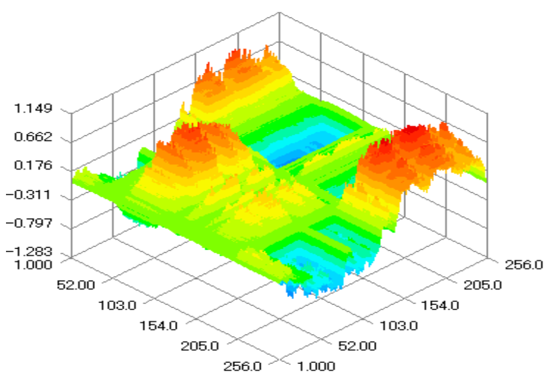


$\psi^{(1)}(x,y)$

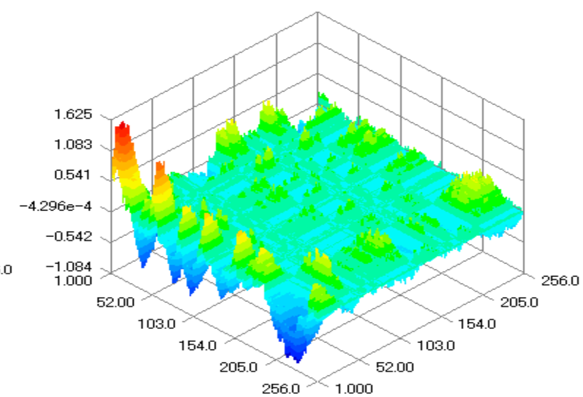
$\psi^{(2)}(x,y)$

$\psi^{(3)}(x,y)$

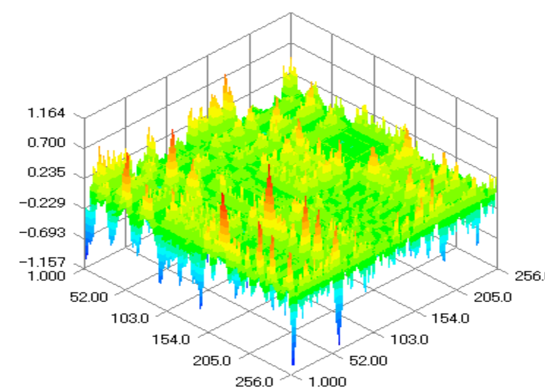
$\psi^{(4)}(x,y)$



$\psi^{(2)}(x,y)$



$\psi^{(4)}(x,y)$



$\psi^{(8)}(x,y)$

Tensor-product construction of scale-invariant systems

Sierpinski carpet → fractal structure

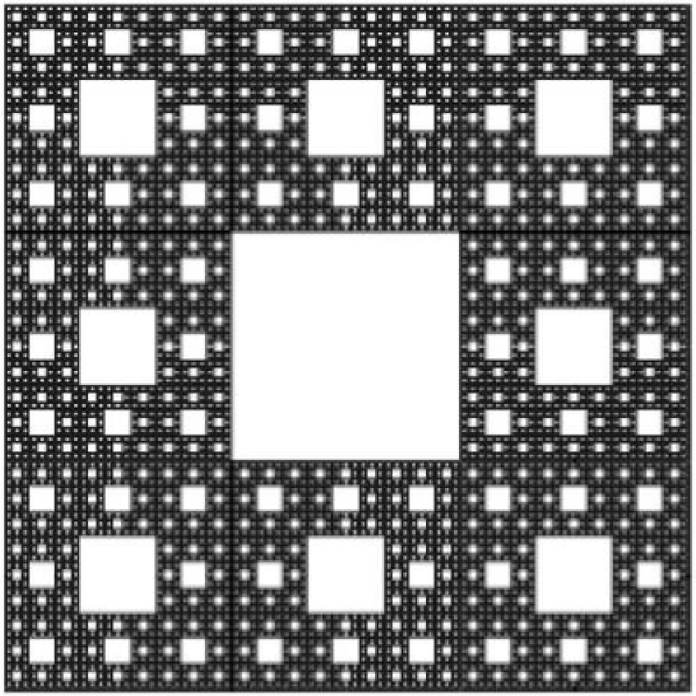
h × h(=3 × 3) unit cell

$$H = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Factorized form

$$M = H \otimes H \otimes \cdots \otimes H \otimes H$$

$$H \otimes H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$



Fractal image

→ L × L matrix

→ N different scales

$$L = h^N$$

SVD spectrum of Sierpinski carpet

$$M = H \otimes H \otimes \cdots \otimes H \otimes H$$

$$\left(-\sum_{i=\pm} \gamma_i \ln \gamma_i \right) \frac{1}{\ln h} = \frac{c}{3}$$

$$M^2 = H^2 \otimes H^2 \otimes \cdots \otimes H^2 \otimes H^2$$

Two non-zero eigenvalues of H² : $\Gamma_{\pm} = 4 \pm 2\sqrt{3}$

Normalization of Γ : $\gamma_{\pm} = \frac{1}{2} \pm \frac{\sqrt{3}}{4}$ $\quad \gamma_{-} = 1 - \gamma_{+}$

Eigenvalues of M² : $\lambda_j = \gamma_+^j \gamma_-^{N-j} = \gamma_+^j (1-\gamma_+)^{N-j}$ (Degeneracy : ${}_N C_j$)

Snapshot entropy ⇔ entanglement entropy of 1D free fermions

$$S = -\sum_{j=0}^{N} {}_N C_j \left(\lambda_j \ln \lambda_j \right) = \left(-\sum_{i=\pm} \gamma_i \ln \gamma_i \right) N = \left(-\sum_{i=\pm} \gamma_i \ln \gamma_i \right) \frac{\ln L}{\ln h}$$

C.H.Lee, Y.Yamada, T.Kumamoto, and HM, JPSJ 84, 013001 (2015)

臨界点の画像情報を非臨界的情報の和で表すこと

行列の特異値分解

$$M(x,y)=\sum_n U_n(x)\sqrt{\Lambda_n}V_n(y)$$

臨界点では
$\Lambda_n \propto n^{\eta-1}$

密度行列(相関関数)

$$\rho(x,x')=\sum_n U_n(x)U_n(x')\Lambda_n$$

連続極限での分解公式(Mellin変換)

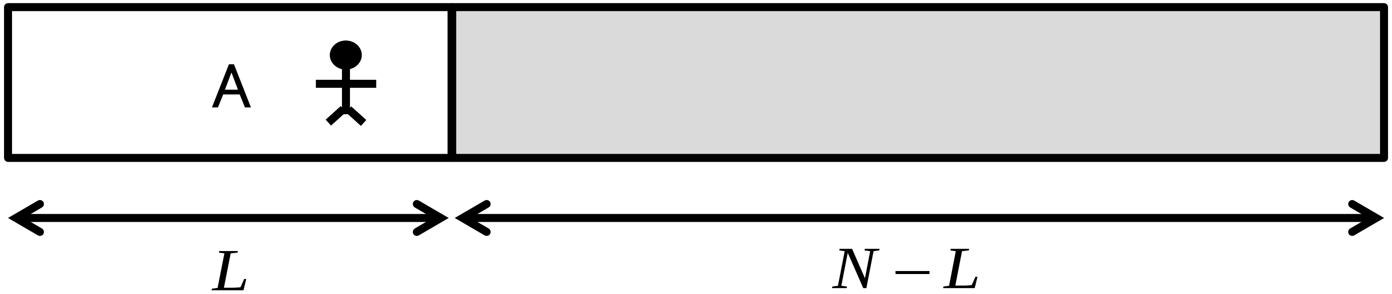
$$\lim_{\eta'\to\eta-0}\frac{A}{|x-x'|^{\eta}}\Gamma(\eta-\eta')=\lim_{\eta'\to\eta-0}\int_0^{\infty}dz\,\frac{Ae^{-z|x-x'|}}{(z|x-x'|)^{\eta'}}z^{\eta-1}$$

$$U_n(x)U_n(x')\propto\frac{e^{-n|x-x'|}}{(n|x-x'|)^{\eta'}}$$

$$n\propto\frac{1}{\xi}$$

How to identify the canonical parameters ?

Is the exponential family form really a reasonable assumption → yes!



$a=1$

Tight-binding model on 1D lattice

$$H=-\sum_{i=1}^{N}\left(c_i^{+}c_{i+1}+c_{i+1}^{+}c_i\right)=\sum_k \varepsilon_k c_k^{+}c_k$$

$$\varepsilon_k=-\cos(ka)\approx -1+\frac{1}{2}(ka)^2$$

Carrier density δ

Total carrier number n

$$\delta=\frac{n}{N}$$

(usual CFT: δ →0)

The ground-state properties of this model are completely characterized by L and δ.

→ L and δ are relevant model parameters.

(Be careful that they are 'not' canonical parameters)

Partial density matrix and entanglement spectrum (t=0)

$$\rho_A \propto \exp\left\{-\sum_{l=1}^{L}\varphi_l(L,\delta)n_l\right\}$$

S.-A. Cheong and C. L. Henley, PRB69, 075111 (2004)

Scaling relations for the entanglement spectrum

$$\varphi_l(L,\delta) = Lf(\delta,x)$$

$$x = \frac{l - l_F}{L} \qquad l_F = \delta L + \frac{1}{2}$$

$$\begin{cases} f(\delta,0) = 0 \\ f'(\delta,0) > 0 \\ f(\delta,-x) = -f(1-\delta,x) \end{cases}$$

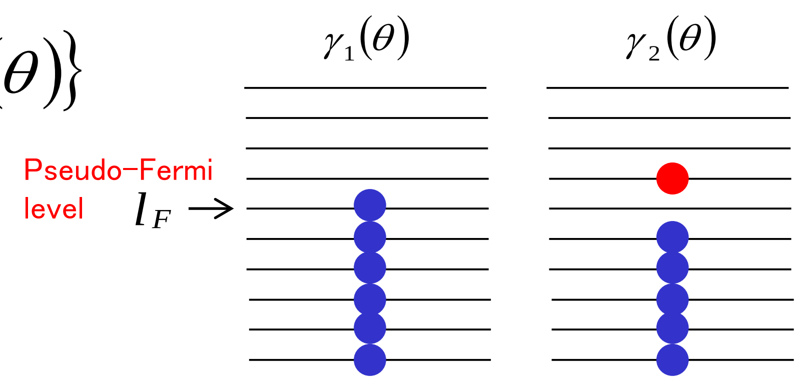
$$\lambda_n(\theta) = e^{-\gamma_n(\theta)} = \exp\{\theta^\mu F_{n,\mu} - \psi(\theta)\}$$

$$\gamma_n(\theta) = \psi(\theta) - \theta^\mu F_{n,\mu}$$

$$\gamma_1(\theta) \le \gamma_2(\theta) \le \cdots$$

$$\gamma_2(\theta) - \gamma_1(\theta) = \theta^\mu (F_{1,\mu} - F_{2,\mu})$$

$$\gamma_2(\theta) - \gamma_1(\theta) = Lf(\delta,1/L) = f'(\delta,0) + \frac{f''(\delta,0)}{2L} + \frac{f'''(\delta,0)}{6L^2} + \cdots$$



Numerical results suggest

very small constant

$$\gamma_2(\theta)-\gamma_1(\theta)=Lf(\delta,1/L)=\boxed{f'(\delta,0)}+\frac{f''(\delta,0)}{2L}+\frac{\boxed{f'''(\delta,0)}}{6L^2}+\cdots$$

only weak δ dependence

We can identify two of canonical parameters as

$$\left(\theta^1,\theta^2\right)=\left(\frac{f''(\delta,0)}{L},\frac{1}{L^2}\right)$$

Resolution of entanglement spectrum for finite L systems

Truncated quantum state

$$\left|\psi\right\rangle \approx \left|\psi_{\chi}\right\rangle = \sum_{n=1}^{\chi} \sqrt{\lambda_n} \left|n\right\rangle_A \otimes \left|n\right\rangle_{\bar{A}} \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\chi$$

Two scaling relations (area law and finite-entanglement scaling) for the entanglement entropy (ξ : finite-entanglement exponent)

$$S \approx \varsigma \log \chi = a L^{d-1} \qquad \varsigma(c) \log \chi = \frac{c}{6} \log \xi = \frac{c}{6} \log L$$

$$\Rightarrow \chi \approx \exp\left(\frac{a}{\varsigma} L^{d-1}\right) \qquad \Rightarrow \xi = L \Rightarrow \theta \approx \xi^{-1}$$

The parameter χ is related to how many states are necessary to keep numerical accuracy of the optimization of Ψ.

Thus, the inverse of χ is roughly the resolution of the entanglement spectrum.

$$\theta \approx \chi^{-1} \approx \exp\left(-\frac{a}{\varsigma} L^{d-1}\right) \qquad \theta = e^{-aL/\kappa} \left(d = 2\right)$$

$$\varsigma \approx \kappa$$

$$g_{\mu\nu}(\theta) = \langle \partial_\mu \partial_\nu \gamma \rangle \approx \partial_\mu \partial_\nu S(\theta)$$

One of θ control the energy scale of the entanglement spectrum, and this should be related to L.

Owing to the positivity of the Fisher metric, we require (d=1)

$$S(\theta) \approx -\kappa \log \theta, \theta = \frac{1}{L^\nu} \Rightarrow S = \kappa \nu \log L, g_{\theta\theta} \approx \frac{\kappa}{\theta^2}$$

Then, the warp factor of AdS naturally appears and the entropy coincides with the logarithmic violation formula.

d=2 case → area law scaling can be reproduced

$$S(\theta) \approx -\kappa \log \theta, \theta = e^{-aL/\kappa} \Rightarrow S = aL, g_{\theta\theta} \approx \frac{\kappa}{\theta^2}$$