Chiral Random Matrix Theory
and
(non-) QCD like theory

— matrix model as a 1-dim lattice simulation —

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based on on-going work with C.-J. David Lin (NCTU)

Sep. 19, 2017, APCTP, Pohang:
“Discrete Approaches to the Dynamics of Fields and Space-Time”

I.K. is supported by Priority Issue 9 to be Tackled by Using Post K Computer

Outline

1. Motivation: Random Matrix Theory(RMT) and QCD
   conformal window, calculation of chial condensate...
2. Matrix Model as 1-dim Field Theory
   1-dim lattice simulation
3. Chiral Susceptibility — How far can RMT go?
4. Summary and Discussions

Motivation: QCD like or non QCD like

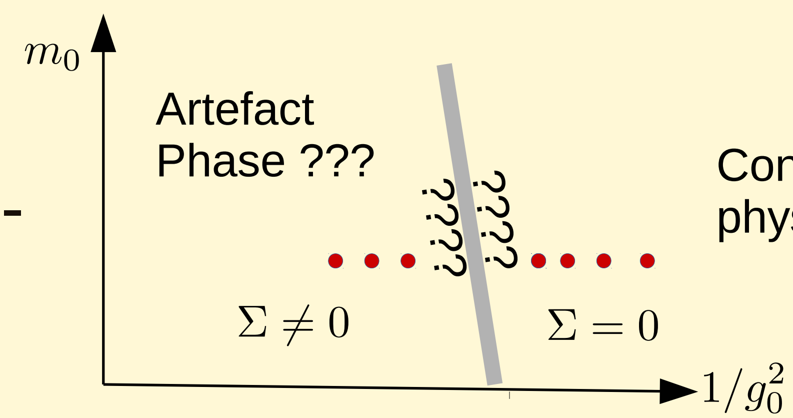
QCD

- Spontaneous breaking of chiral symmetry
  order parameter: chiral condensate
  $\Sigma = \frac{\partial}{\partial m} \ln Z|_{m=0} = \lim_{m\to 0} \lim_{V\to\infty} \langle \bar{\psi}\psi \rangle$
- finite Σ provides a typical scale of the theory

Non-QCD like (conformal window)

- No scale, No SSB of chiral symmetry
- may describe composite Higgs models
- lattice simulation: must be done in parameter region with vanishing Σ

Motivatoin 1:
to find a proper parameter region for SU(2) $n_f = 8$ system



Random Matrix Theory and QCD

we use RMT to extract finite chiral condensate

QCD in ϵ-regime : can be described Random Matrix Theory

- SSB of the chiral symmetry
- finite box: only the zero mode of the Goldstone boson (pion) is relevant

λ_i: eigenvalue of (massless) Dirac op.
ζ_i: eigenvalue of $N \times N$ random matrix

$$\boxed{Z_{\text{QCD}}(m_a, \{\lambda_i\}) = Z_{\text{RMT}}(\mu_a = V\Sigma m_a, \{\zeta_i = V\Sigma\lambda_i\})}$$

Correlation function of ζ_i
Correlation function of λ_i
} They are the same

useful to determine value of Σ

Random Matrix Theory and QCD, cont'd

Problem: not always easy to obtain the distribtuion of ζ_i

cf. P. H. Damgaard, S. M. Nishigaki RRD 63(2001) 045012

their anaylitic expression does no apply to our case

$\Rightarrow$we use numerical method

Question:

how much RMT (+ our numerical method) is useful?

(Our motivation 2)

Matrix Model as 1-dim Field Theory

Setup

$$Z_\nu^{(\beta)} = \int dW e^{-\beta \operatorname{tr} WW^\dagger} \prod_{a=1}^{n_f} \det \begin{pmatrix} m_a & W \\ -W^\dagger & m_a \end{pmatrix}$$

β: Dyson index, ν: topological charge, m_a: fermion mass
W: $N \times (N+\nu)$ matrix

Matrix integration $\Rightarrow$ integration of eigenvalues

$$Z_\nu^{(\beta)} = \int_0^\infty dx_1 \cdots \int_0^\infty dx_N \, \rho_\nu^{(\beta)}(\{x_i\}, \{m_a\})$$

$$\rho_\nu^{(\beta)}(\{x_i\}, \{m_a\}) = \left(\prod_{a=1}^{N_f} m_a\right)^\nu \prod_{i=1}^{N} \left[x_i^{\beta\frac{\nu+1}{2}-1} e^{-\beta x_i} \prod_{a=1}^{N_f} (x_i + m_a^2) \right] \prod_{i>j}^{N} \left| x_i - x_j \right|^\beta$$

microscopic limit

$N \to \infty$ with $x_i \to 0$, $m_a \to 0$ keeping the following finite

- $\zeta_i = \sqrt{8Nx_i} = V\Sigma\sqrt{x_i}$
- $\mu_a = \sqrt{8N}\, m_a = V\Sigma m_a$

(V: 4-volume, Σ: chiral condensate)

concrete expression: P. H. Damgaard, S. M. Nishigaki RRD 63(2001) 045012

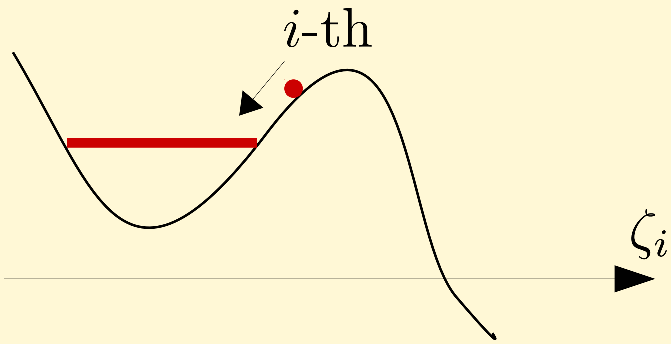
I am a Lattician...

physics: dynamics of eigenvalues

standard view point

cf. quantum mechanics

- ζ_i: location of the i-th charged particle in the given potentail
  Vandermond: $\ln|\zeta_i - \zeta_j|$ Coulomb potential in 2-dim.



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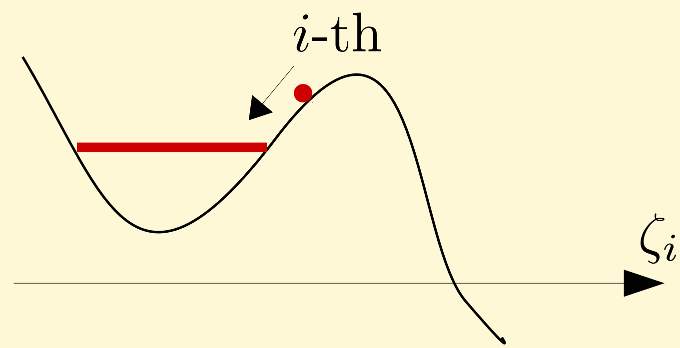
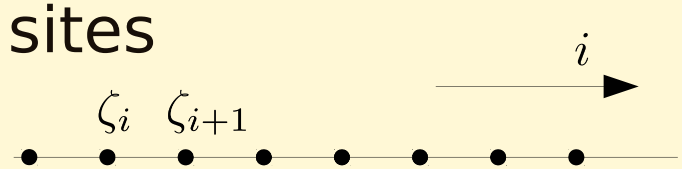


Lattician's view — cf. 1-dim field theory

- value of the real scalar field at site *i*
- field theory on 1-dim lattice with *N* sites



Lattice Simulation

action: $\rho = \exp(-S_\nu^{(\beta)})$

$$S_\nu^{(\beta)} = \sum_{i=1}^{N}\left[\frac{\beta}{8N}\zeta_i^2 - \frac{\beta(\nu+1)-1}{2}\ln\zeta_i^2 - \sum_{a=1}^{N_f}\ln(\zeta_i^2+\mu_a^2) - \beta\sum_{j=1}^{i-1}\ln|\zeta_i^2-\zeta_j^2|\right].$$

- no sign problem
- additional constraints: $0 < \zeta_1 < \zeta_2 < \cdots < \zeta_N$
  infinity high potential barrier to flip this ordering
  $\ln|\zeta_i - \zeta_j|$, $\ln|\zeta_i|$
- straightforward to implement standard Hybrid Monte Carlo (HMC)
- [for HMC experts] if the above ordering is violated in the Molecular Dynamics, retry the same trajectory with a smaller MD step $\delta\tau$ (if the violation survives after several retrying, reject the trial configuration)

Observables

We know distribution of the *all* eigenvalues!

- distribution of $^\forall i$-th eigenvalue
- any function made of eigenvalues
  - examples: later
- brute force but applicable to other matrix models

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take home message

Numerically, it is easy to study (one-)matrix model in quantum level

Sanity check with Chiral Condensate

$$Z_{\text{QCD}}(m_a, \lambda_i) = Z_{\text{RMT}}(\mu_a = V\Sigma_{\text{param.}} m_a, \zeta_i = V\Sigma_{\text{param.}} \lambda_i)$$

is $\Sigma_{\text{param.}}$ the same as Σ for QCD ?

Σ for QCD form the partition func.:

$$\Sigma_a^{(\text{QCD})} = -\frac{1}{V}\frac{\partial}{\partial m_a} \ln Z_{\text{QCD}} = \Sigma_{\text{param.}} \underbrace{\left(-\frac{\partial}{\partial \mu_a} \ln Z_{\text{RMT}}\right)}_{\text{is this 1?}}$$

Sanity check with Chiral Condensate, cont'd

we want check: $-\frac{\partial}{\partial\mu_a}\ln Z_{\mathrm{RMT}} = \left\langle \sum_i \frac{2\mu_a}{\zeta_i^2+\mu^2} \right\rangle_{\mathrm{RMT}} \overset{??}{=} 1$

delicate but important point: eingenvalue density at 0

Sanity check with Chiral Condensate, cont'd

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Banks-Casher relation

$\Sigma^{\rm (QCD)} = \lim_{m\to 0}\lim_{V\to\infty}\int_0^\infty d\lambda\rho(\lambda)\frac{2m}{\lambda^2+m^2} = \pi\rho(0)$

Ordering of the limits is crucial to obtain "$\rho(0)$":

$$\rho(0) = \lim_{\epsilon\to 0}\rho_{V=\infty}(\epsilon) = \lim_{\epsilon\to 0}\lim_{V\to\infty}\rho_V(\epsilon) = \text{finite}$$

cf. $\lim_{\epsilon\to 0}\rho_V(\epsilon) = 0$

regularization (QCD):

$\int_0^\infty d\lambda\rho(\lambda)\frac{2m}{\lambda^2+m^2}$

$$\Rightarrow \int_0^\infty d\lambda\rho(\lambda+\epsilon)\frac{2m}{\lambda^2+m^2} = \int_\epsilon^\infty d\lambda\rho(\lambda)\frac{2m}{(\lambda-\epsilon)^2+m^2} = \pi\rho(\epsilon)$$

Sanity check with Chiral Condensate, cont'd

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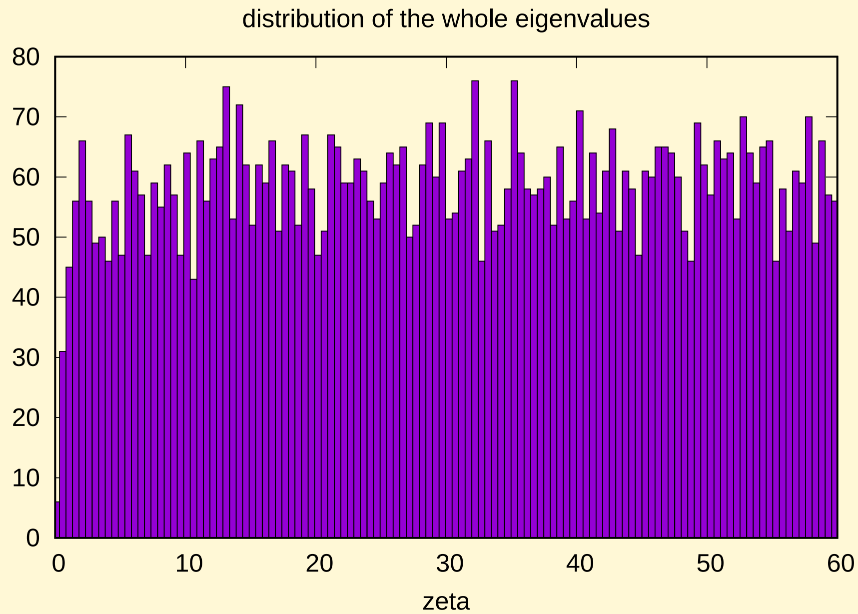
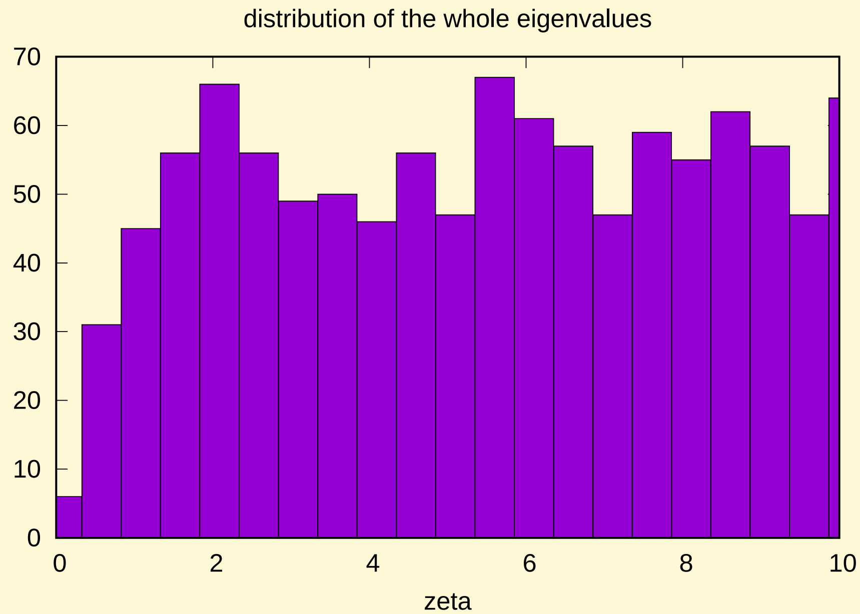
natural regularization for RMT

$$-\frac{\partial}{\partial\mu_a} \ln Z_{\mathrm{RMT}} = \left\langle \sum_i \frac{2\mu_a}{\zeta_i^2+\mu^2} \right\rangle_{\mathrm{RMT}} \Rightarrow \left\langle \sum_{\zeta_i \geq \epsilon} \frac{2\mu_a}{(\zeta_i-\epsilon)^2+\mu^2} \right\rangle_{\mathrm{RMT}} \stackrel{??}{=} 1$$

Sanity check with Chiral Condensate: result

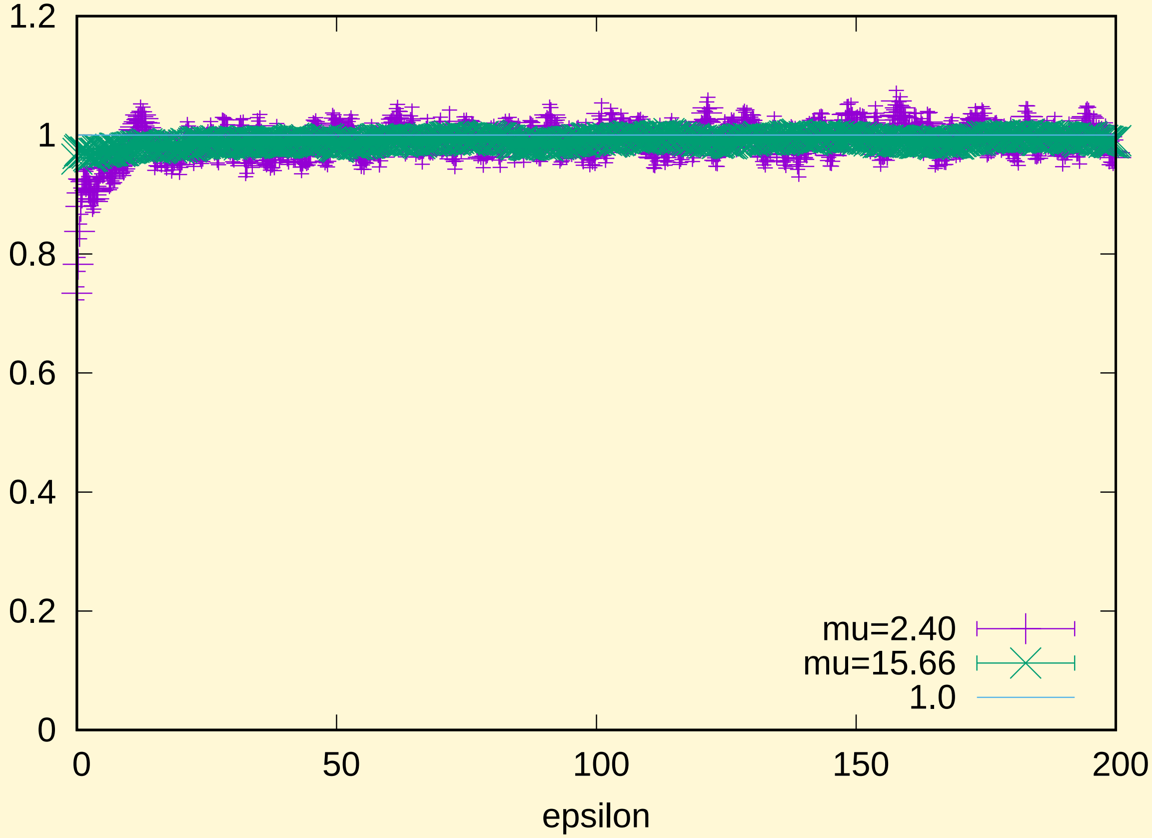
example of $\rho(\zeta_i)$

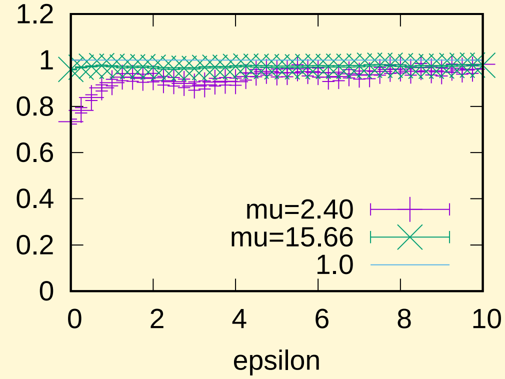




- $N = 1000$: $\lambda_i = \zeta_i/\sqrt{8000} = 0.011 \times \zeta_i$

Sanity check with Chiral Condensate: result





- almost one
- slighitly off for small ϵ

Chiral Susceptibility — How far can RMT go?

Use the equivalence of the partition function:

$$Z_{\mathrm{QCD}}(m_a,\lambda_i)=Z_{\mathrm{RMT}}(\mu_a=V\Sigma_{\mathrm{param.}}m_a,\zeta_i=V\Sigma_{\mathrm{param.}}\lambda_i)$$

susceptibility :

$$\chi_a=\frac{\partial}{\partial m_a}n_f^{(a)}\Sigma_{\mathrm{QCD}}^{(a)}=n_f^{(a)}V\Sigma_{\mathrm{param}}^2\frac{\partial}{\partial\mu_a}\left\langle\sum_{\zeta_i\geq\epsilon}\frac{2\mu_a}{(\zeta_i-\epsilon)^2+\mu_a^2}\right\rangle_{\mathrm{RMT}}$$

$$=n_f^{(a)}V\Sigma_{\mathrm{param}}^2\Big\{\langle A(\epsilon)\rangle_{\mathrm{RMT}}$$

$$+n_f^{(a)}(\langle B(\epsilon)B(\epsilon)\rangle_{\mathrm{RMT}}-\langle B(\epsilon)\rangle_{\mathrm{RMT}})\langle B(\epsilon)\rangle_{\mathrm{RMT}})\Big\}$$

with

$$A(\epsilon)\equiv\sum_{\zeta_i\geq\epsilon}\left[\frac{2}{(\zeta_i-\epsilon)^2-\mu_a^2}-\frac{4\mu_a^2}{\left((\zeta_i-\epsilon)^2+\mu_a^2\right)^2}\right]$$

$$B(\epsilon)\equiv\sum_{\zeta_i\geq\epsilon}\frac{2\mu_a}{(\zeta_i-\epsilon)^2+\mu_a^2}$$

cf. another path is possible:

equivalence of the correlation func. of eigenvalues

field theoretical expression $\Rightarrow$ rewrite with λ_i

reproducing QCD data

lattice data: A. Bazavov et al. (HotQCD collab) PRD 85, 054503 (2012)

Lattice: $48^3 \times 12$ latt., $n_f = 2 + 1$ staggered quarks
disconnected part of the susc. for light quark (dominant cotnrib., no additive renorm.)

Input for RMT

- $\beta = 2$
- $\nu = 0$ (may not be very correct but $\nu = 0$ becomes dominant in high temp.)
- lattice volume
- quark masses (light (=u, d) and s)
- chiral condensate for light quark: $\langle \bar{\psi}_l \psi_l \rangle$

Output

- chiral susceptibility χ_l

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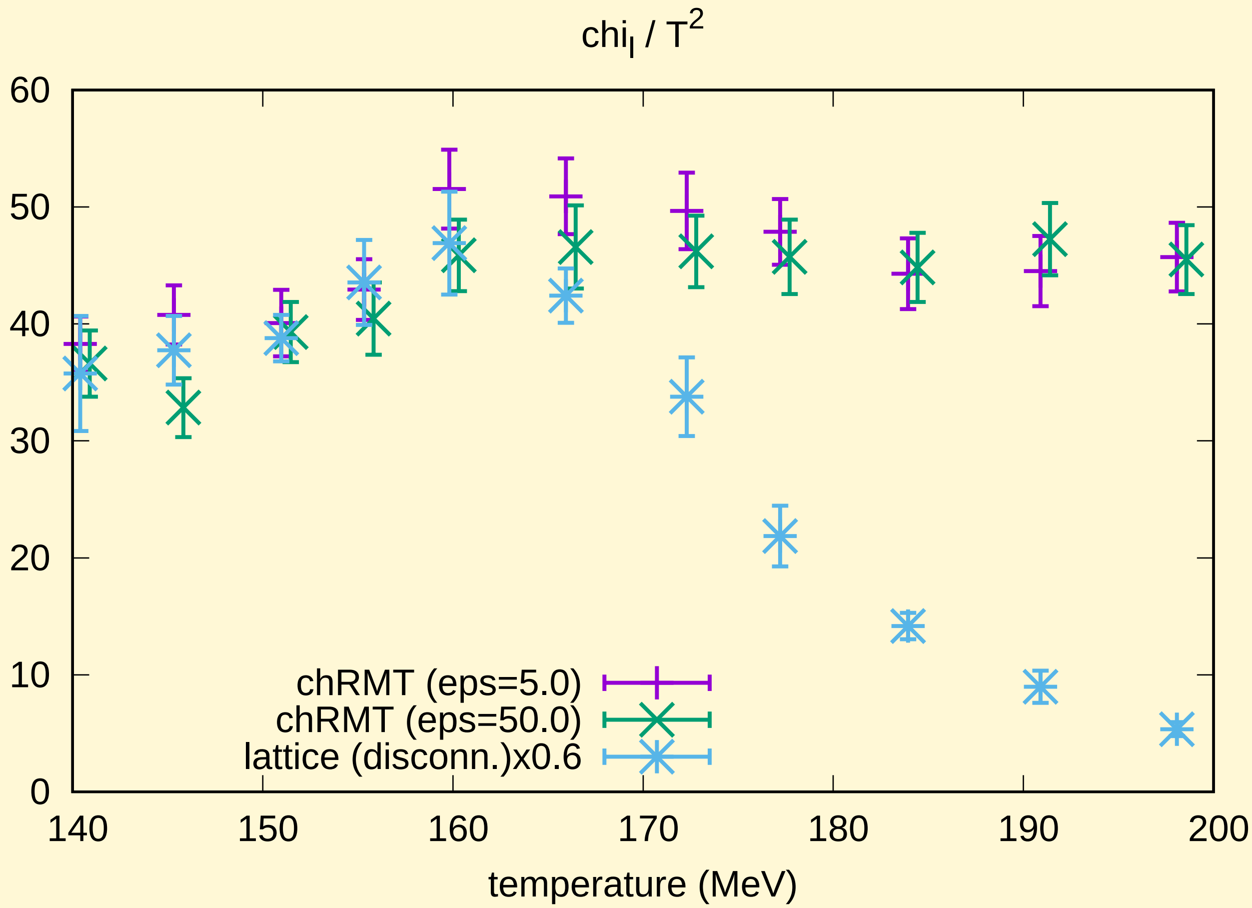

in high temp., chiral sym. is restored so RMT does not apply.

$SU(2)$ $n_f = 8$ system: estimate of the condensate

C. Y.-H. Huang, I.K, C.-J. D. Lin, K. Ogawa, H. Ohki, A. Ramos and E. Rinaldi

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Use the equivalence of the *eigenvalue distribution*

$$\underbrace{\rho_{\mathrm{QCD}}(\lambda_i; m)}_{\text{known from lattice sim.}} = \rho_{\mathrm{RMT}}(\zeta_i; \mu)\Big|_{\zeta_i = \Sigma V \lambda_i, \mu = \Sigma V m}$$

- known: V, m
- unknown: Σ — 1 param. fit

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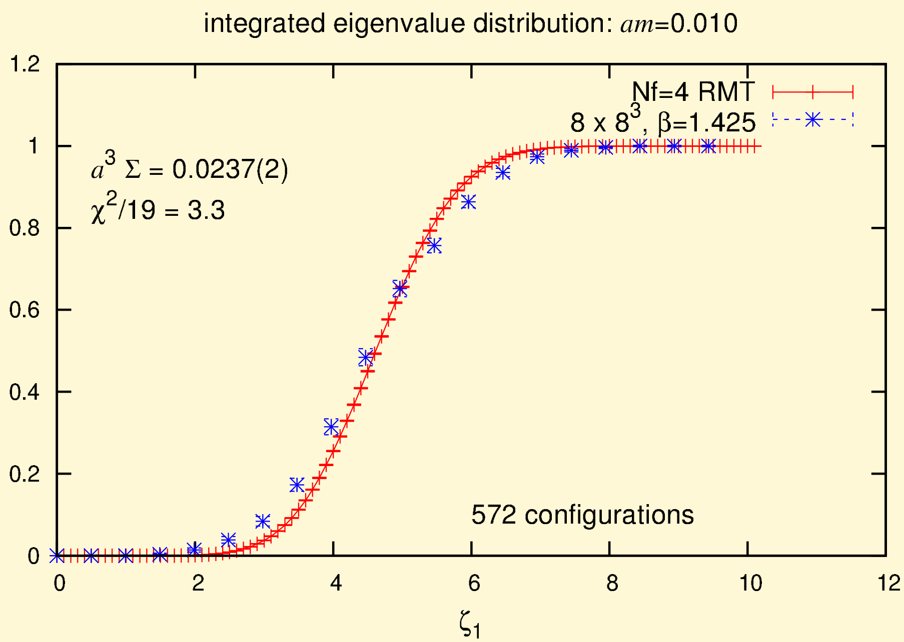
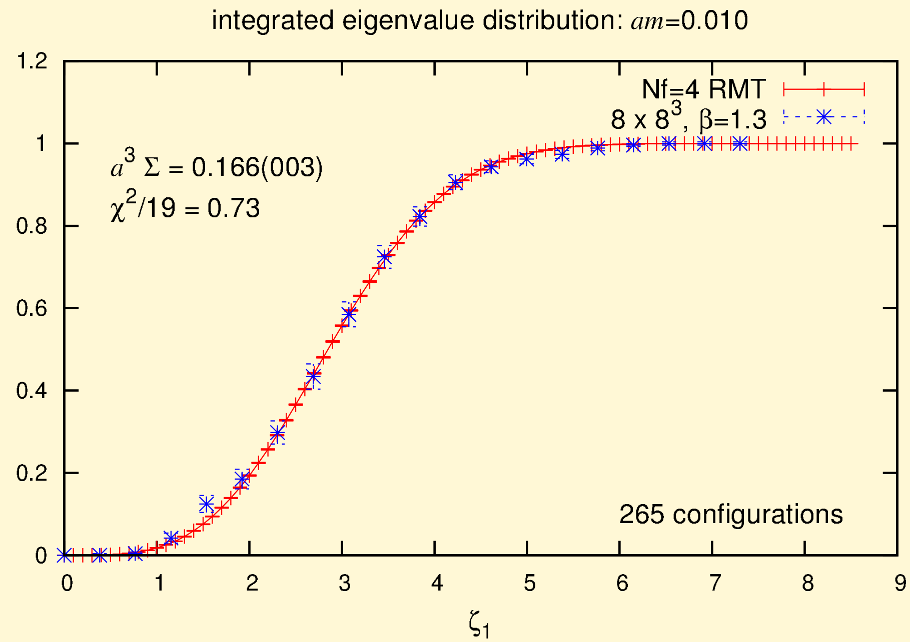
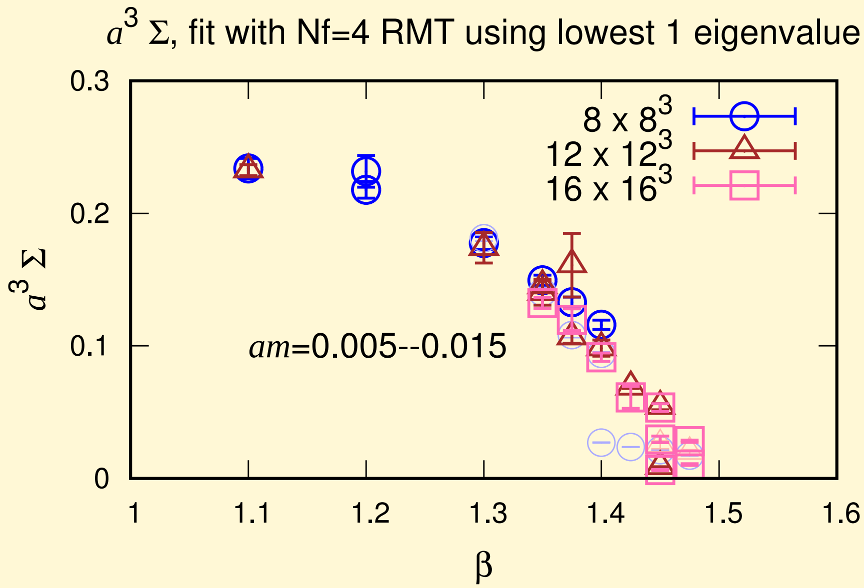




bad fit: chiral symmetry restored, $\Sigma \sim 0$

good fit: broken phase, $\Sigma \neq 0$

$SU(2)$ $n_f = 8$ system: chiral condensate and susceptibility



- Σ disappears for $\beta = 4/g_0^2 \gtrsim 1.45$
- almost no volume (inc. "temperature") dep.
  $\Rightarrow$bulk transition? order?
  (V dep. of susc. gives information for the order)

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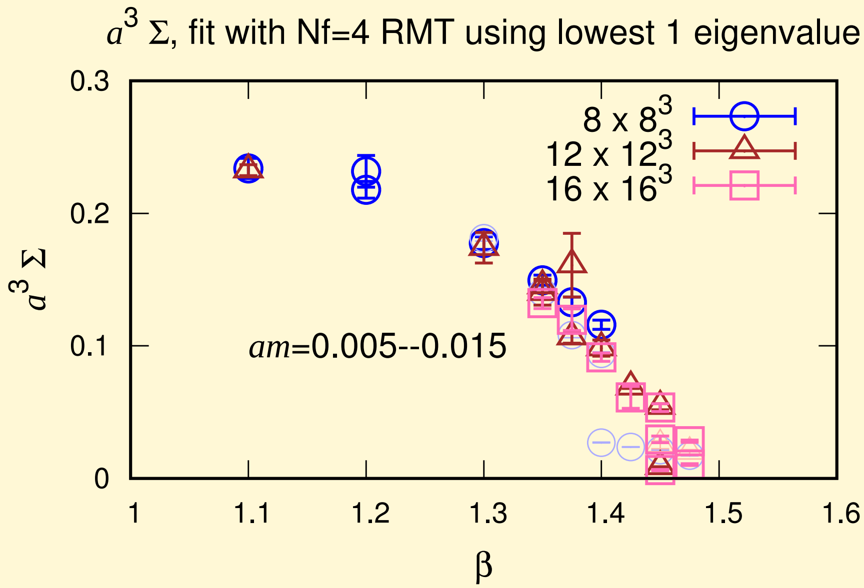

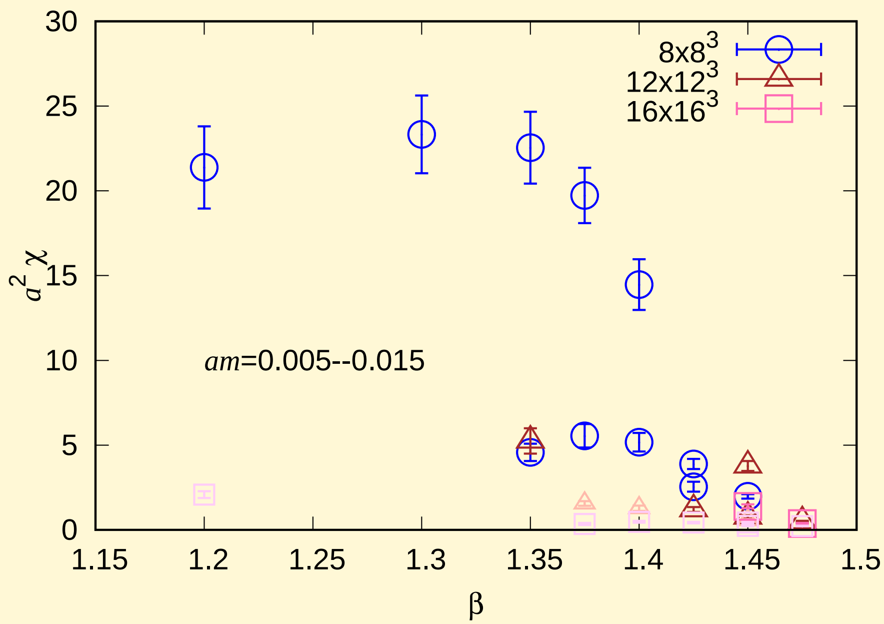

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- almost no volume (inc. "temperature") dep.
  $\Rightarrow$bulk transition? order? (V dep. of susc. gives information for the order)
- susc. disappears in $V \to \infty$ limit!? strange scaling....
- only small μ corresponds to the epsilon-regime (small symb. has $\mu > 15$)

Summary and Discussions

Summary and Discussions

- treatment of RMT as a 1-dim field theory
- chiral susceptibility: applicable if $\Sigma \neq 0$

Discussions — simulation of matrix model

- applicable to general (1-) matrix model
- potential should be bounded from below (does not work with ϕ^3)
- free energy without classical approximation etc.