THERMALIZATION OF ENTANGLEMENT ENTROPY

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@Discrete Approaches to the Dynamics of Fields and Space-Time

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Motivation

The entanglement entropy has been investigated holographically well in a UV regime.

- It showed the expected area law.
- It produced the correct central charge and free energy.
- It plays a crucial role to prove the c- and F-theorem along the RG flow.

- It satisfies the thermodynamics-like law, which allows us to reconstruct the (linearized) AdS geometry from the CFT data only.

Questions:

What is the IR entanglement entropy?

What is the relation between the entanglement and thermal entropies?

(Recently, it has been shown in quantum information theory that quantum information can evolve into the thermal entropy.)

1. Review of the holographic entanglement entropy

2. Known results in two-dimensional CFT vs. holographic entanglement entropy

3. Universality of the holographic entanglement entropy

4. Discussion

The extract of the holographic entanglement entropy and groups are entitled as a given group of the holographic entanglement entropy and α *given ground states and* α

The entanglement entropy measures

- Reduced density matrix of the subsystem A: $\rho_B = \text{Tr}_A \rho_{tot}$ \sim Kenteent entropy matrix of the subsystem λ
	- The entanglement entropy (EE)

$$
S_B=-\text{Tr}\,{}_B\rho_B\log\rho_B
$$

S^B = Tr *^B*⇢*^B* log ⇢*^B* which is proportional to the area of the entangling surface (∂A)

describes the quantum entanglement detected by an observer who is only *S^B* describes the quantum entanglement detected by an observer, who is only accessible to the subsystem B and can not receive any signal from A.

This is similar to the Bekenstein-Hawking entropy of the black hole. Since an observer sitting in the outside of the horizon, B, can not receive any information from A, we can regard A as a black hole and the boundary of A as the black hole horizon.

1. The area law of the entanglement entropy is also similar to that of the black hole entropy

> 2. The entanglement entropy is utilized to figure out the black hole entropy

Due to the similarity to the black hole,

Ryu and Takayanagi [2006] proposed the holographic entanglement entropy (hEE) following the AdS/CFT correspondence

the EE of a d-dimensional CFT can be evaluated by the area of the minimal surface in the d+1-dimensional dual AdS gravity

$$
S_E = \frac{Area(\gamma_A)}{4G}
$$

Example 6 General features of the holographic entanglement entropy is. Here, the total system is divided into two subsystems

General properties of the entanglement entropy subsystem a obtained by the antique of the continuous the particle of the second continuous of the continuous over the continuous of the continuous continuous of the continuous continuous continuous continuous continuous c Substrame B of the total ground theory

1) Area law of the entanglement entropy $\frac{1}{2}$ are calculated the entropy for an observer. η Area law of the entanglement entropy

The leading term of the entanglement entropy is provided by the short distance interaction between two subsystems near the boundary. In the continuum limit, this term causes a UV divergence and its coefficient is proportional to the area of the entangling surface ∂A (UV cutoff sensitive, $a:$ UV cutoff). receive any signals from B. In this sense, the subsystem between two subsystems near the boundary. In the commutant mint, this term causes $(TN_{\text{cutoff}}^{\text{c}})$ (UV cutoff sensitive, $a:$ UV cutoff). n
The distance interact ce intera $\lim_{n\to\infty}$

$$
S_A \sim \frac{\text{Area}(\partial A)}{a^{d-1}} + \text{subleading finite terms}
$$

2) Subleading finite terms

There exists the terms not relying on a UV cutoff, which can provide an important physical information associated with the long range correlations. $\frac{1}{2}$ or $\frac{1}{2}$ of the entropy for $\frac{1}{2}$ is $\frac{1}{2}$ interressociated with the iong range correlations. 'area law' of the entanglement entropy in quantum field information associate

In general, the entanglement entropy depends on the shape and size of the entangling surface. I_n respected the enters and further generalized to higher dimensional cases [9, 10]. We start with summarizing those

In
$$
AdS_{d+1}
$$

\n
$$
ds^2 = R^2 z^{-2} (dz^2 - dx_0^2 + \sum_{i=1}^d dx_i^2)
$$

let us take into account the HEE of a strip (a) and disk (b) *,* (5)

(a) Area of a strip In a *d*-dimensional CFT, when a subsystem lies in a thin strip with a width *l*, the holographic entanglement entropy is given by the area of a minimal surface extended in AdS*d*+1

$$
A = \frac{2R^{d-1}}{d-2} \left(\frac{L}{\epsilon}\right)^{d-2} - \frac{2^{d-1}\pi^{(d-1)/2}R^{d-1}}{d-2} \left[\frac{\Gamma\left(\frac{d}{2(d-1)}\right)}{\Gamma\left(\frac{1}{2(d-1)}\right)}\right]^{d-1} \left(\frac{L}{l}\right)^{d-2}
$$

exa L and andicate the size of the total system and a UV cutoff respectively. radius is denoted by *R*. From now on, we set *R* = 1 for simplicity. This result expresses where *L* and andicate the size of the total system and a UV cutoff respectively.

- This result represents the entanglement entropy of vacuum states.
- Γ here is no logarithmic term excent for $d=2$ - There is no logarithmic term except for $d=2$.

(b) Area of a disk entanglement entropy. In general, the entanglement entropy of a thin strip has no logarithmic

which depends on the dimensionality

 (i) for $d = odd$

$$
A = \Omega_{d-2} \left[\frac{1}{d-2} \left(\frac{l}{\epsilon} \right)^{d-2} + F + \mathcal{O} \left(\frac{\epsilon}{l} \right) \right]
$$

- No logarithmic term
	- **1** α ter $\overline{1}$ ², ², ² **l**
⊥l ✏ α ^{*l*} is identified - There exists a constant term, F, which is identified with a free energy of a CFT for d=3.
- For d=3,

 Γ is the game of the fuse energy of Ω , dim, CET which has been checked by the If is the same as the free energy of stamm, of I which has occur encenced by the comparison with the localization result. F is the same as the free energy of 3-dim. CFT which has been checked by the

(ii) for d =even $\frac{r}{1-\epsilon}$ d=even

$$
A = \Omega_{d-2} \left[\frac{1}{d-2} \left(\frac{l}{\epsilon} \right)^{d-2} + a' \log \left(\frac{l}{\epsilon} \right) + \mathcal{O}(1) \right]
$$

with

$$
a' = (-)^{d/2 - 1} \frac{(d-3)!!}{(d-2)!!}
$$

- is related to the central charge of the dual CFT. In a higher dimensional CFT, *a*⁰ is related to - There exists a universal logarithmic term because it is independent of the renormalization cheme. are a consequence, the logarithmic term related to the logarithmic term related to the central charge. scheme. erm because it is independent of the renormalization Interval 2D CFTs, the CFTs, the Weyl and STS. There exists a universal logarithmic term because it is independent of
	- The coefficient of the logarithmic term is independent of the entangling surface area, which is related to the a-type anomaly. 12R, (4.18) (12R, (4.18) (1 \mathbb{R} is the scalar curvature. We can regard this as a definition of this as a definition of the central charge c in 2011.00
	- The Weyl anomaly of 4-dim. CFT, $The W_{ov}l anomaly of 4 dim CFT$ be written as

$$
\langle T^{\alpha}_{\alpha} \rangle = - \frac{c}{8 \pi} W_{\mu \nu \rho \sigma} W^{\mu \nu \rho \sigma} + \frac{a}{8 \pi} \tilde{R}_{\mu \nu \rho \sigma} \tilde{R}^{\mu \nu \rho \sigma}
$$

in a curve density background ground g $\mathcal{U}/\mathcal{U}/\mathcal{U}/\mathcal{U}$ with

$$
W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2,
$$

$$
\tilde{R}_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2.
$$

2. Known results in two-dimensional CFT vs. holographic entanglement entropy

Even in higher dimensions and in the strong coupling regime, one can easily apply the Ryu-Takayanagi formula

 γ_A is given by a co-dimension 2 surface

In the AdS/CFT context,

the entanglement entropy is geometrized as a minimal surface area.

2-dim. CFT result [Calabrese-Cardy, 2004] In order to understand the holographic entropy and its extension, we first need to understand the holographic entropy and its extension, we first need to understand the second to understand the second term of the second t aspects of string theory is the microscopic derivation of $\overline{41}$ $\frac{1}{\sqrt{2}}$

FOL a finite system of length L with a periodic boundary condition, the ground state entanglement entropy in an interval 1 is given by 1) For a finite system of length L with a periodic boundary condition, the ground state entanglement entropy in an interval l is given by $\frac{1}{100}$ $\frac{1}{100}$ *boundary condition*, the ground state $\begin{bmatrix} 1 & \sqrt{\pi} \end{bmatrix}$

$$
S_E = \frac{c}{3} \log \left[\frac{L}{\pi \epsilon} \sin \left(\frac{\pi l}{L} \right) \right]
$$

2) For an infinite system without any boundary, of boundary for the entangling surface. For a single interval *A* = 2. The above result is symmetric $\frac{2}{3}$ for an infinite system while the theory cutoff (lattice spacing); c is the central charge of the central charge of the CFT. *^S^E* ⁼ *^c* $\frac{1}{2}$ *ary*,

the ground entanglement entropy of a single interval reduces to equivalent to the $\frac{1}{2}$ dimensional anti- $\mathcal{H}(\mathcal{M})$ from critical terms of $\mathcal{H}(\mathcal{M})$ is replaced by replaced by replaced by replaced by $\mathcal{H}(\mathcal{M})$

$$
S_E = \frac{c}{3} \log\left(\frac{l}{\epsilon}\right)
$$

S
E E 2 letteral during the set of the set + *Bg* + *c* 3) Away from criticality with a correlation length <u>3) Away from criticality with a corre</u> $\frac{n \text{ length}}{\xi}$ boundary points of A (e.g. A (e.g. A \sim 2 in the setup of (1.3)). The setup of (1.3) \sim (*a*), the experimential entire set-up ϵ of a single interval reduces to ϵ ³ log ✓ *^l* ◆

the entanglement entropy is replaced by

the entanglement entropy is replaced by
\n
$$
S_E = \frac{c}{3} \log \frac{\xi}{\epsilon},
$$

4) At finite temperature,

the entanglement entropy is $\overline{}$

the entanglement entropy is
\n
$$
S_E = \frac{c}{3} \log \left[\frac{\beta}{\pi \epsilon} \sin \left(\frac{\pi l}{\beta} \right) \right]
$$

AdS(3)/CFT(2) *ds*² ⁼ *^R*² introduce a new coordinate, *z* = *R*2*/r*. Then, the AdS metric in the Poicare patch reduces to

3-dimensional AdS metric is called a the Euclidean version of this metric is called a thermal AdS metric

$$
ds^{2} = -\left(\frac{r^{2}}{R^{2}} + k\right)dt^{2} + \frac{r^{2}}{R^{2}}d\Omega_{k}^{2} + \frac{1}{r^{2}/R^{2} + k}dr^{2}
$$

Relying on the boundary topology, *r*₂*<i>k*ely *Relying on the boundary topology,* changes into S*x*. The Einstein equation, which the above AdS metric satisfies, allows and α $\frac{1}{2}$ solution called a black hole ca

by ^R*^t* ⇥ ^S✓ where ^R*^t* indicates a real line for time, *^t*. If *^k* = 0, on the other hand, *^d*⌦² ⁼ *dx*² with 1) k=0 (Poincare patch) Let us further study the generalization of the above AdS metric for *k* = 0. For later convenience, we where the one-dimensional metric crucially depends on the value of *k*. When *k* = 1, for instant, the $d\Omega^2 = dx^2$ with $-\infty < x < \infty$ where the one-dimensional metric crucially depends on the value of \mathbf{h} μ μ α β α β β β and β $d\Omega^2 = dx^2$ with $-\infty < x < \infty$ $\begin{array}{|c|c|c|}\n\hline\n\end{array}\n\quad\n\begin{array}{c}\n\mathbf{R}_t \times \mathbf{R}_x\n\end{array}$ \overline{a} *f*(*z*)*dt*² ⁺ *dx*² ⁺ After the Wick rotation, the absence of a conical singularity at the horiozn (*z* = *zh*) determines the

 $2\left(\frac{1}{\alpha} \right)$ $k=1$ (global natch) introduce a new coordinate, *z* = *R*2*/r*. Then, the AdS metric in the Poicare patch reduces to $2)$ k=1 (global patch) After the Wick rotation, the absence of a conical singularity at the horiozn (*z* = *zh*) determines the

$$
d\Omega^2 = d\theta^2 \qquad \text{with} \qquad 0 \le \theta \le 2\pi
$$

$$
\mathbf{R}_t \times \mathbf{S}_\theta
$$

• I of a finite system of length *L* with a periodic boundary condition, the ground state 1) For a finite system of length L with a periodic boundary condition, the ground state 1) For a finite system of length L with a periodic boundary condition *c* boundary condition the ground state

entanglement entropy in an interval l is given by
 $C = \frac{c}{\ln \pi} \left[L \sin \left(\frac{\pi l}{2} \right) \right]$

$$
S_E = \frac{c}{3} \log \left[\frac{L}{\pi \epsilon} \sin \left(\frac{\pi l}{L} \right) \right]
$$

¹ is a constant crucailly relying on the regularization scheme and *A* indicates the number In order to describe the periodic finite system, performal field system, and on to decompte the nominate finite order to describe the performal finite. $t_{\rm cm}$ \mathfrak{r} $\frac{1}{2}$ results in a Rt^{op}y on Rt^t _{*x*} *r x r* eriodic finite *d*✓ tem *R*² + A the definition of the entropy is manifest, in general, it is not an easy work to in general, it is not an easy work to in general, it is not an easy work to in general, it is not an easy work to in general, it is not

we consider the global patch $(k=1)$. equivalent to the (super)gravity on d+2 dimensional anti- ϵ consider the global patch ($k=1$). (-1) we consider the global patch $(K=1)$.

$$
ds^{2} = \frac{1}{z^{2}} \left(-(1+z^{2})dt^{2} + \frac{dz^{2}}{1+z^{2}} + d\theta^{2} \right)
$$

At $t=0$, the entanglement entropy is governed by t ational interpretation of the entanglement entropy has entanglement entropy has entrop ϵ is associated with the area of the dual geometric extended to the dual geometric ϵ

1) For a finite system of length L with a periodic boundary condition, the ground
entanglement entropy in an interval 1 is given by

$$
S_E = \frac{c}{3} \log \left[\frac{L}{\pi \epsilon} \sin \left(\frac{\pi l}{L} \right) \right]
$$
In order to describe the periodic finite system,
we consider the global patch (k=1).

$$
ds^2 = \frac{1}{z^2} \left(-(1 + z^2) dt^2 + \frac{dz^2}{1 + z^2} + d\theta^2 \right)
$$
At t=0, the entanglement entropy is governed by

$$
S_E = \frac{1}{4G} \int_{-\theta_0/2}^{\theta_0/2} d\theta \frac{1}{z} \sqrt{R^2 + \frac{z'^2}{1 + z^2}}
$$

 W_{max} CFT max 14 r_{H} and r_{H} is subsequently sense, the substitution r_{H} hich reproduces the known CET result which reproduces the known CFT result **•** It is the system of length *L* with a periodic condition, the ground state entanglement of $\frac{1}{2}$ which reproduces the Know $\overline{\text{CFT}}$ result

$$
S_E = \frac{c}{3} \log \left[\frac{L}{\pi \epsilon} \sin \left(\frac{\pi l}{L} \right) \right]
$$

$$
\text{with} \quad c = \frac{3R}{2G}
$$

2) For an infinite system without any boundary, For an infinite system without any boundary $\frac{1}{101}$ an infinite system writiout any boundary,

the ground entanglement entropy of a single interval reduces to

$$
S_E = \frac{c}{3} \log \left(\frac{l}{\epsilon}\right)
$$

3-dim. AdS metric in the Poincare patch

$$
ds^{2} = \frac{1}{z^{2}} \left(-dt^{2} + dx^{2} + dz^{2} \right)
$$

The entanglement entropy is given by $\frac{1}{l} \int_0^{l/2} x^{l/2} dx$

The entanglement entropy is given by
\n
$$
S_E = \frac{1}{4G} \int_{-l/2}^{l/2} dx \frac{\sqrt{z'^2 + 1}}{z}
$$

q*l*² Γ minimal sympace satisfies the following equation rite minimar surface

$$
0 = zz'' + z'^2 + 1
$$

 $X₁$ *z*(*x*) $X₂$ *z*(*x*) $X₁$ *z*(*x*) $X₂$ *z*(*x*) $X₁$ $X₂$ $X₃$ $X₄$ $X₅$ $X₆$ $X₇$ $X₈$ $X₉$ $X₉$ $X₁$ $X₁$ $X₂$

Solution
\n
$$
z = \sqrt{\frac{l^2}{4} - x^2} \quad \text{with} \quad l_* = \frac{2\epsilon^2}{l}
$$
\nThe entanglement entropy for $\epsilon \to 0$

Research Program through the National Research Foundation of Korea funded by the Ministry of Education (NRF-2013R1A1A2A10057490). *l/*2 *l/*2 Acknowledgement $\epsilon \to 0$
and $\epsilon \to 0$ The entanglement entropy for $\epsilon \to 0$

$$
S_E = \frac{1}{2G} \log \frac{l}{\epsilon} \qquad \text{with} \qquad c = \frac{3R}{2G}
$$

3) Away from criticality with a correlation length α <u>describing such an RG flow nonperturbative interesting</u> and important problem in the set of the set of the set o

the entanglement entropy is replaced by the entanglement entropy is replaced by ξ aced by ξ

$$
S_E = \frac{c}{3} \log \frac{\xi}{\epsilon},
$$

Introducing the IR cutoff, is. Hurbauchig the two substitutions in U_{r} regime and the entanglement entropy can not be a mass ϵ on θ

 \overline{u} \overline{u} \overline{u} Introducing the IR cutoff, ξ (non-conformal with mass gap) $($ non-conformal with mass gap $)$ μ discussions from different viewpoints can be found in e.g. μ low energy physics in the hard wall model. On the other hand, if *zmin* = *zir* (*zir < z*0), it describes

The Similar calculation, the National Research Foundation of Research Foundation of Korea (NRF) grant funded by the NRF (NRF) grant substitution and the complete original trace of the particle trace of the particle trace of the particle trace of the set o

who is one thangeled to the subsystem ϵ and ϵ and ϵ *z* intervals the entanglement entropy is governed by

$$
S_E = \frac{1}{2G} \int_{\epsilon}^{\xi/2} dz \; \frac{z_0}{z\sqrt{z_0^2 - z^2}}.
$$

The entanglement entropy is of growing importance

$$
S_E \approx \frac{c}{3} \log \frac{\xi}{\epsilon}
$$

with $c = \frac{3R}{2G}$

- Brief review on the AdS/CFT correspondence 4) At finite temperature, *•* For a time temperature,

the entanglement entropy is

$$
S_E = \frac{c}{3} \log \left[\frac{\beta}{\pi \epsilon} \sin \left(\frac{\pi l}{\beta} \right) \right]
$$

For the entanglement entropy $\frac{1}{2}$ External entanglement entropy <u>For the emanglement entropy</u>

we can derive the entanglement entropy of an excited state by using the AdS/CFT correspondence *R*2 *R*2 $\frac{1}{2}$ such small corrections of ³ log ✓ *^l*

For the three-dimensional AdS (BTZ) black hole (dual to a 2-dim. CFT) extensive quantity proportional to the volume of the subsystem [1]

$$
ds^{2} = -\frac{R^{2}}{z^{2}}f(z)dt^{2} + \frac{R^{2}}{z^{2}f(z)}dz^{2} + \frac{R^{2}}{z^{2}}dx^{2}, \text{ with } f(z) = 1 - \frac{z^{2}}{z_{h}^{2}}
$$

 $\frac{1}{2}$ *z* thermodynamic quantities are given by the state of th

$$
T_H = \frac{1}{2\pi} \frac{1}{z_h},
$$

\n
$$
S_{th} = \frac{1}{4G} \frac{l}{z_h},
$$

\n
$$
E = \frac{1}{16\pi G} \frac{l}{z_h^2}.
$$

and satisfy the first law of thermodynamics $dE = T_H dS_{th}$ ω the correlation distance is given by the mass, ω ω \mathbf{r}_{H} we can

In the holographic context $\frac{1}{2}$. The interpretation of a two-dimensional CFT $\sqrt{1/\sqrt{1-\epsilon^2}}$

the entanglement entropy can be evaluated as the area α surface extended in the dual geometry. ϵ in the holographic contexts, the entropy contexts, the entropy can be represented as a minimal $(l$ *x z*(*x*)

$$
A = \int_0^{l/2} dx \frac{R}{z} \sqrt{1 + \frac{z'^2}{f}}
$$

Then, the subsystem size and the entanglement entropy \overline{h} ns of th .
... can be rewritten in terms of the turning point ✏ *z* ⁰ *z*² $T_{\text{HUI}, \text{ the subsystem size and the triangular entropy} = P$ *P* entanglement entropy *z*₂ *Figure 1, Then, the subsystem size and the entanglement entropy* can be rewritten in terms of the turning point \overline{y} (32) \overline{y} (32) \overline{y} *p*_{*i*} (3)</sub> $\frac{1}{2}$ *z* tropy \overline{A}

$$
z_0 = z_h \tanh\left(\frac{l}{2z_h}\right),
$$

\n
$$
S_E = \frac{1}{2G} \log \frac{2z_0}{\epsilon} - \frac{1}{4G} \log \left(1 - \frac{z_0^2}{z_h^2}\right).
$$

divergence. The coecient of the logarithmic term is associated with the central charge of the dual charge o

This is an exact and analytic result. *^z*⁰ ⁼ *^z^h* tanh ✓ *^l* ⁴*^G* log ✓ *z* an exact and analy

When $z_h \to \infty$, When $z_h \to \infty$,

$$
z_0 = l/2
$$

$$
S_E^0 = \frac{1}{2G} \log \frac{l}{\epsilon}
$$
 (ground state entanglement entropy, UV divergence)

In the UV region Holographic renormalization and entanglement entanglement entropy in the second entropy of \mathbb{R}^n

Thermodynamics-like law of the entanglement entropy

 $T_E S_E = E$ with $T_E = \frac{6}{\pi l}$. *.* (40) features of strongly interacting systems. In this lecture, I will discuss how to extract various informa-

- This relation is defined in the UV region with neglecting higher order corrections.
- *h h* - It reproduces the linearized Einstein equation of the dual geometry.
- It is not valid in the IR region. the entanglement entropy and entanglement temperature in the UV region have nothing to with *^Sth* ⁼ ¹

linearized geometry only from CFT data. To go beyond the linearized level, it would be interesting In order to go beyond the linearized lever and to describe the RG flow correctly, consents involving all higher order corrections, we need generalized concepts involving all higher order corrections. *^E* ⁼ ¹ *.* (1)

In order to account for the connection between the entanglement and thermal entropies, we should We define a generalized thermodynamics-like law and generalized entanglement α all becomes ander compation and optioning in the outing nacion temperature involving all higher order correction and satisfying in the entire region $\frac{1}{2}$ *l z*2 law and generalized entanglement This internal energy satisfies

$$
\bar{T}_E \,\,\bar{S}_E = \bar{E}
$$

Define a renormalized entanglement entropy (subtracting the ground state EE) In order to get rid of the UV divergence, we introduce a renormalized entanglement entropy by \mathbf{S} subtracting the ground state entanglement entropy \mathbf{S} *^S*¯*^E* ⌘ *^S^E ^S*⁰

$$
\bar{S}_E \equiv S_E - S_E^0.
$$

Then, the exact renormalized EE and a generalized entanglement temperature $\frac{1}{2}$ \mathcal{C} renormalized entanglement entropy becomes in terms of the subsystem size in terms of the t renormalized EE and a generalized entanglement temperature

$$
\bar{S}_E = \frac{1}{2G} \log \left(\frac{2z_h}{l} \sinh \left(\frac{l}{2z_h} \right) \right)
$$

$$
\frac{1}{\bar{T}_E} \equiv \frac{1}{2} \frac{\bar{S}_E}{\bar{E}} = \frac{4\pi z_h^2}{l} \log \left(\frac{2z_h}{l} \sinh \left(\frac{l}{2z_h} \right) \right)
$$

$$
S_E = \frac{1}{2G} \log \left(\frac{\beta}{\pi \epsilon} \sinh \left(\frac{\pi l}{\beta} \right) \right)
$$

In the UV region (
$$
l/z_h \ll 1
$$
),
\n
$$
\bar{S}_E = \frac{1}{48G} \frac{l^2}{z_h^2} \left(1 - \frac{l^2}{120z_h^2} + \cdots \right)
$$
\nwith the previously temperature
\n
$$
\frac{1}{\bar{T}_E} = \frac{1}{T_E} \left(1 - \frac{l^2}{120z_h^2} + \cdots \right)
$$
\n
$$
\text{defined entanglement} \qquad T_E = \frac{6}{\pi l}.
$$

Ignoring l^2 order corrections, they are reduced to the known results. **EXPLICITE:** Note that the generalized entanglement temperature was defined to satisf \overline{a} $\overline{}$ *E E <i>E* ly with involving all high Ignoring l^2 order corrections, they are reduced to the known results. There that the generalized entanglement temperature was defined to satisfy the thermodynamics-like law exactly with involving all higher order correction. Therefore, the canal apply the thermodynamics-like law to the IR entanglement entropy. ² when the subsystem size increases, whereas the thermal entropy the entanglement entropy and entanglement temperature in the UV region have nothing to with Note that the generalized entanglement temperature was defined to satisfy the H_h grows by *l*. Comparing these result with the black hole quantities in (26), one can easily see that we can apply the incrinou ynannes-like law to the IR changlement we can apply the thermodynamics-like law to the IR entanglement entropy. $\frac{1}{2}$ order corrections they are reduced to the known results In order to remove the UV divergence, we introduce a renormalized entanglement entropy by s row that the generalized entanglement temperature *S*^E *E*^{*E*} *S*^E *S*² *S*² *C*_{*E*} *S*² *C*_{*E*} *S*² *S*² *C*_{*E*} *S*² *C*_{*E*} *S*²

tion of a strongly interactive of the holographic entanglement entropy reviewing the holographic technique. After briefly reviewing the holographic entanglement entropy 2. Universality, of the halographic entered was at entered universal features of strongly interacting systems. In this lecture, I will discuss how to extract various informa-Holographic renormalization and entanglement entropy **3. Universality of the holographic entanglement entropy 3.** Universality of the holographic entanglement entropy

Consider a Lifshitz geometry holographic renormalization and entanglement entropy, I will discuss their relation and RG flow. The rest of a strongly rest of the paper is organized as follows: In Sec. 2, we represent the paper is organized as follows: In Sec. 2, we represent the paper is organized as follows: In Sec. 2, we represent the paper is o $\mathcal{P}(\mathcal{A}|\mathcal{A})$ and the electric property of the non-relativistic medium significant lifetimes $\mathcal{P}(\mathcal{A})$ Chanyong Park*a,b*⇤

$$
ds^2=\frac{1}{z^2}\left(-\frac{1}{z^{2(\bar z-1)}}dt^2+\delta_{ij}dx^idx^j+dz^2\right)
$$

which is dual of a Lifshitz field theory and allows the following <u>scale symmetry</u>
 $Q_{\vec{k}}$ \vec{k} \vec{k} \vec{k} $Q_{\vec{k}}$ \vec{k} $1e$ which is dual of a Ensing flow theory and anows the foll which is dual of a Lifshitz field theory and allows the following seale symmetry which is data of a Lifshitz meta theory and anows the renowing <u>beare symmetry</u> which is dual of a Lifshitz field theory and allows the follow

$$
z\to \Omega z\,\,,\,\, t\to \Omega^{\bar z} t\,\,,\,\, x^i\to \Omega x^i
$$

- When the dynamical critical exponent is 1 ($\bar{z} = 1$), the Lifshitz geometry reduces to the AdS one. $\mathcal{L}^{\mathcal{L}}$ - When the dynamical critical exponent is $1(z = 1)$, the Lifs There exists we have the Lorentz invariant field the Lorentz invariant field $\frac{1}{2}$
	- For $\bar{z} = 2$, the dual field theory corresponds to a non-relativistic Lifshitz theory $S[\chi] = \int d^3x$ $\left(\partial_t \chi\right)^2 - K \left(\nabla^2 \chi\right)^2$ \sim For $z = 2$, the dual field theory corresponds to a non-relativistic Effshite ✓ ◆ $\frac{1}{2}$.
Te $\overline{\mathbf{u}}$
	- Due to this scale symmetry, the entanglement temperature behaves as - Due to this scale symmetry, the entanglement temperature

$$
T_E = \frac{dE}{dS_E} \sim \frac{1}{l^{\bar{z}}}
$$

l : subsystem size

In the IR region $(z_0 \approx z_h)$, λ (1) \sim 2h λ ,

Regardless of the dimensionality and microscopic detail of the dual field theory, *<u>Isionality and micr</u> z*2 ess of the dimensionality and microscopic detail of the dual field theory,

the IR entanglement entropy reduces to *T*
the IR entanglement entropy reduces to the first law of thermodynamics, *dE* = *THdSth*, leads to the following internal energy

$$
\bar{S}_E=S_{th}+S_q
$$

 S_{th} : Universal

: depending on the dual theory \overline{e} $\overline{$ $\frac{1}{2}$ **l** $\frac{1}{2}$ S_q : depending on the dual theory

For a two-dimensional scale invariant theory *S*¯*^E* = *Sth* + *Scorrection* (3) *^RMN* ¹

$$
S_q \sim -\log S_{th}
$$

 \mathbf{S} **S** \mathbf{S} (\mathbf{D} anton class out outgoing proposed from the hele of Intriguingly, the universality of the IR entanglement entropy proposed from the holography *R R* entanglement entropy proposed from the holog *n* $\frac{d}{dx}$ *R R a H R entanglement entropy proposed from the holog*

$$
\bar{S}_E \approx S_{th}
$$

 \overline{a} occurs in the real space renormalization group flow of the lattice theory (Ising model).

Extending Universality of the IR entanglement entropy with the theory with t near ↵² = 0.

A general (d+1)-dimensional gravity theory

$$
ds^{2} = \frac{1}{z^{2}} \left(-e^{2A(z)} f(z) dt^{2} + e^{2B(z)} \delta_{ij} dx^{i} dx^{j} + \frac{e^{2C(z)}}{f(z)} dz^{2} \right)
$$

we can set $e^{2C(z)} = 1$ without loss of generality $\mathbf v$

- previous sections. The dual field theory is **conformal**. • For $e^{2A(z)} = e^{2B(z)} = 1$, the metric reduces to that of the AdS black hole studied in the
- For $e^{2A(z)} \neq e^{2B(z)} = 1$, it reduces to the Lifshitz black hole which breaks the boost symmetry in the $t - x^i$ plane. The resulting dual field theory is <u>a non-relativistic field</u> theory with a scale invariance. $\frac{1}{2}$ $\frac{1}{2}$ invariance [56, 57, 58].
- For $e^{2A(z)} = e^{2B(z)} \neq 1$, it leads to a black hole on the hyperscaling violation geometry which has no scale symmetry. The dual field theory can be identified with a relativistic quantum field theory without a scale symmetry. has no scale symmetry. The dual field theory can be identified with a relativistic $\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt{5}}$
- For $e^{2A(z)} \neq 1$, $e^{2B(z)} \neq 1$ and $e^{2A(z)} \neq e^{2B(z)}$, it is the combination of the previous two cases. In this case, the scale and boost symmetry are broken and the dual field theory is given by <u>a non-relativistic theory without a scale symmetry</u>. • For $e^x \neq 1$, $e^x \neq 1$ and $e^x \neq e^x$, it is the combination of the previous cases. In this case, the scale and boost symmetry are broken and the dual field the In this case, the scale and boost symmetry are broken and the dual field theory given by <u>a non-relativistic theory with</u>

the black hole horizon. For convenience, the black hole factor can be further rewritten as the following *<u><i>For a black hole with a simple root</u>*</u> Using the conserved the conserved by the translation of the translation of the *x*-direction, the width of width o

$$
f(z) = \left(1 - \frac{z}{z_h}\right) F(z),
$$

where *F*(*z*) must be regular for 0 *z z^h* and approaches to 1 as *z* ! 0. The other unknown the Bekenstein-Hawking entropy from the area law

$$
S_{th} = \frac{V_{d-1}}{4G} \frac{e^{(d-1)B(z_h)}}{z_h^{d-1}}
$$

The entanolement entropy in a strip-shaped region **The entanglement entropy in a strip-shaped region**

$$
l = 2 \int_0^{z_0} dz \frac{z^{d-1} e^{(d-1)B_0}}{e^B \sqrt{f} \sqrt{e^{2(d-1)B} z_0^{2(d-1)} - e^{2(d-1)B_0} z^{2(d-1)}}},
$$

$$
S_E = \frac{L^{d-2}}{2G} \int_0^{z_0} dz \frac{z_0^{d-1} e^{(2d-3)B_0}}{z^{d-1} \sqrt{f} \sqrt{e^{2(d-1)B} z_0^{2(d-1)} - e^{2(d-1)B_0} z^{2(d-1)}}},
$$

In the IR region where *B*⁰ implies the value of *B*(*z*) at *z* = *z*0. Here the range of the turning point is restricted to 0 *z*⁰ *z^h* and 1*/z* corresponds to the energy scale of the dual QFT. This relations imply that *z*⁰ = 0 In the IR region

$$
S_E = \frac{lL^{d-2}}{4G} \frac{e^{(d-1)B_0}}{z_0^{d-1}} + \frac{L^{d-2}}{2Gz_0^{d-1}} \int_{\epsilon}^{z_0} dz \frac{\sqrt{e^{2(d-1)B}z_0^{2(d-1)} - e^{2(d-1)B_0}z^{2(d-1)}}}{z^{d-1}e^B\sqrt{f}}.
$$

The entanglement entropy in a ball-shaped region **The entanglement entropy in a ball-shaped region** entries and the entries of the state of t

$$
S_E = \frac{\Omega_{d-2}}{4G} \int_0^l d\rho \; \frac{e^{(d-2)B} \rho^{d-2} \sqrt{e^{2B} f + z'^2}}{z^{d-1} \sqrt{f}}.
$$

In the IR limit In the IR limit when we rewrite the state of \mathbb{R} in the following as the followin

$$
S_E = \frac{\Omega_{d-2}e^{(d-1)B_0}}{4Gz_0^{d-1}} \int_0^l d\rho \rho^{d-2} \n+ \frac{\Omega_{d-2}}{4Gz_0^{d-1}} \int_0^l d\rho \frac{\rho^{d-2} \left(z_0^{d-1} e^{(d-1)B} \sqrt{f + e^{-2B}z'^2} - z^{d-1} e^{(d-1)B_0} \sqrt{f}\right)}{z^{d-1}\sqrt{f}}.
$$

 $N_A = 2$ $N_A = 2$ with $V_{d-2} = \Omega_{d-2} \int_0^l d\rho \rho^{d-2}$

In the IR limit, the entanglement entropy reduces to the thermal entropy regardless of International results in the International results in the IR entangling surface Intriguingly, all results studied in this work show that the IR entanglement entropy reduces to the microscopic detail and the shape of the entangling surface.

4. Discussion

We showed that the quantum entanglement entropy evolves into the thermal entropy along the RG flow regardless of the microscopic details

- The thermodynamics-like law of the entanglement entropy leads to the exact thermodynamic law in the IR limit.
- The universal feature of the IR entanglement entropy has been shown in the holographic setup. Therefore, it would be interesting to check this IR universality also occurs in a quantum field theory (or lattice model).

