Lattice Theory and Graph Theory

- Supersymmetric Gauge Theory on the Graph -

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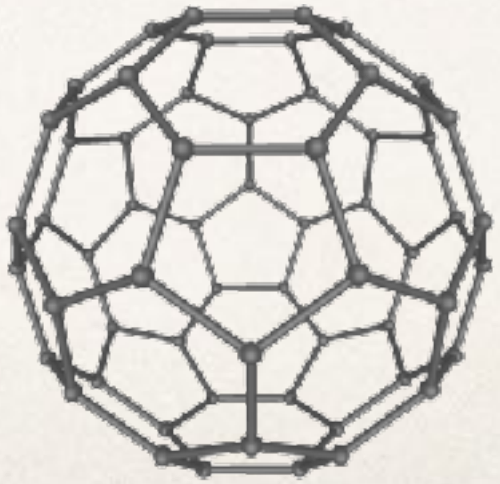
Based on:
S. Matsuura and T. Misumi, PTEP (2014) 123B01; PTEP (2015) 033B07,
S. Kamata, S. Matsuura and T. Misumi [arXiv:1607.01260]
and work in progress

Introduction

We show that a discretization of the supersymmetric 2d gauge theory can be constructed on generic graphs (polygons) ⇒ a generalization of the supersymmetric lattice gauge theory (the so-called Sugino model)

The graph theory is useful for formulating and manipulating this kind of the discretized gauge theory

on S^2



on a simplicial complex with the same Euler characteristics

$\chi_h = 2$

Introduction

We show:

- Graph theory is useful to formulate and analyze the model
- The zero mode and anomaly play important roles on the graph
- The integrable structure (localization property) still holds in the discretized theory

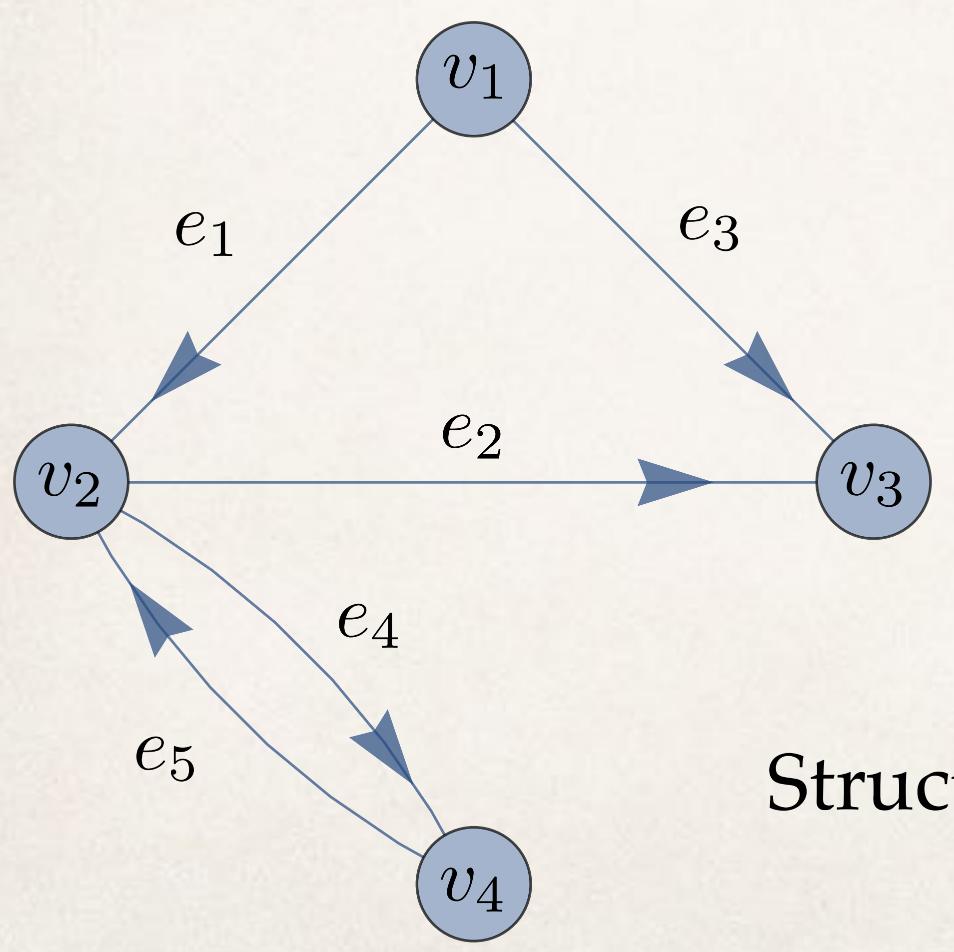
Today, I will explain how to apply localization method to the discretized gauge theories and give some exact results.

Quiver matrix model of the generic graph

= gauge theory on the discretized space-time

What is graph theory?

- The graph Γ consists of vertices (sites) and edges (links)
- We consider connected graphs with oriented edges (quiver diagram)



$$V(\Gamma) = \{v_1, v_2, v_3, v_4\} : \text{ a set of the vertices}$$
$$E(\Gamma) = \{e_1, e_2, e_3, e_4, e_5\} : \text{ a set of the edges}$$

$$(n_v = |V(\Gamma)| \text{ and } n_e = |E(\Gamma)|)$$

vertices ⇔ gauge groups
edges ⇔ bi-fundamental matters

Structure of the graph ⇒

- Adjacency matrix
- Incidence matrix
- Laplacian matrix

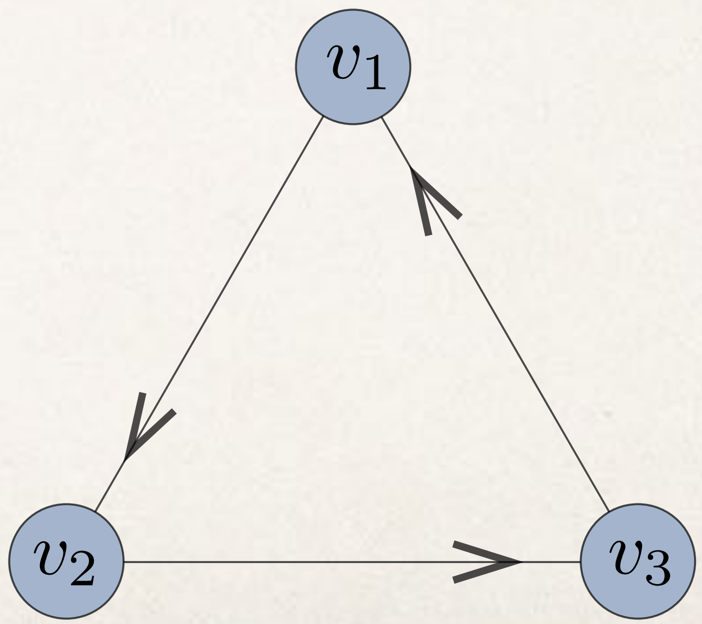
Adjacency matrix

- Adjacency matrix A: $V(\Gamma) \to V(\Gamma)$ ($n_v \times n_v$ matrix)

$$A_{ij} = \begin{cases} \sharp \text{ of edges from } v_i \text{ to } v_j \\ 0 \text{ otherwise} \end{cases}$$

e.g.

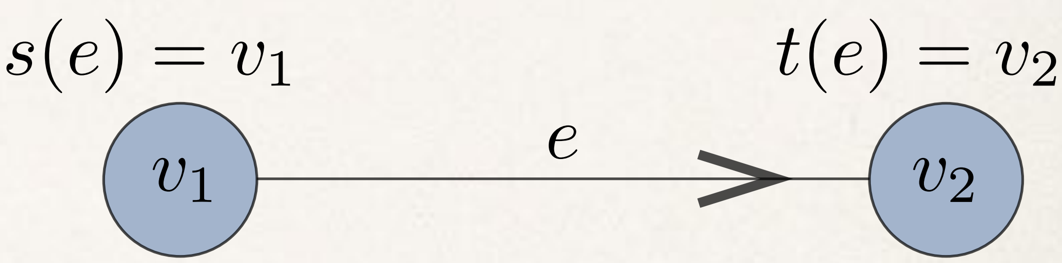
$$A(\Gamma) = \begin{matrix} & v_1 & v_2 & v_3 \\ v_1 & 0 & 1 & 0 \\ v_2 & 0 & 0 & 1 \\ v_3 & 1 & 0 & 0 \end{matrix}$$



Incidence matrix

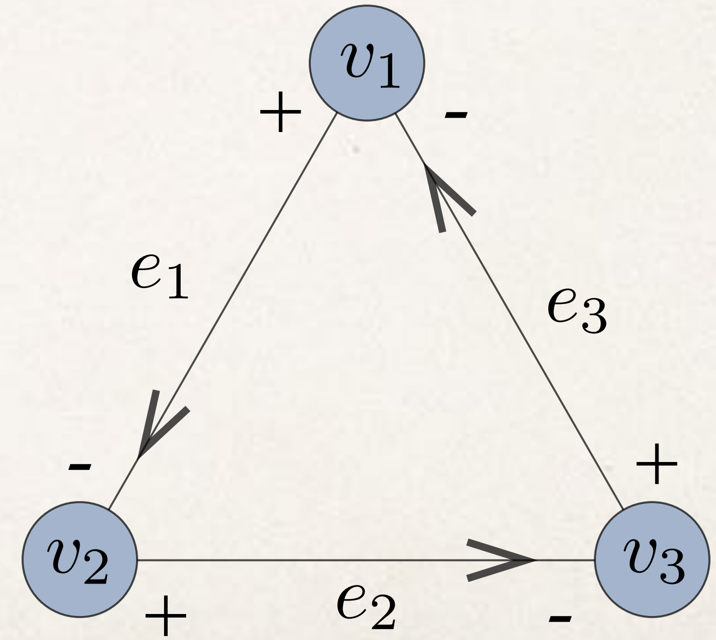
- Incidence matrix L: $V(\Gamma) \to E(\Gamma)$ ($n_e \times n_v$ matrix)

$$L_{li} = \begin{cases} +1 & \text{if } s(e_l) = v_i \\ -1 & \text{if } t(e_l) = v_i \\ 0 & \text{others} \end{cases}$$



e.g.

$$L(\Gamma) = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{pmatrix} +1 & -1 & 0 \\ 0 & +1 & -1 \\ -1 & 0 & +1 \end{pmatrix} \end{array}$$

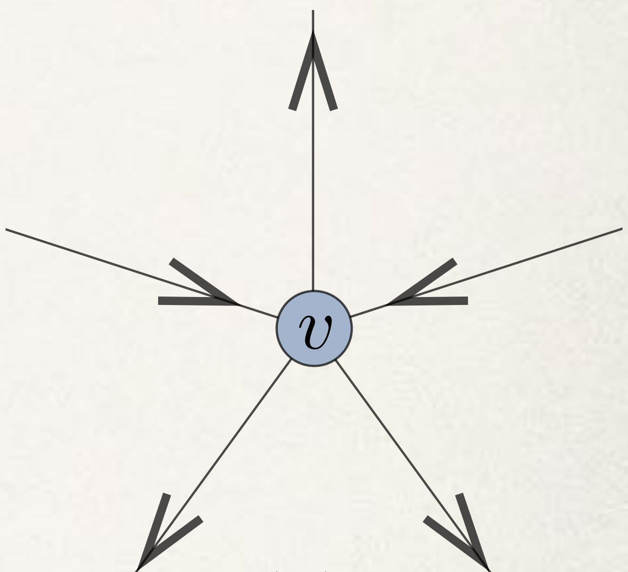


Known as charge matrix (toric data) for the bi-fundamental matters

Laplacian matrix

- Laplacian matrix $\Delta(\Gamma)$: $V(\Gamma) \to V(\Gamma)$ ($n_v \times n_v$ matrix)

$$\Delta_{ij} = \begin{cases} \deg(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$



$\deg(v) = 5$

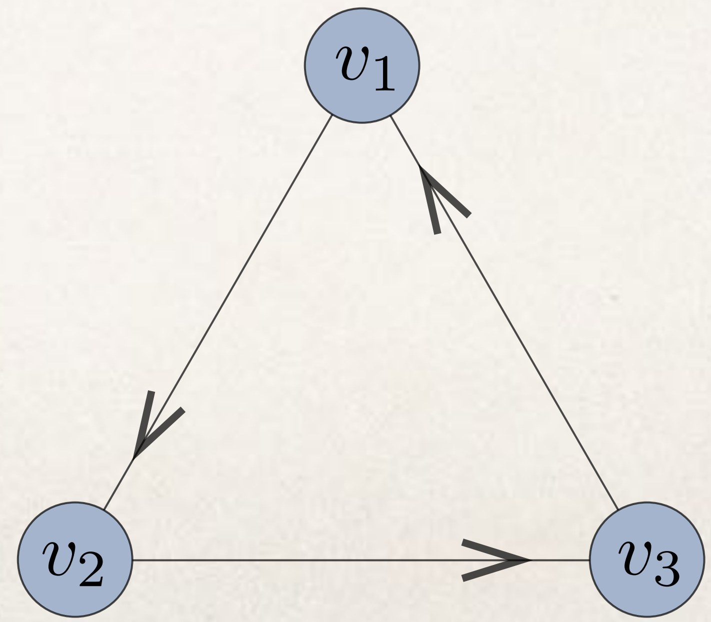
For $\vec{x}^T = (x_1, x_2, \ldots, x_{n_v})$

2nd order difference op.

$$\vec{x}^{\mathsf{T}} \Delta \vec{x} = \sum_{e \in E(\Gamma)} \left(x_{t(e)} - x_{s(e)}\right)^2$$

e.g.

$$\Delta(\Gamma) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \end{array}$$

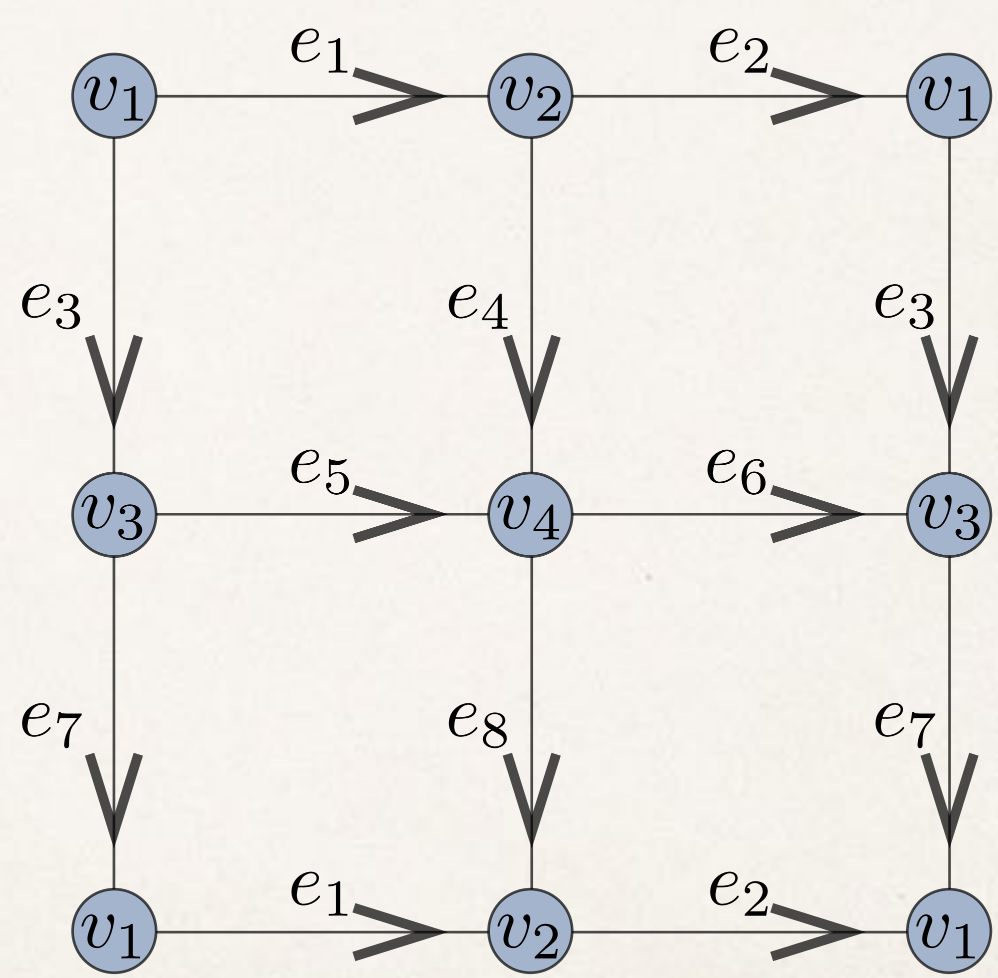


Examples: square lattice (torus)

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

$$\Delta = \begin{pmatrix} 4 & -2 & -2 & 0 \\ -2 & 4 & 0 & -2 \\ -2 & 0 & 4 & -2 \\ 0 & -2 & -2 & 4 \end{pmatrix}$$



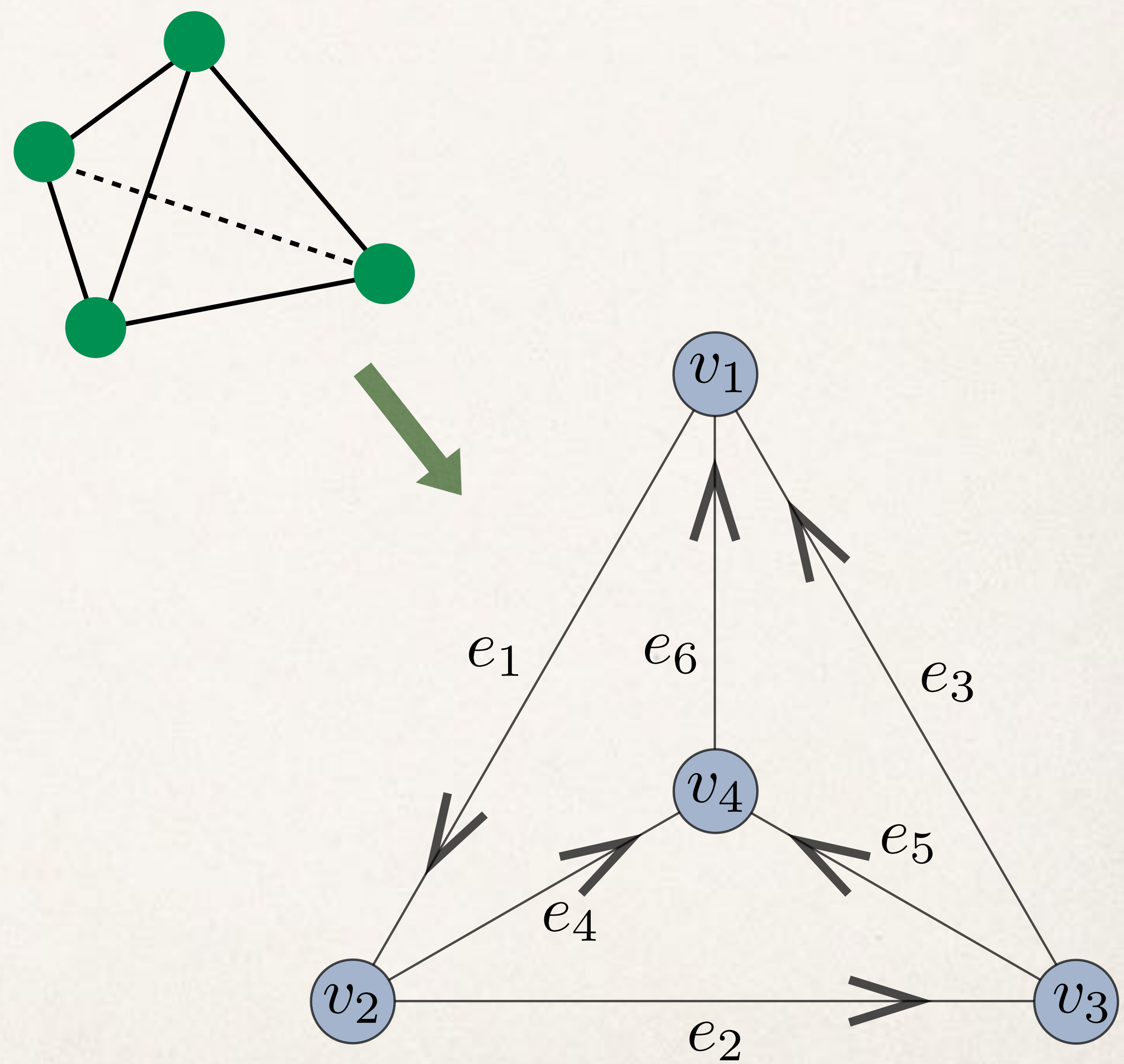
The same topology as T^2

Examples: tetrahedron (sphere)

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$\Delta = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$



The same topology as S^2

Relation of the matrices

- The adjacency, incidence and Laplacian matrices are related with each other by

degree matrix

$$\Delta = D - A - A^{\intercal}$$

adjacency matrix

Laplacian matrix

$$\Delta = L^{\intercal} L$$

incidence matrix

Abelian gauge theory

Let us now consider a simple Abelian gauge theory on the graph Γ. We define the following bosonic and fermionic vectors:

- Bosons on $V(\Gamma)$

$$\boldsymbol{\phi}^\intercal = (\phi_1, \phi_2, \ldots, \phi_{n_v})$$

$$\bar{\boldsymbol{\phi}}^\intercal = (\bar{\phi}_1, \bar{\phi}_2, \ldots, \bar{\phi}_{n_v})$$

- Bosons on $E(\Gamma)$

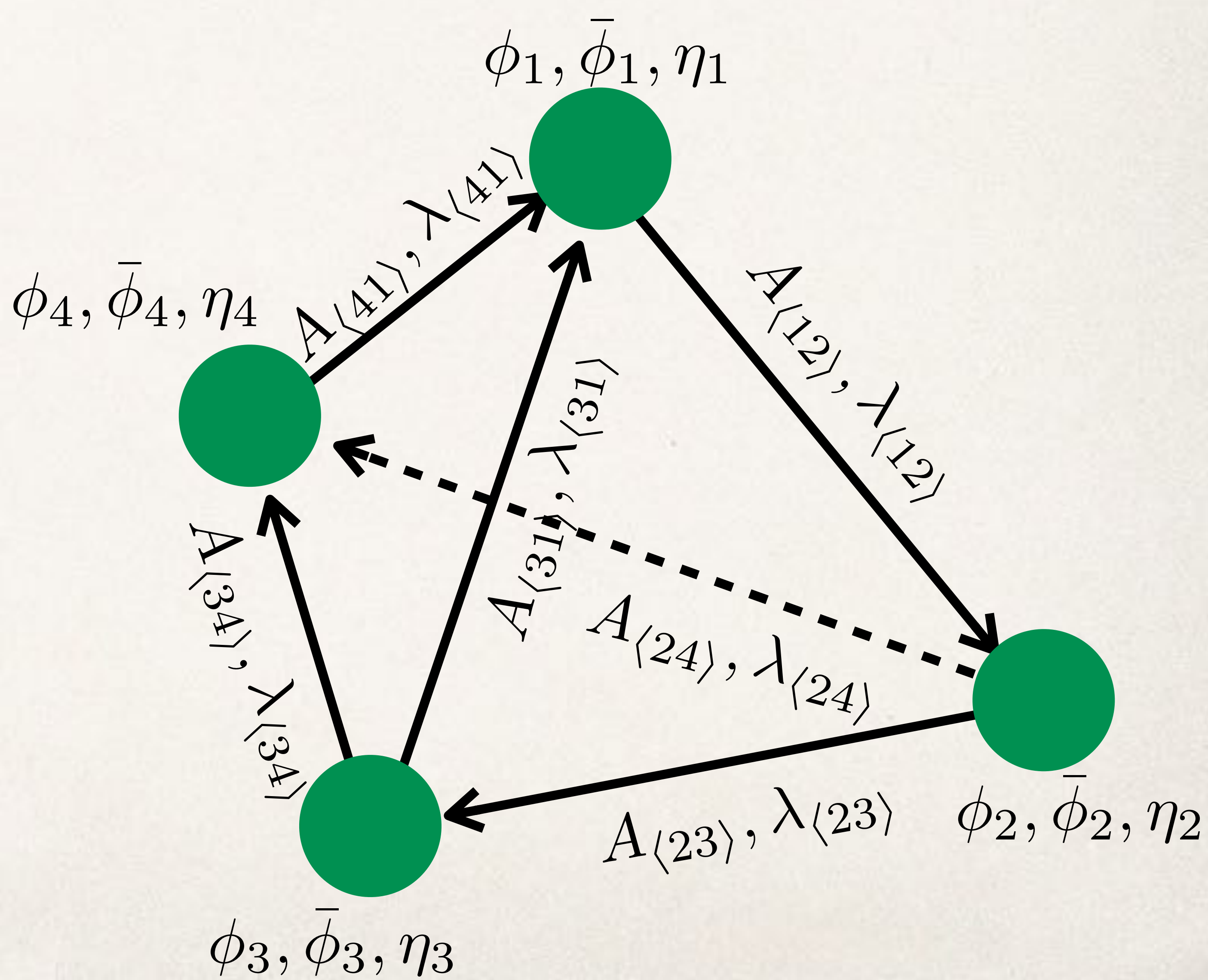
$$\boldsymbol{A}^\intercal = (A_1, A_2, \ldots, A_{n_e})$$

- Fermions on $V(\Gamma)$

$$\boldsymbol{\eta}^\intercal = (\eta_1, \eta_2, \ldots, \eta_{n_v})$$

- Fermions on $E(\Gamma)$

$$\boldsymbol{\lambda}^\intercal = (\lambda_1, \lambda_2, \ldots, \lambda_{n_e})$$



Faces and Plaquettes

We also need variables associated with faces (cycles, loops) $F(\Gamma)$:

$$F(\Gamma) = \{f_1, f_2, \ldots, f_{n_f}\}$$

- Bosons on $F(\Gamma)$

Plaquette variables

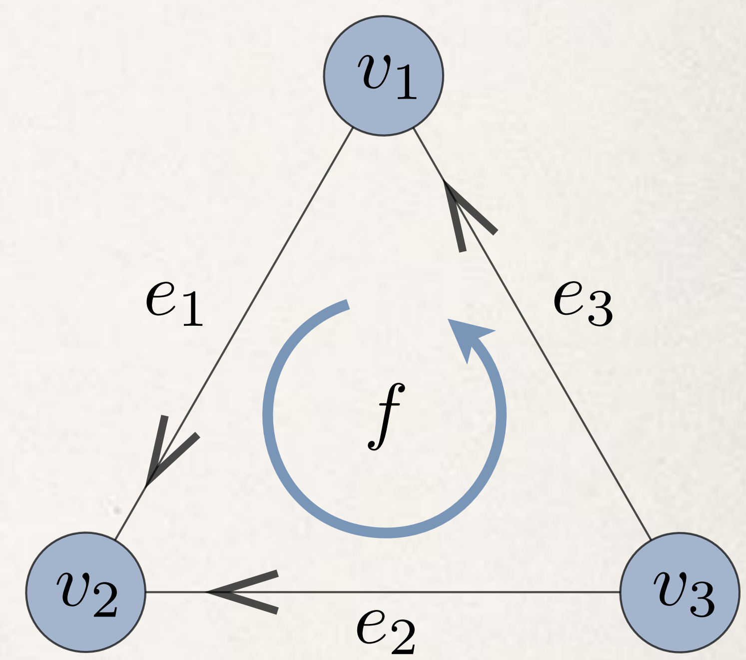
$$\boldsymbol{P}^{\mathsf{T}} = (P_1, P_2, \ldots, P_{n_f})$$

$$\boldsymbol{Y}^{\mathsf{T}} = (Y_1, Y_2, \ldots, Y_{n_f})$$

Lagrange multipliers (→ D-term field)

- Fermions on $F(\Gamma)$

$$\boldsymbol{\chi}^{\mathsf{T}} = (\chi_1, \chi_2, \ldots, \chi_{n_f})$$



$$f = \{e_1, \bar{e}_2, e_3\}$$

$$P_f = U_1 U_2^{-1} U_3$$
$$= e^{i(A_1 - A_2 + A_3)}$$

Supersymmetry and Action

- We define the following Q-transformations (SYSY) for the vectors:

$$
\begin{array}{ll}
Q\boldsymbol{\phi} = 0 & \\
Q\bar{\boldsymbol{\phi}} = \boldsymbol{\eta}, & Q\boldsymbol{\eta} = 0 \\
Q\boldsymbol{A} = \boldsymbol{\lambda}, & Q\boldsymbol{\lambda} = -L\boldsymbol{\phi} \\
Q\boldsymbol{Y} = 0, & Q\boldsymbol{\chi} = \boldsymbol{Y}
\end{array}
$$

- The action is written in the following Q-exact form

$$S = \frac{1}{2g^2} QV$$

where

$$V = -(L\bar{\boldsymbol{\phi}})^{\mathsf{T}} \cdot \boldsymbol{\Lambda} - \boldsymbol{\chi}^{\mathsf{T}} \cdot (\boldsymbol{Y} - 2\boldsymbol{\mu})$$

$$\boldsymbol{\mu}^{\mathsf{T}} = \big(\mu(P_1), \mu(P_2), \ldots, \mu(P_{n_f})\big)$$

moment map constraints
(D-term potential)

→

$\mu(P_f) = 0 \Rightarrow P_f = 1$

and

$\mu(P_f) \sim -ig^{z\bar{z}} F_{z\bar{z}}$

Localization

- After eliminating the auxiliary fields, bosonic part of the action becomes

$$S_B = \frac{1}{2g^2}\left\{|L\phi|^2 + \frac{1}{2}|\boldsymbol{\mu}|^2\right\}$$

$$= \frac{1}{2g^2}\left\{\phi\Delta\bar{\phi} + \frac{1}{2}\underset{\substack{\wr \\ F_{12}^2}}{|\boldsymbol{\mu}|^2}\right\} \qquad (\Delta = L^{\intercal}L)$$

Laplacian

Q-exact action ⇒ The path integral is independent of the coupling g

⇒ 1-loop (WKB) exact

⇒ localized at the fixed points:

$$L\phi = 0$$

$$\boldsymbol{\mu} = 0 \quad (\rightarrow \text{flat connection condition})$$

Action for fermions

- The fermionic part of the action becomes

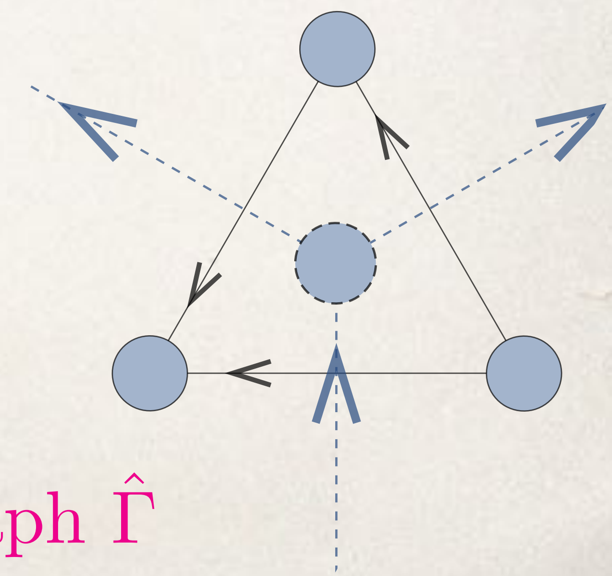
$$S_F = -\frac{1}{2g^2}\left\{\boldsymbol{\eta}^\intercal L^\intercal \boldsymbol{\lambda} + \boldsymbol{\chi}^\intercal \frac{\delta\boldsymbol{\mu}}{\delta\boldsymbol{U}}\boldsymbol{\lambda}\right\}$$
$$= -\frac{1}{2g^2}\{\boldsymbol{\Psi}^\intercal D^\intercal \boldsymbol{\lambda}\}$$

where

$$\boldsymbol{\Psi}^\intercal \equiv (\boldsymbol{\eta}^\intercal, \boldsymbol{\chi}^\intercal) = (\eta_1, \ldots, \eta_{n_v}, \chi_1, \ldots, \chi_{n_f}) \in V(\Gamma) \oplus F(\Gamma)$$

$$D^\intercal \equiv \underbrace{\begin{pmatrix} L^\intercal \\ \frac{\delta\boldsymbol{\mu}}{\delta\boldsymbol{A}} \end{pmatrix}}_{n_e} \begin{matrix} \}n_v \\ \}n_f \end{matrix}$$

"Dirac operator" on Γ



Note that $\frac{\delta\boldsymbol{\mu}}{\delta\boldsymbol{A}} \propto \hat{L}^\intercal \longrightarrow$ incidence matrix for the dual graph $\hat{\Gamma}$

1-loop determinant

- Using the 1-loop (WKB) approximation, we obtain the following 1-loop determinant under the above gauge fixing:

$$\Delta_{\text{1-loop}} = \underset{\phi,\bar{\phi}}{\overset{c,\bar{c}}{\frac{\det'_V(L^\intercal L)}{\left|\det'_V(L^\intercal L)\right|}}} \times \sqrt{\frac{\overset{\text{fermions}}{\det'_E(DD^\intercal)}}{\underset{A}{\det'_E\left(LL^\intercal + \frac{\delta\boldsymbol{\mu}}{\delta\boldsymbol{\lambda}}^\intercal\frac{\delta\boldsymbol{\mu}}{\delta\boldsymbol{\lambda}}\right)}}} = 1$$

where ʻ stands for omitting zero modes (zero eigenvalues) from the determinants

Cohomology on the graph

- The Laplacian matrix Δ_V has rank n_v - 1 on the simply connected graph

$$\Rightarrow \dim \mathrm{Harm}^V(\Gamma) = 1 \qquad (\boldsymbol{x} \in \mathrm{Harm}^V(\Gamma) \quad \text{if} \quad \Delta_V \boldsymbol{x} = 0)$$

$$\Rightarrow \dim H^V(\Gamma) = 1$$

- Similarly, we have

$$\dim H^F(\Gamma) = \dim H^V(\check{\Gamma}) = 1$$

- Using the definition of the Euler characteristic:

$$\dim H^V(\Gamma) - \dim H^E(\Gamma) + \dim H^F(\Gamma) = \chi_h = 2 - 2h$$

we find

$$\dim H^E(\Gamma) = 2h$$

Zero modes and index theorem

- $\dim\ker\Delta_V = \dim \mathrm{Harm}^V(\Gamma)$

  $\Rightarrow$ one complex bosonic zero mode ϕ_0

- The fermionic zero modes are solutions to the following equations:

$$D\mathbf{\Psi}_0 = 0$$

$$D^\intercal \mathbf{\Lambda}_0 = 0$$

thus we find

# of handles (genus)

$$(\sharp \text{ of zero modes of } \mathbf{\Psi}) = \dim\ker D = n_v + n_f - \mathrm{rank}\, D (= 2)$$

$$(\sharp \text{ of zero modes of } \boldsymbol{\lambda}) = \dim\ker D^\intercal = n_e - \mathrm{rank}\, D^\intercal (= 2h)$$

Euler characteristic of the graph with genus h

$$\mathrm{ind}\, D = \dim\ker D - \dim\ker D^\intercal = n_v - n_e + n_f = \chi_h$$

Hirzebruch-Riemann-Roch theorem on the graph Γ

Anomaly

- The modes is invariant under the following U(1)$_A$ symmetry

$$\boldsymbol{\phi} \to e^{2i\theta}\boldsymbol{\phi}, \quad \bar{\boldsymbol{\phi}} \to e^{-2i\theta}\bar{\boldsymbol{\phi}}$$
$$\boldsymbol{\eta} \to e^{-i\theta}\boldsymbol{\eta}, \quad \boldsymbol{\chi} \to e^{-i\theta}\boldsymbol{\chi}, \quad \boldsymbol{\lambda} \to e^{i\theta}\boldsymbol{\lambda}$$

but the integral measure has the anomaly since

$$\prod_{v\in V} d\eta_v \prod_{e\in E} d\lambda_e \prod_{e\in F} d\chi_f \supset \prod_i d\eta_0^{(i)} \prod_j d\lambda_0^{(j)} \prod_k d\chi_0^{(k)}$$
$$\to e^{i\chi_h\theta} \prod_i d\eta_0^{(i)} \prod_j d\lambda_0^{(j)} \prod_k d\chi_0^{(k)}$$

This agrees with the number of the existing zero modes (index theorem, similar to Fujikawa's method)

Non-Abelian case

- The U(N) gauge theory can be obtained by an extension of the variables to $N \times N$ matrices

- The SUSY transformations are:

$$
\begin{aligned}
&Q\Phi_v = 0, \\
&Q\bar{\Phi}_v = \eta_v, && Q\eta_v = i[\Phi_v, \bar{\Phi}_v], \\
&QU_e = i\Lambda_e, && Q\Lambda_e = -(L_U \mathbf{\Phi})_e, \\
&QY_f = i[\Phi_f, \chi_f], && Q\chi_f = Y_f,
\end{aligned}
\qquad
\begin{pmatrix} U_e \equiv e^{iA_e} \\ \Lambda_e \equiv e^{i\lambda_e} \end{pmatrix}
$$

where

$$(L_U \mathbf{\Phi})_e \equiv U_e \Phi_{t(e)} - \Phi_{s(e)} U_e$$

and Φ_f is a representative (average) value of Φ_v on the face f

Fixed point equation

- The action is again written in the *Q*-exact form

- The path integral is localized at the fixed points which are solutions to the following equations:

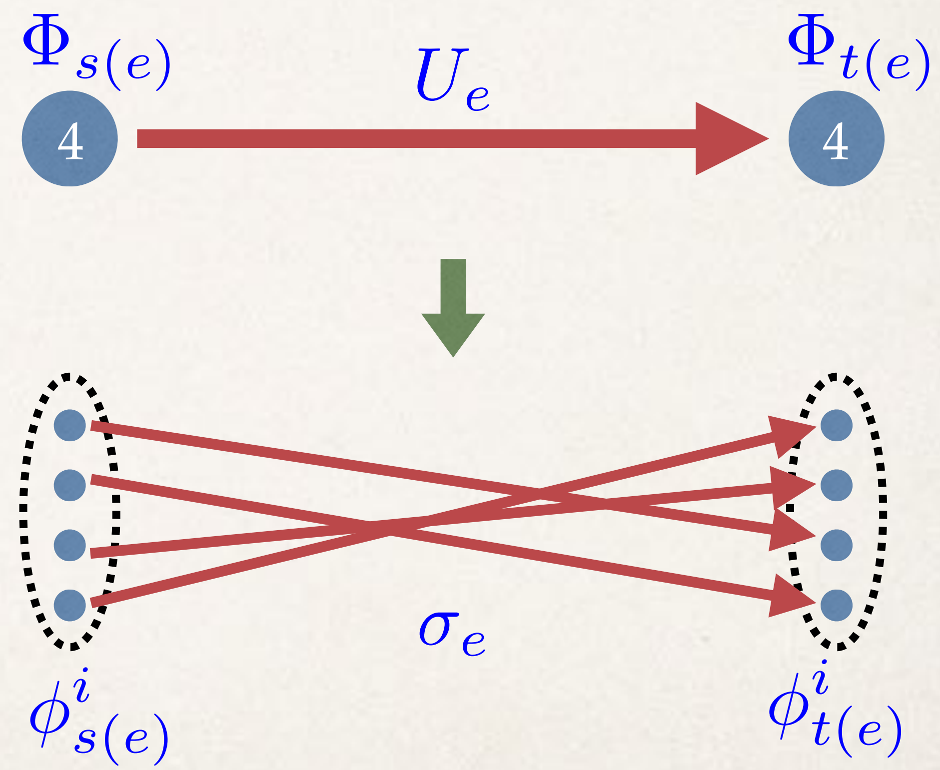
$$[\Phi_v, \bar{\Phi}_v] = 0$$

$$(L_U \mathbf{\Phi})_e = U_e \Phi_{t(e)} - \Phi_{s(e)} U_e = 0$$

$$\Phi_v = \mathrm{diag}(\phi_v^1, \phi_v^2, \ldots, \phi_v^N)$$

$$U_e = \sigma_e \in \mathfrak{S}_N$$

$$\Phi_{t(e)} = \sigma_e^{-1} \Phi_{s(e)} \sigma_e$$



G=U(*N*) reduces to $\mathrm{U}(1)^N$ around the fixed points (Abelianization)

Exact partition function

- Using the 1-loop approximation around the fixed points, we can evaluate the partition function exactly in terms of the summation over the possible permutations and residue integrals

$$Z = \sum_{\{\sigma_e\}} \int_C \prod_{i=1}^{N} d\boldsymbol{\phi}^i \, \Delta_{\text{1-loop}}(\boldsymbol{\phi})$$

where

$$\Delta_{\text{1-loop}}(\boldsymbol{\phi}) = \prod_{i<j} \frac{\prod_{v\in V}(\phi_v^i - \phi_v^j)\prod_{f\in F}(\phi_f^i - \phi_f^j)}{\prod_{e\in E}(\phi_{t(e)}^{\sigma_e(i)} - \phi_{s(e)}^j)}$$

Finally, we obtain (except for the fermionic zero mode integral)

$$Z = \mathcal{N} \int \prod_{i=1}^{N} d\phi_0^i \prod_{i<j} (\phi_0^i - \phi_0^j)^{n_v + n_f - n_e}$$

$$n_v + n_f - n_e = \chi_h$$

agrees with the continuum limit

Zero modes for non-Abelian theory

- There are zero modes on each decomposed Abelian graph (only in the Cartan part)

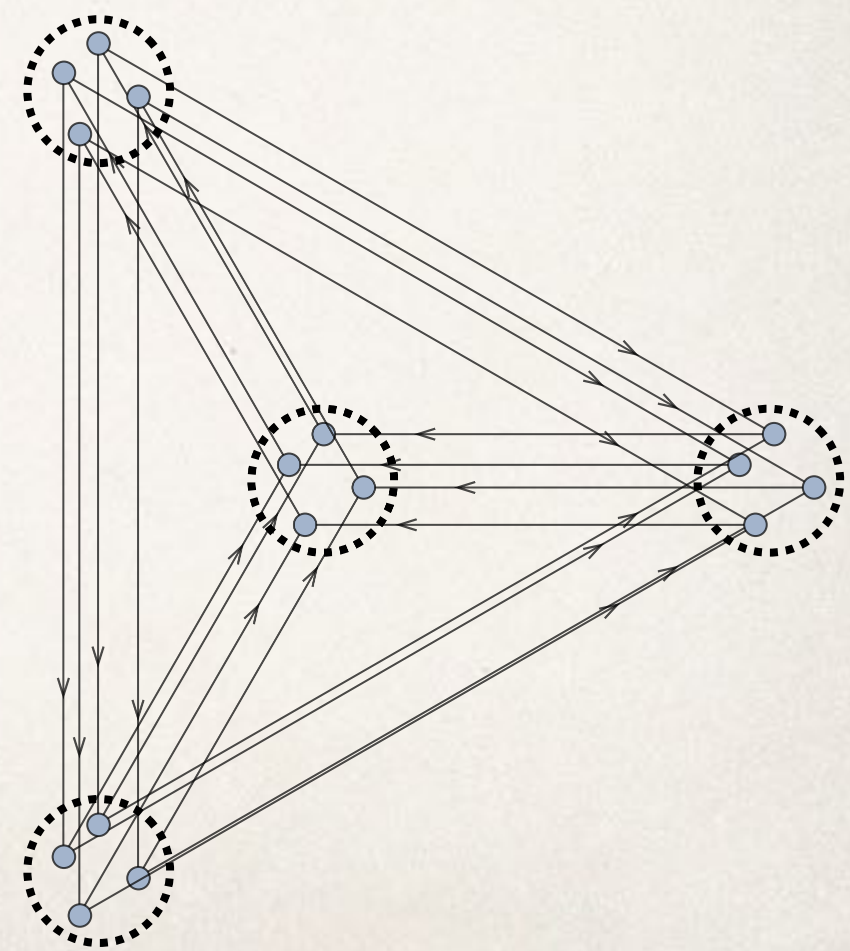
(# of the zero modes of η_v and χ_f)= $N \times 2$
(# of the zero modes of Λ_e) = $N \times 2h$

= # of handles (genus)

$$\prod_{v\in V} d\Phi_v d\bar{\Phi}_v d\eta_v \prod_{e\in E} d\lambda_e \prod_{f\in F} d\chi_f$$

$$\prod_{i=1}^{N} d\phi_0^i \prod_{l=1}^{2h} d\lambda_{0,l}^i \prod_{i<j} (\phi_0^i - \phi_0^j)^{\chi_h}$$

has the same $U(1)_A$ anomaly as the original one: $N^2 \times \chi_h$



The compensator

- Due to the existence of the zero modes and anomaly, we consider inserting Q-closed operators to compensate (cancel) the phases from the integral measure in the numerical simulation

e.g.

$$\mathcal{O}_{\mathrm{Tr}} = \frac{1}{n_v} \sum_{v \in V} \left(\frac{1}{N} \mathrm{Tr}\, \Phi_v^2 \right)^{-\frac{N^2}{4}\chi_h}$$

$$\mathcal{O}_{\mathrm{IZ}} = \frac{1}{n_e} \sum_{e \in E} \left(\frac{1}{N} \mathrm{Tr}\, \left(2U_e \Phi_{t(e)} U_e^{-1} \Phi_{s(e)} - (U_e \Phi_{t(e)} U_e^{-1} - \Phi_{s(e)}) \Lambda_e U_e^{-1} \Lambda_e U_e^{-1} \right) \right)^{-\frac{N^2}{4}\chi_h}$$

where

$$Q\mathcal{O} = 0, \text{ but } \mathcal{O} \neq Q\{\star\}$$

The evidence of the zero modes and phase from the anomaly are checked numerically in [Kamata-Matuura-Misumi-KO (2016)], see also Matsuura-san's talk

Conclusion and Discussion

Results:

- We formulate the discretized gauge theory on the generic graphs
- The graph theory is useful (beautiful) to formulate, analyze and solve the model
- The zero modes and anomaly are also important

Outlook:

- Relation to (or realization in) string/M theory or gravity (topological invariants, etc. in mathematics)
- A possibility of the emergence (deconstruction) of the space-time geometry from the generalized quiver or matrix models