THERMALIZATION OF ENTANGLEMENT ENTROPY

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@Discrete Approaches to the Dynamics of Fields and Space-Time

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Collaboration with K. Kim (POSTECH)

Based on : C. Park, Phys. Rev. D93 (2016), 086003,
K. S. Kim and C. Park, Phys. Rev. D95 (2017), 106007,
arXiv:1610.07312

Motivation

The entangelement entropy has been investigated holographically well in a UV regime.

- It showed the expected area law.
- It produced the correct central charge and free energy.
- It plays a crucial role to prove the c- and F-theorem along the RG flow.
- It satisfies the thermodynamics-like law, which allows us to reconstruct the (linearized) AdS geometry from the CFT data only.

Questions:

What is the IR entanglement entropy?

What is the relation between the entanglement and thermal entropies?

(Recently, it has been shown in quantum information theory that quantum information can evolve into the thermal entropy.)

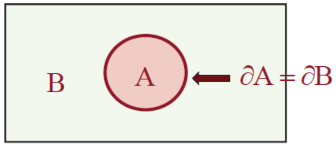
Contents

1. Review of the holographic entanglement entropy

The entanglement entropy measures
how closely and quantumly a given wave function is entangled.

Definition of EE (entanglement entropy)

- Divide a quantum system into two parts, A and B.



$$H_{tot} = H_A \otimes H_B \quad .$$

- Reduced density matrix of the subsystem A : $\rho_B = \mathrm{Tr}_A \rho_{tot}$

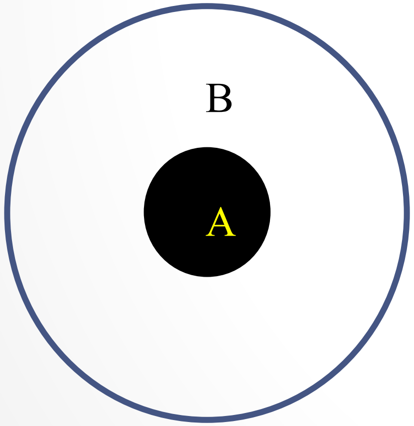
- The entanglement entropy (EE)

$$S_B = -\mathrm{Tr}_B \rho_B \log \rho_B$$

which is proportional to the area of the entangling surface (∂A)

S_B describes the quantum entanglement detected by an observer who is only accessible to the subsystem B and can not receive any signal from A.

This is similar to the Bekenstein-Hawking entropy of the black hole.
Since an observer sitting in the outside of the horizon, B, can not receive any information from A, we can regard A as a black hole and the boundary of A as the black hole horizon.



1. The area law of the entanglement entropy is also similar to that of the black hole entropy

2. The entanglement entropy is utilized to figure out the black hole entropy

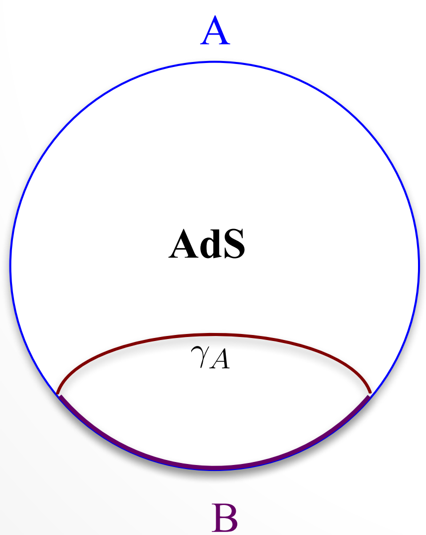
Due to the similarity to the black hole,

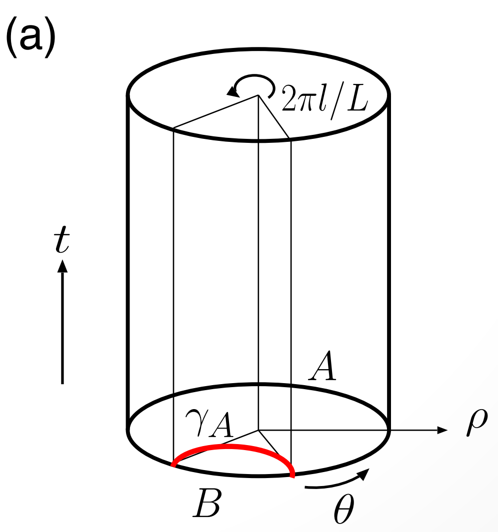
Ryu and Takayanagi [2006] proposed the holographic entanglement entropy (hEE)

following the AdS/CFT correspondence

the EE of a d-dimensional CFT can be evaluated by the area of the minimal surface in the d+1-dimensional dual AdS gravity

$$S_E = \frac{Area(\gamma_A)}{4G}$$





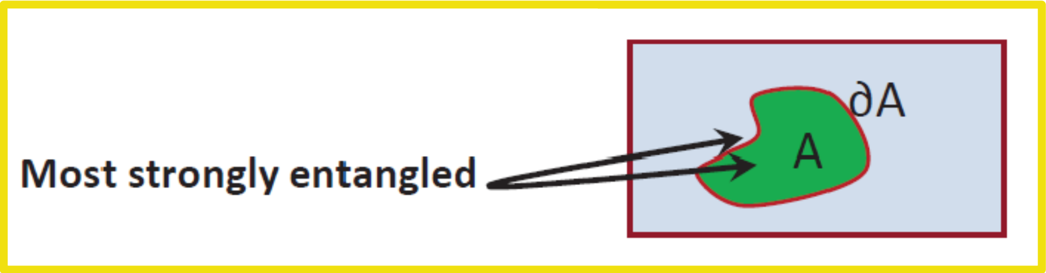
General features of the holographic entanglement entropy

General properties of the entanglement entropy

1) Area law of the entanglement entropy

The leading term of the entanglement entropy is provided by the short distance interaction between two subsystems near the boundary. In the continuum limit, this term causes a UV divergence and its coefficient is proportional to the area of the entangling surface ∂A (UV cutoff sensitive, a : UV cutoff).

$$S_A \sim \frac{\mathrm{Area}(\partial A)}{a^{d-1}} + \text{subleading finite terms}$$



2) Subleading finite terms

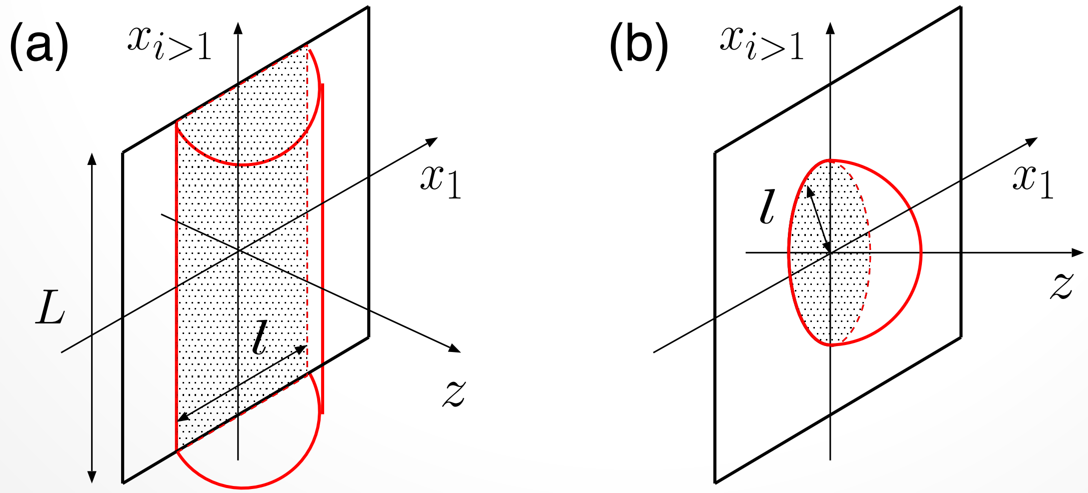
There exists the terms not relying on a UV cutoff, which can provide an important physical information associated with the long range correlations.

In general, the entanglement entropy depends on the shape and size of the entangling surface.

In AdS_{d+1}

$$ds^2 = R^2 z^{-2}(dz^2 - dx_0^2 + \sum_{i=1}^{d} dx_i^2)$$

let us take into account the HEE of a strip (a) and disk (b)



(a) Area of a strip

$$A = \frac{2R^{d-1}}{d-2}\left(\frac{L}{\epsilon}\right)^{d-2} - \frac{2^{d-1}\pi^{(d-1)/2}R^{d-1}}{d-2}\left[\frac{\Gamma\left(\frac{d}{2(d-1)}\right)}{\Gamma\left(\frac{1}{2(d-1)}\right)}\right]^{d-1}\left(\frac{L}{l}\right)^{d-2}$$

where L and ϵ indicate the size of the total system and a UV cutoff respectively.

- This result represents the entanglement entropy of vacuum states.

- There is no logarithmic term except for d=2.

(b) Area of a disk

which depends on the dimensionality

(i) for d=odd

$$A = \Omega_{d-2}\left[\frac{1}{d-2}\left(\frac{l}{\epsilon}\right)^{d-2} + F + \mathcal{O}\left(\frac{\epsilon}{l}\right)\right]$$

- No logarithmic term

- There exists a constant term, F, which is identified with a free energy of a CFT for d=3.

- For d=3,

  F is the same as the free energy of 3-dim. CFT which has been checked by the comparison with the localization result.

(ii) for d=even

$$A = \Omega_{d-2}\left[\frac{1}{d-2}\left(\frac{l}{\epsilon}\right)^{d-2} + a' \log\left(\frac{l}{\epsilon}\right) + \mathcal{O}(1)\right]$$

with

$$a' = (-)^{d/2-1}\frac{(d-3)!!}{(d-2)!!}$$

- There exists a universal logarithmic term because it is independent of the renormalization scheme.

- The coefficient of the logarithmic term is independent of the entangling surface area, which is related to the a-type anomaly.

- The Weyl anomaly of 4-dim. CFT,

$$\langle T^{\alpha}_{\alpha}\rangle = -\frac{c}{8\pi}W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} + \frac{a}{8\pi}\tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}$$

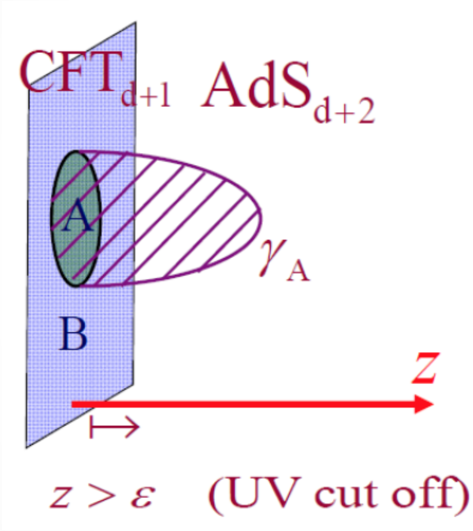
with

$$\begin{aligned} W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2, \\ \tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \end{aligned}$$

2. Known results in two-dimensional CFT vs. holographic entanglement entropy

Even in higher dimensions and in the strong coupling regime,

one can easily apply the Ryu-Takayanagi formula



γ_A is given by a co-dimension 2 surface

In the AdS/CFT context,

the entanglement entropy is geometrized as a minimal surface area.

2-dim. CFT result [Calabrese-Cardy, 2004]

1) For a finite system of length L with a periodic boundary condition, the ground state entanglement entropy in an interval l is given by

$$S_E = \frac{c}{3}\log\left[\frac{L}{\pi\epsilon}\sin\left(\frac{\pi l}{L}\right)\right]$$

2) For an infinite system without any boundary,

the ground entanglement entropy of a single interval reduces to

$$S_E = \frac{c}{3}\log\left(\frac{l}{\epsilon}\right)$$

3) Away from criticality with a correlation length ξ

the entanglement entropy is replaced by

$$S_E = \frac{c}{3}\log\frac{\xi}{\epsilon}.$$

4) At finite temperature,

the entanglement entropy is

$$S_E = \frac{c}{3}\log\left[\frac{\beta}{\pi\epsilon}\sin\left(\frac{\pi l}{\beta}\right)\right]$$

AdS(3)/CFT(2)

3-dimensional AdS metric

$$ds^2 = -\left(\frac{r^2}{R^2} + k\right) dt^2 + \frac{r^2}{R^2}\, d\Omega_k^2 + \frac{1}{r^2/R^2 + k} dr^2$$

Relying on the boundary topology,

1) k=0 (Poincare patch)

$d\Omega^2 = dx^2$ with $-\infty < x < \infty$

⇒ $\mathbf{R}_t \times \mathbf{R}_x$

2) k=1 (global patch)

$d\Omega^2 = d\theta^2$ with $0 \leq \theta \leq 2\pi$

⇒ $\mathbf{R}_t \times \mathbf{S}_\theta$

1) For a finite system of length L with a periodic boundary condition, the ground state entanglement entropy in an interval l is given by

$$S_E = \frac{c}{3}\log\left[\frac{L}{\pi\epsilon}\sin\left(\frac{\pi l}{L}\right)\right]$$

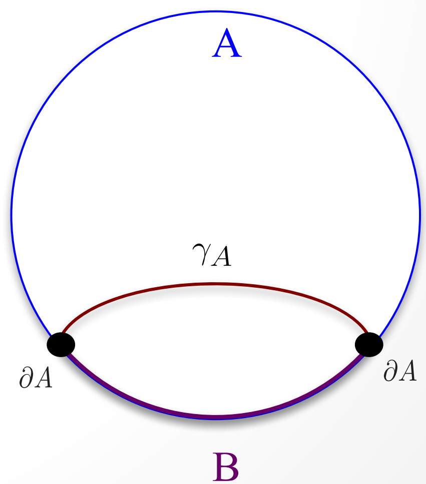
In order to describe the periodic finite system,

we consider the global patch (k=1).

$$ds^2 = \frac{1}{z^2}\left(-(1+z^2)dt^2 + \frac{dz^2}{1+z^2} + d\theta^2\right)$$

At t=0, the entanglement entropy is governed by

$$S_E = \frac{1}{4G}\int_{-\theta_0/2}^{\theta_0/2} d\theta \frac{1}{z}\sqrt{R^2 + \frac{z'^2}{1+z^2}}$$



which reproduces the known CFT result

$$S_E = \frac{c}{3}\log\left[\frac{L}{\pi\epsilon}\sin\left(\frac{\pi l}{L}\right)\right]$$

with $\quad c = \dfrac{3R}{2G}$

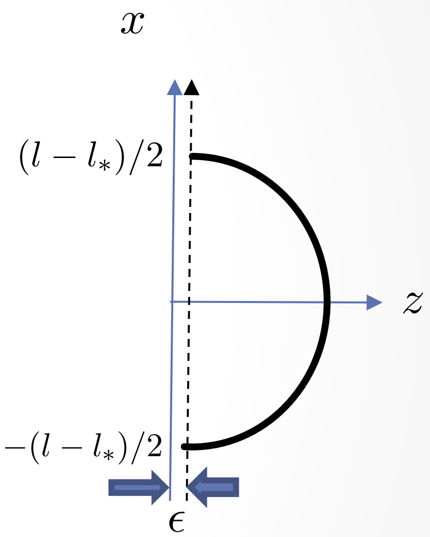
2) For an infinite system without any boundary,

the ground entanglement entropy of a single interval reduces to

$$S_E = \frac{c}{3} \log \left(\frac{l}{\epsilon} \right)$$

3-dim. AdS metric in the Poincare patch

$$ds^2 = \frac{1}{z^2} \left(-dt^2 + dx^2 + dz^2 \right)$$



The entanglement entropy is given by

$$S_E = \frac{1}{4G} \int_{-l/2}^{l/2} dx \frac{\sqrt{z'^2 + 1}}{z}$$

The minimal surface satisfies the following equation

$$0 = zz'' + z'^2 + 1$$

Solution

$$z = \sqrt{\frac{l^2}{4} - x^2} \qquad \text{with} \qquad l_* = \frac{2\epsilon^2}{l}$$

The entanglement entropy for $\epsilon \to 0$

$$S_E = \frac{1}{2G} \log \frac{l}{\epsilon} \qquad \text{with} \qquad c = \frac{3R}{2G}$$

3) Away from criticality with a correlation length

the entanglement entropy is replaced by ξ

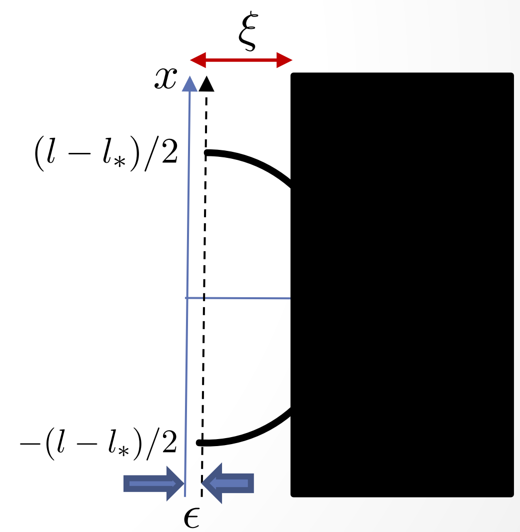
$$S_E = \frac{c}{3} \log \frac{\xi}{\epsilon}$$

Introducing the IR cutoff, ξ (non-conformal with mass gap)

Using the similar calculation,

the entanglement entropy is governed by

$$S_E = \frac{1}{2G} \int_{\epsilon}^{\xi/2} dz \, \frac{z_0}{z\sqrt{z_0^2 - z^2}}.$$



the resulting entanglement entropy becomes

$$S_E \approx \frac{c}{3} \log \frac{\xi}{\epsilon}$$

with $c = \frac{3R}{2G}$

4) At finite temperature,

the entanglement entropy is

$$S_E = \frac{c}{3} \log \left[\frac{\beta}{\pi \epsilon} \sin \left(\frac{\pi l}{\beta} \right) \right]$$

For the entanglement entropy

we can derive the entanglement entropy of an excited state by using the AdS/CFT correspondence

For the three-dimensional AdS (BTZ) black hole (dual to a 2-dim. CFT)

$$ds^2 = -\frac{R^2}{z^2} f(z) dt^2 + \frac{R^2}{z^2 f(z)} dz^2 + \frac{R^2}{z^2} dx^2, \qquad \text{with} \qquad f(z) = 1 - \frac{z^2}{z_h^2}$$

thermodynamic quantities are given by

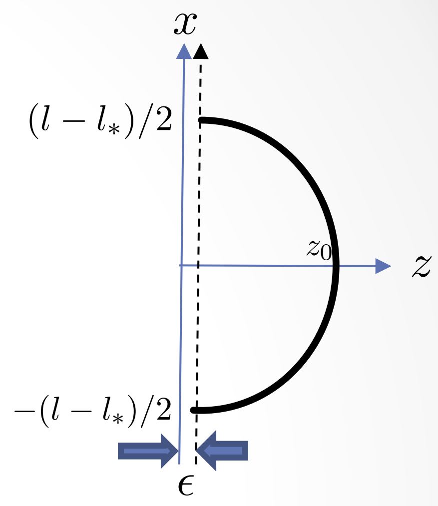
$$\begin{aligned} T_H &= \frac{1}{2\pi} \frac{1}{z_h}, \\ S_{th} &= \frac{1}{4G} \frac{l}{z_h}, \\ E &= \frac{1}{16\pi G} \frac{l}{z_h^2}. \end{aligned}$$

and satisfy the first law of thermodynamics $dE = T_H dS_{th}$

In the holographic context

the entanglement entropy can be evaluated as the area of the minimal surface extended in the dual geometry.

$$A = \int_0^{l/2} dx \, \frac{R}{z} \sqrt{1 + \frac{z'^2}{f}}$$



Then, the subsystem size and the entanglement entropy can be rewritten in terms of the turning point

$$z_0 = z_h \tanh\left(\frac{l}{2z_h}\right),$$

$$S_E = \frac{1}{2G} \log \frac{2z_0}{\epsilon} - \frac{1}{4G} \log\left(1 - \frac{z_0^2}{z_h^2}\right).$$

This is an exact and analytic result.

When $z_h \to \infty$,

$$z_0 = l/2$$

$$S_E^0 = \frac{1}{2G} \log \frac{l}{\epsilon}$$ (ground state entanglement entropy, UV divergence)

In the UV region

Thermodynamics-like law of the entanglement entropy

$$T_E \; S_E = E \qquad \text{with} \qquad T_E = \frac{6}{\pi l}.$$

- This relation is defined in the UV region with neglecting higher order corrections.
- It reproduces the linearized Einstein equation of the dual geometry.
- It is not valid in the IR region.

In order to go beyond the linearized lever and to describe the RG flow correctly, we need generalized concepts involving all higher order corrections.

We define a generalized thermodynamics-like law and generalized entanglement temperature involving all higher order correction and satisfying in the entire region

$$\bar{T}_E \; \bar{S}_E = \bar{E}$$

Define a renormalized entanglement entropy (subtracting the ground state EE)

$$\bar{S}_E \equiv S_E - S_E^0.$$

Then, the exact renormalized EE and a generalized entanglement temperature

$$\bar{S}_E = \frac{1}{2G} \log\left(\frac{2z_h}{l} \sinh\left(\frac{l}{2z_h}\right)\right)$$

$$\frac{1}{\bar{T}_E} \equiv \frac{1}{2}\frac{\bar{S}_E}{\bar{E}} = \frac{4\pi z_h^2}{l} \log\left(\frac{2z_h}{l} \sinh\left(\frac{l}{2z_h}\right)\right)$$

$$S_E = \frac{1}{2G} \log\left(\frac{\beta}{\pi\epsilon} \sinh\left(\frac{\pi l}{\beta}\right)\right)$$

In the UV region ($l/z_h \ll 1$),

$$\bar{S}_E = \frac{1}{48G}\frac{l^2}{z_h^2}\left(1 - \frac{l^2}{120 z_h^2} + \cdots\right)$$

$$\frac{1}{\bar{T}_E} = \frac{1}{T_E}\left(1 - \frac{l^2}{120 z_h^2} + \cdots\right)$$

with the previously temperature
defined entanglement $T_E = \frac{6}{\pi l}$.

Ignoring l^2 order corrections, they are reduced to the known results.

Note that the generalized entanglement temperature was defined to satisfy the thermodynamics-like law exactly with involving all higher order correction. Therefore, we can apply the thermodynamics-like law to the IR entanglement entropy.

3. Universality of the holographic entanglement entropy

Consider a Lifshitz geometry

$$ds^2 = \frac{1}{z^2}\left(-\frac{1}{z^{2(\bar{z}-1)}}dt^2 + \delta_{ij}dx^i dx^j + dz^2\right)$$

which is dual of a Lifshitz field theory and allows the following scale symmetry

$$z \to \Omega z \ , \ t \to \Omega^{\bar{z}} t \ , \ x^i \to \Omega x^i$$

- When the dynamical critical exponent is 1 ($\bar{z} = 1$), the Lifshitz geometry reduces to the AdS one.
- For $\bar{z} = 2$, the dual field theory corresponds to a non-relativistic Lifshitz theory

$$S[\chi] = \int d^3x \left[(\partial_t \chi)^2 - K\left(\nabla^2 \chi\right)^2\right]$$

- Due to this scale symmetry, the entanglement temperature behaves as

$$T_E = \frac{dE}{dS_E} \sim \frac{1}{l^{\bar{z}}}$$

l : subsystem size

- Macroscopic effective theory

- Microscopic QFT

When thermalization occurs

- Thermodynamics law

$$dE = T\ dS$$

- E and S are extensive quantities.
- T is independent of the system size.
- Universal

(independent of microscopic details)

- Thermodynamics-like law

$$dE = T_E\ dS_E$$

- S_E is not an extensive quantity.
- T_E relies on the subsystem size.
- (partially) universal

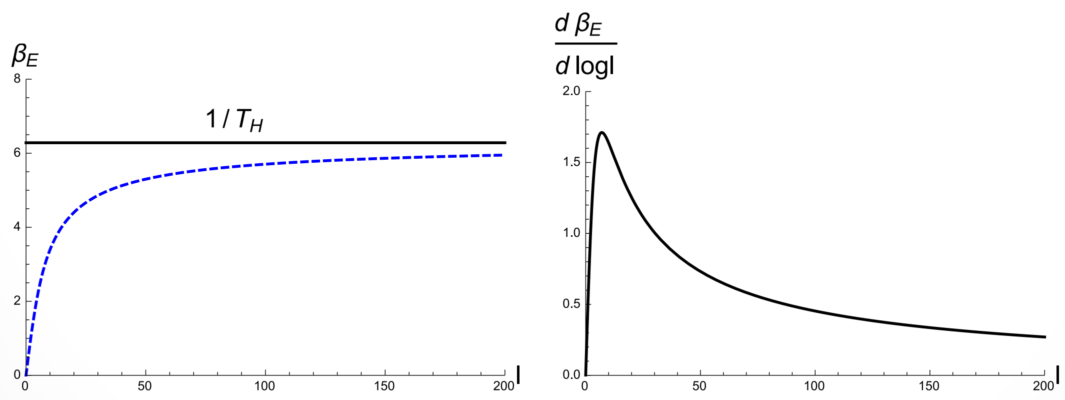
In the IR region ($z_0 \approx z_h$),

Reexpressing it in terms of the black hole entropy involved in the volume we reach to the similar result obtained from the black hole and CFT calculations

$$\bar{S}_E = S_{th} - \frac{1}{2G} \log S_{th} + \mathcal{O}(1)$$

Since $S_{th} \to \infty$ in the IR limit,

the IR entanglement entropy reduces to the thermal entropy with small quantum corrections. Also, we can see that the generalized entanglement temperature reduces to the real temperature.



(a) (b)

$$\beta_E = 2\pi z_h + \frac{4\pi z_h^2}{l} \log\left(\frac{z_h}{l}\right) + \cdots, \qquad l \frac{d\beta_E}{dl} = 2\pi z_h \left[\coth \frac{l}{2z_h} - \frac{2z_h}{l}\left\{1 + \log\left(\frac{2z_h}{l} \sinh \frac{l}{2z_h}\right)\right\}\right].$$

Regardless of the dimensionality and microscopic detail of the dual field theory,

the IR entanglement entropy reduces to

$$\bar{S}_E = S_{th} + S_q$$

S_{th} : Universal

S_q : depending on the dual theory

For a two-dimensional scale invariant theory

$$S_q \sim -\log S_{th}$$

Intriguingly, the universality of the IR entanglement entropy proposed from the holography

$$\bar{S}_E \approx S_{th}$$

occurs in the real space renormalization group flow of the lattice theory (Ising model).

Universality of the IR entanglement entropy

A general (d+1)-dimensional gravity theory

$$ds^2 = \frac{1}{z^2}\left(-e^{2A(z)} f(z) dt^2 + e^{2B(z)} \delta_{ij} dx^i dx^j + \frac{e^{2C(z)}}{f(z)} dz^2\right)$$

we can set $e^{2C(z)} = 1$ without loss of generality

- For $e^{2A(z)} = e^{2B(z)} = 1$, the metric reduces to that of the AdS black hole studied in the previous sections. The dual field theory is conformal.
- For $e^{2A(z)} \neq e^{2B(z)} = 1$, it reduces to the Lifshitz black hole which breaks the boost symmetry in the $t - x^i$ plane. The resulting dual field theory is a non-relativistic field theory with a scale invariance.
- For $e^{2A(z)} = e^{2B(z)} \neq 1$, it leads to a black hole on the hyperscaling violation geometry which has no scale symmetry. The dual field theory can be identified with a relativistic quantum field theory without a scale symmetry.
- For $e^{2A(z)} \neq 1$, $e^{2B(z)} \neq 1$ and $e^{2A(z)} \neq e^{2B(z)}$, it is the combination of the previous two cases. In this case, the scale and boost symmetry are broken and the dual field theory is given by a non-relativistic theory without a scale symmetry.

For a black hole with a simple root

$$f(z) = \left(1 - \frac{z}{z_h}\right) F(z),$$

the Bekenstein-Hawking entropy from the area law

$$S_{th} = \frac{V_{d-1}}{4G} \frac{e^{(d-1)B(z_h)}}{z_h^{d-1}}$$

The entanglement entropy in a strip-shaped region

$$\begin{aligned} l &= 2\int_0^{z_0} dz \, \frac{z^{d-1} \; e^{(d-1)B_0}}{e^B \sqrt{f} \sqrt{e^{2(d-1)B} z_0^{2(d-1)} - e^{2(d-1)B_0} z^{2(d-1)}}}, \\ S_E &= \frac{L^{d-2}}{2G} \int_0^{z_0} dz \, \frac{z_0^{d-1} \; e^{(2d-3)B_0}}{z^{d-1} \sqrt{f} \sqrt{e^{2(d-1)B} z_0^{2(d-1)} - e^{2(d-1)B_0} z^{2(d-1)}}}, \end{aligned}$$

In the IR region

$$S_E = \frac{lL^{d-2}}{4G} \frac{e^{(d-1)B_0}}{z_0^{d-1}} + \frac{L^{d-2}}{2G z_0^{d-1}} \int_\epsilon^{z_0} dz \, \frac{\sqrt{e^{2(d-1)B} z_0^{2(d-1)} - e^{2(d-1)B_0} z^{2(d-1)}}}{z^{d-1} e^B \sqrt{f}}.$$

The entanglement entropy in a ball-shaped region

$$S_E = \frac{\Omega_{d-2}}{4G} \int_0^l d\rho \, \frac{e^{(d-2)B} \rho^{d-2} \sqrt{e^{2B} f + z'^2}}{z^{d-1} \sqrt{f}} .$$

In the IR limit

$$\begin{aligned} S_E &= \frac{\Omega_{d-2} e^{(d-1)B_0}}{4G z_0^{d-1}} \int_0^l d\rho \, \rho^{d-2} \\ &+ \frac{\Omega_{d-2}}{4G z_0^{d-1}} \int_0^l d\rho \, \frac{\rho^{d-2} \left(z_0^{d-1} e^{(d-1)B} \sqrt{f + e^{-2B} z'^2} - z^{d-1} e^{(d-1)B_0} \sqrt{f} \right)}{z^{d-1} \sqrt{f}} . \end{aligned}$$

with $V_{d-2} = \Omega_{d-2} \int_0^l d\rho \, \rho^{d-2}$

In the IR limit, the entanglement entropy reduces to the thermal entropy regardless of the microscopic detail and the shape of the entangling surface.

4. Discussion

- We showed that the quantum entanglement entropy evolves into the thermal entropy along the RG flow regardless of the microscopic details

- The thermodynamics-like law of the entanglement entropy leads to the exact thermodynamic law in the IR limit.

- The universal feature of the IR entanglement entropy has been shown in the holographic setup. Therefore, it would be interesting to check this IR universality also occurs in a quantum field theory (or lattice model).

Thank you！