# Topological Operators in 2D N=(2,2) Lattice SYM

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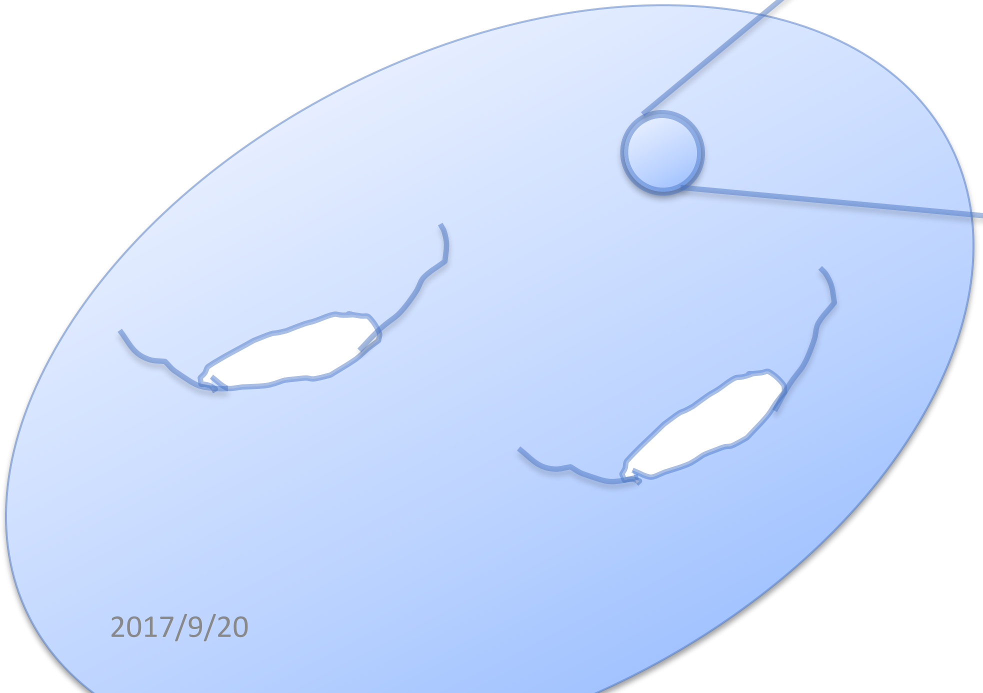


based on the work with
S. Kamata(Fudan Univ.),
T. Misumi(Akita Univ.),
K. Ohta(Meiji Gakuin Univ.)

# Simple 2D SUSY gauge theory on a square lattice

# (Sugino model)

Sugino 2003

## Continuum theory

4D N=1 SYM

$A_\mu, \quad (\mu = 1, 2)$
$A_3, \ A_4$

dimensional → reduction

2D N=(2,2) SYM

$A_\mu,$
$\Phi, \ \bar{\Phi}$ (scalar fields)

$$S = \frac{1}{g_4^2} \int d^4x Tr \left( \frac{1}{4} F_{MN}^2 + \frac{i}{2} \bar{\Psi} \Gamma^M D_M \Psi \right)$$

$$S = \frac{1}{g^2} \int d^2x Tr \left( \frac{1}{4} F_{\mu\nu}^2 + |D_\mu \Phi|^2 + \cdots \right)$$

## supersymmetry

4D SUSY transformation

$$\delta A_M = -i\bar{\xi}\Gamma_M \Psi$$

$$\delta \Psi = -\frac{1}{2} F_{MN} \Gamma_{MN} \xi$$

dimensional → reduction

2D SUSY transformation

$$\delta A_\mu = -i\bar{\xi}\Gamma_\mu \Psi$$

$$\delta \Phi = -i\bar{\xi}\Gamma_+ \Psi$$

etc…

## specific transformation

Transformation by $\xi = (0,0,0,-\epsilon)^T$: $\delta = i\epsilon Q$

$$
\begin{aligned}
QA_\mu &= \lambda_\mu, & Q\lambda_\mu &= iD_\mu\Phi, \\
Q\bar{\Phi} &= \eta, & Q\eta &= [\Phi,\bar{\Phi}], \\
Q\chi &= Y, & QY &= [\Phi,\chi], \quad Q\Phi = 0.
\end{aligned}
$$

Fermions in component
$\Psi = (\lambda_1, \lambda_2, \chi, \eta/2)^T$

$$Q^2 = \delta_\Phi$$

**Q is nilpotent up to gauge transformation**

## action in Q–exact form

$$S = \frac{1}{2g^2}\int d^2x\, Q\, Tr\left(\frac{1}{4}\eta[\Phi,\bar{\Phi}] + \chi\left(Y - 2iF_{12}\right) - i\lambda_\mu D_\mu\bar{\Phi}\right)$$

**Manifestly Q-invariant!**

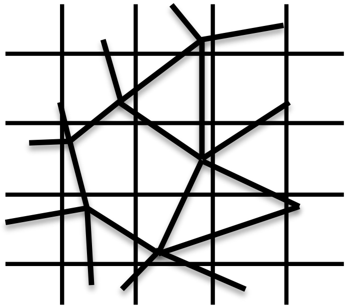
## idea

**Can we put this Q-invariant action on lattice with keeping Q-symmetry?**

**YES！→ Sugino model**

# Geometrical structure of Sugino model

$$QU_\mu(x) = \Lambda_\mu(x), \quad Q\Lambda_\mu(x) = \Phi(x)U_\mu(x) - U_\mu(x)\Phi(x+\hat{\mu})$$
$$Q\bar{\Phi}(x) = \eta(x), \quad Q\eta(x) = [\Phi(x), \bar{\Phi}(x)],$$
$$Q\chi(x) = Y(x), \quad QY(x) = [\Phi(x), \chi(x)], \quad Q\Phi(x) = 0$$



**Fields**

$\bar{\Phi}(x)$ ● $\quad$ $\eta(x)$ ●

$U_\mu(x)$ $\quad$ $Q$ ⟷ one-to-one $\quad$ $\Lambda_\mu(x)$

$Y(x)$ $\quad$ $\chi(x)$

**Action**

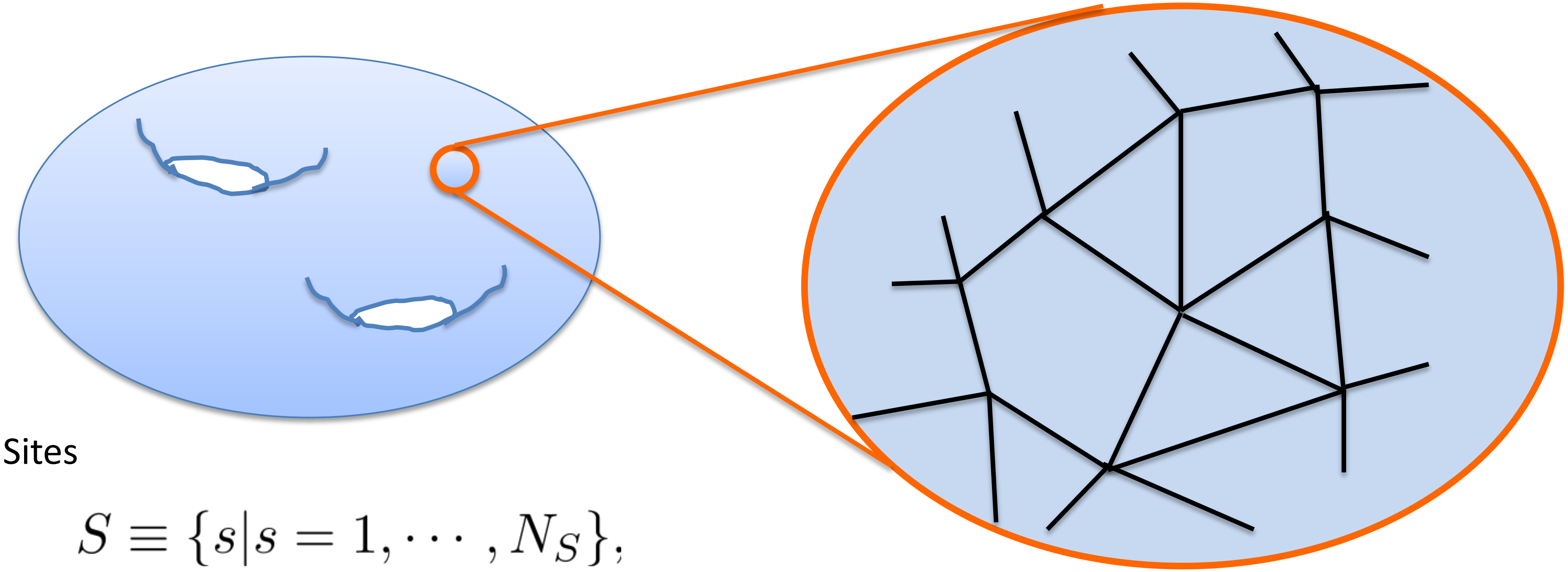
$$S_{\text{lat}} = \frac{1}{2g^2}\sum_x QTr\left(\frac{1}{4}\eta[\Phi(x), \bar{\Phi}(x)]\right. \quad \text{: site}$$
$$- i\lambda_\mu(x)D_\mu\bar{\Phi}(x) \quad \text{: link}$$
$$\left. + \chi(x)\left(Y(x) - i\mu(x)\right)\right) \quad \text{: face}$$

**Can we extend it to a general lattice (simplicial complex)?**

# N=(2,2) Topological SYM on Discretized Riemann Surface

Misumi-Ohta-S.M. 2014

cf) Ohta-san's talk



Sites

$$S \equiv \{s | s = 1, \cdots, N_S\},$$

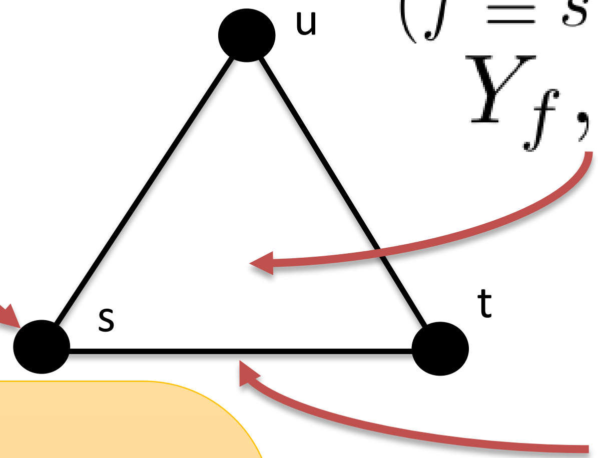
Links

$$L \equiv \{\langle st \rangle | s, t \in S\},$$

Faces

$$F \equiv \{(s_1, \cdots, s_n) | s_1, \cdots, s_n \in S,\ (s_i, s_{i+1}) \in L \text{ or } (s_{i+1}, s_i) \in L\},$$

# Fields on the discretized surface




$\Phi_s, \bar{\Phi}_s, \eta_s$

$(f \equiv s :\ \text{representative point})$

$Y_f, \chi_f$

$U_{st}, \Lambda_{st} \ \ (\Lambda_{st} \equiv \lambda_{st} U_{st})$

**Action**

$$S = Q\left\{\sum_{s\in S}\alpha_s \Xi_s + \sum_{l\in L}\alpha_l \Xi_l + \sum_{f\in F}\alpha_f \Xi_f\right\}$$

$$\begin{cases} \Xi_s \equiv \dfrac{1}{2g_0^2}\mathrm{Tr}\left\{\dfrac{1}{4}\eta_s[\Phi_s, \bar{\Phi}_s]\right\}, \\ \Xi_{\langle st\rangle} \equiv \dfrac{1}{2g_0^2}\mathrm{Tr}\left\{-i\lambda_{st}\left(U_{st}\bar{\Phi}_t U_{st}^{-1} - \bar{\Phi}_s\right)\right\}; \\ \Xi_f \equiv \dfrac{1}{2g_0^2}\mathrm{Tr}\left\{\chi_f\left(Y_f - i\beta_f \mu(U_f)\right)\right\}, \end{cases}$$

**SUSY transformation**

$$Q\bar{\Phi}_s = \eta_s, \quad Q\eta_s = [\Phi_s, \bar{\Phi}_s],$$

$$QU_l = \Lambda_l, \quad Q\Lambda_l = \Phi_{org(l)}\Lambda_l - \Lambda_l\Phi_{tip(l)},$$

$$QY_f = [\Phi_f, \chi_f], \quad Q\chi_f = Y_f, \quad Q\Phi_s = 0$$

## facts

- The action becomes 2D N=(2,2) Supersymmetric Yang-Mills theory on a Riemann surface in the continuum limit.

- There remains the U(1) symmetry,

$$\Phi \to e^{2i\alpha}\Phi, \quad \bar{\Phi} \to e^{-2i\alpha}\bar{\Phi}, \quad A_\mu \to A_\mu,$$
$$\eta \to e^{-i\alpha}\eta, \quad \lambda_\mu \to e^{i\alpha}\lambda_\mu, \quad \chi \to e^{-i\alpha}\chi.$$

- An appropriate continuum limit reproduces the continuum action.

- Because of the Q-symmetry and the U(1) symmetry, there is no relevant operator which breaks other symmetries in the continuum limit.

# One result: Partition function using localization

Misumi-Ohta-S.M. 2014
Kamata-Misumi-Ohta-S.M. 2016

$$Z = \mathcal{N} \int d\phi_i \prod_{i<j} (\phi_i - \phi_j)^{\chi}$$

- It reproduces the partition function of the continuum theory.
- It depends only on the topology of the network.
- Independent of the detail of the discretization.

# $\mathrm{U}(1)_R$ anomaly

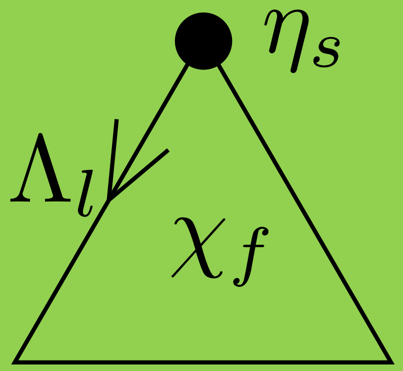
$$\Phi \to e^{2i\alpha}\Phi, \quad \bar{\Phi} \to e^{-2i\alpha}\bar{\Phi}, \quad A_\mu \to A_\mu,$$
$$\eta \to e^{-i\alpha}\eta, \quad \lambda_\mu \to e^{i\alpha}\lambda_\mu, \quad \chi \to e^{-i\alpha}\chi.$$

## Measure in the discretized theory

$$\mathcal{D}\vec{B} = (\prod_{s=1}^{N_S} \mathcal{D}\Phi_s \mathcal{D}\bar{\Phi}_s)(\prod_{l=1}^{N_L} \mathcal{D}U_l)(\prod_{f=1}^{N_F} \mathcal{D}Y_f)$$ : $\mathrm{U}(1)_R$ neutral

$$\mathcal{D}\vec{F} = (\prod_{s=1}^{N_S} \mathcal{D}\eta_s)(\prod_{l=1}^{N_L} \mathcal{D}\lambda_l)(\prod_{f=1}^{N_F} \mathcal{D}\chi_f)$$

$$\to \left(\mathcal{D}\vec{\mathcal{F}}\right) e^{i(N_S - N_L + N_F)(N^2-1)\alpha}$$



# Anomaly–phase–quench method

**vev in the continuum theory**

U(1) charge: $(N^2-1)\chi_h$

$$\langle \mathcal{O} \rangle \equiv \frac{1}{Z_q}\int \mathcal{D}\vec{\mathcal{B}}\mathcal{D}\vec{\mathcal{F}}\, \mathcal{O}\, e^{-S_b-S_f} = \frac{1}{Z_q}\int \mathcal{D}\vec{\mathcal{B}}\, \mathcal{O}\, \mathrm{Pf}(D)\, e^{-S_b}$$

**naïve phase quench**

U(1) charge: ZERO

$$\langle \mathcal{O} \rangle^q \equiv \frac{1}{Z_q}\int \mathcal{D}\vec{\mathcal{B}}'\, \mathcal{O}\, |\mathrm{Pf}(D)|\, e^{-S_b}$$

**NOT A GOOD APPROXIMATION**

**philosophy of the phase quench**

We can (or could) ignore the artificial phase coming from the discretization

**Observation**

$$\mathrm{Pf}(D) = |\mathrm{Pf}(D)| e^{i\theta_A + i\theta}$$

1. $U(1)_R$ phase $\qquad \theta_A$
2. lattice artifact $\qquad \theta$

**We should ignore only θ**

# Compensator

Kamata-Misumi-Ohta-S.M. 2016

$\mathcal{A}$ : an operator with

- $Q\mathcal{A} = 0$
- $[\mathcal{A}] = -(N^2 - 1)\chi_h$
- $\mathcal{A} \equiv |\mathcal{A}| e^{-i\theta_A}$

**anomaly-phase-quench method**

$$\langle \mathcal{O} \rangle^{\hat{q}} \equiv \langle \mathcal{O}\, e^{i\theta_A} \rangle^{q} = \frac{1}{Z_q} \int \mathcal{D}\vec{\mathcal{B}} \mathcal{O} |Pf(\mathcal{D})| e^{i\theta_A}$$

trace type

$$\mathcal{A}_{\rm tr} = \frac{1}{N_S} \sum_{s=1}^{N_S} \left( \frac{1}{N_c} \mathrm{Tr}\,(\Phi_s)^2 \right)^{-\frac{N_c^2-1}{4}\chi_h}$$

determinant type

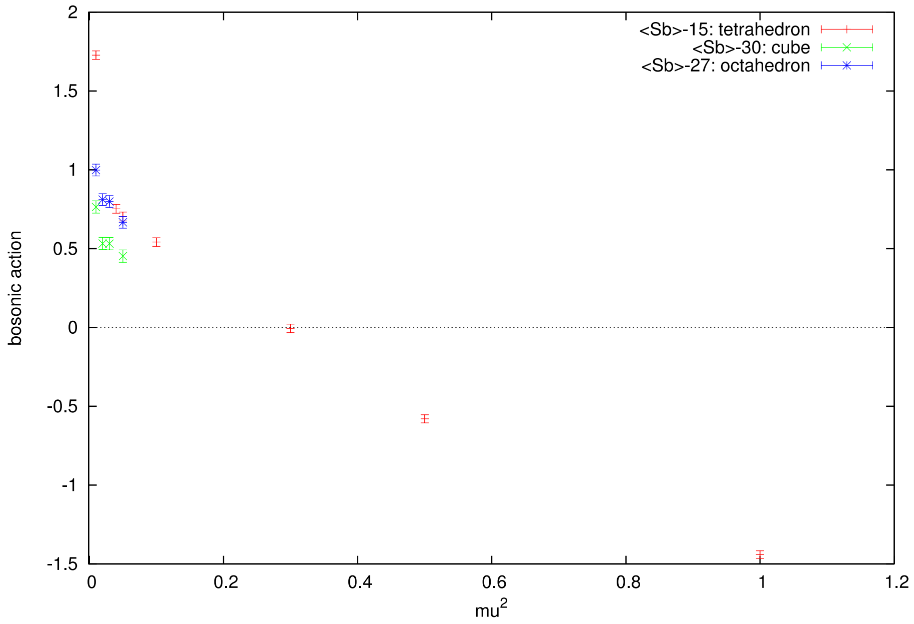
$$\mathcal{A}_{\rm det} = \frac{1}{N_S} \sum_{s=1}^{N_S} (\mathrm{Det}\Phi_s)^{-\frac{N_c^2-1}{2N_c}\chi_h}$$

Izykson-Zuber type

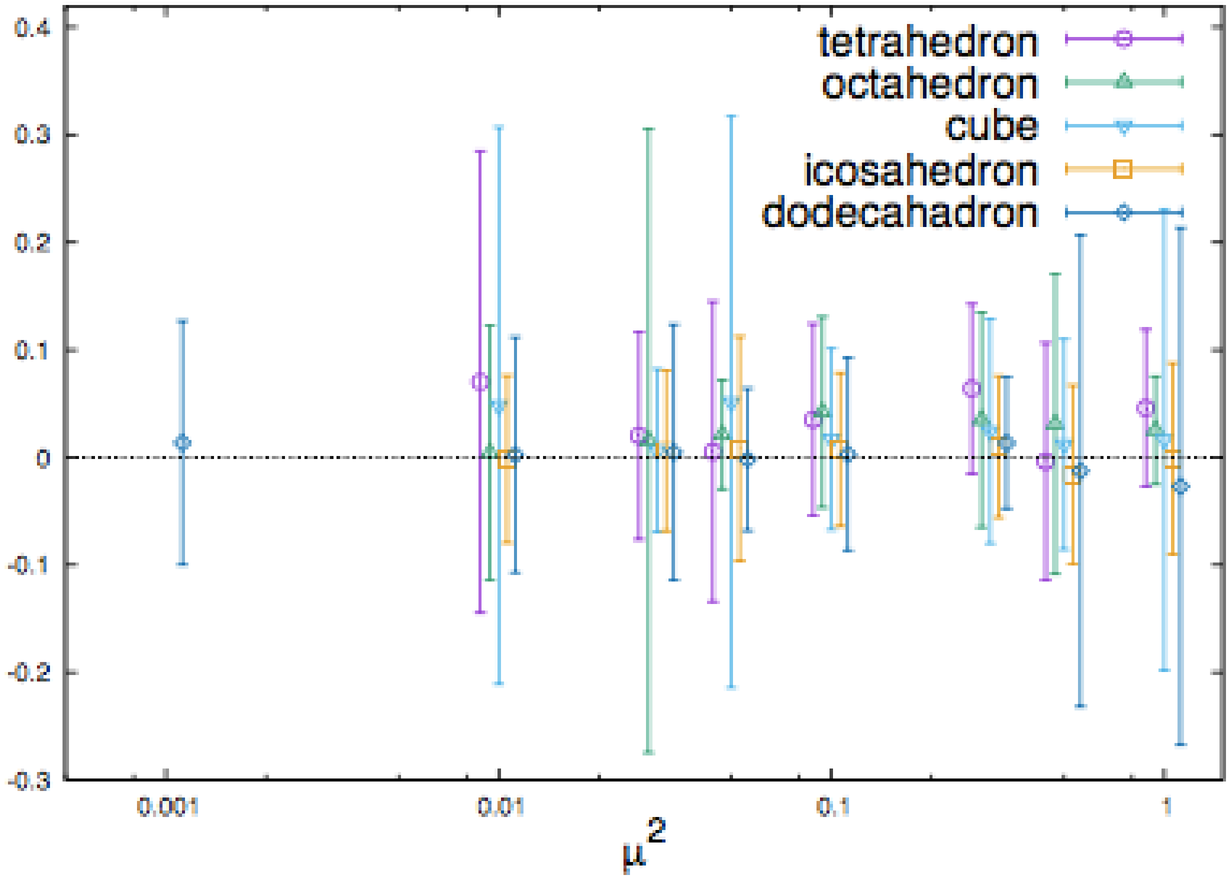
$$\mathcal{A}_{\rm IZ} = \frac{1}{N_l} \sum_{l=1}^{N_l} \left( \frac{1}{N_c} \mathrm{Tr} \left( 2\Phi_{\mathrm{org}(l)} U_l \Phi_{\mathrm{tip}(l)} U_l^\dagger + \lambda_l \lambda_l (U_l \Phi_{\mathrm{tip}(l)} U_l^\dagger + \Phi_{\mathrm{org}(l)}) \right) \right)^{-\frac{N_c^2-1}{4}\chi_h}$$

# A result of numerical simulation in anomaly-phase-quench method

Kamata-Misumi-Ohta-S.M. 2016




**Naïve phase quench**




**anomaly-phase-quench**

## Interesting topological observable in the continuum theory:

Witten 1992

$$\mathcal{O}_{\rm cont.} = \int d^2x {\rm tr}\, \{\phi(x)F_{12}(x) + \lambda_1(x)\lambda_2(x)\}$$

$$Q\mathcal{O}_{\rm cont.} = 0, \quad \mathcal{O} \neq Q[\text{gauge invariant op.}]$$

$$\begin{aligned}\langle e^{t\mathcal{O}_{\rm cont.}+t\int d^2x {\rm tr}\phi(x)^2}\rangle_{\rm SYM} &= \int d\mu e^{-S_{\rm SYM}+t\int d^2x {\rm tr}\{\phi(x)F_{12}(x)+\phi(x)^2+\lambda_1(x)\lambda_2(x)\}} \\ &\simeq \int dA e^{-\int d^2x {\rm tr}(\frac{1}{2}F_{12}^2)} \\ &= \sum_R (\dim R)^{\chi_h} e^{-AC_2(R)} = Z_{2DYM} \;\; !!\end{aligned}$$

**Is there a corresponding operator on the lattice?**

# No-Go Theorem

There is no Q-closed operator whose continuum limit is

$$\mathrm{tr}\left(\phi F_{12} + \lambda_1 \lambda_2\right)$$

under the conditions

1. $Q^2 = \delta_\Phi$
2. The fermion part is made of the link variables $U_l$ and a bilinear of the link fermions $\Lambda_l$

## essence of the proof

- The fermion part is expressed as

$$\mathcal{O}_f = \sum_{k=2}^{n} \alpha_k \mathrm{tr}\left(\Lambda_1 U_2 \cdots \Lambda_k \cdots U_n\right)$$

- In order that the O($\Lambda^3$ )-terms in $QO_f$ cancel with each other, $O_f$ must takes the form,

$$\mathcal{O}_f = \mathrm{tr}\left(\Lambda_1 Q(U_2 \cdots U_n)\right) = \mathrm{tr}\left\{Q(U_1)\, Q(U_2 \cdots U_n)\right\}$$

- The corresponding Q-vanishing operator is uniquely determined as

$$\mathcal{O} = Q\frac{1}{2}\mathrm{tr}\left\{U_1 Q(U_2 \cdots U_n) - Q(U_1) U_2 \cdots U_n\right\}$$

which is Q-exact.

# (may-be incorrect) idea

## observation

The operator of the continuum theory can be expressed in Q-exact form:

$$\int d^2x \mathrm{tr}\left(\phi F_{12} + \lambda_1\lambda_2\right) \simeq \frac{1}{2}\int d^2x Q\mathrm{tr}\left(\epsilon^{\mu\nu}A_\mu \mathcal{D}_\nu\phi\right)$$

**NOT gauge invariant**

In lattice theory, it is easy to make Q-exact operators but it is hard to make non-trivial gauge NON-invariant operator.

If there is a lattice counterpart, it will be Q-exact but very non-trivial.

## Idea

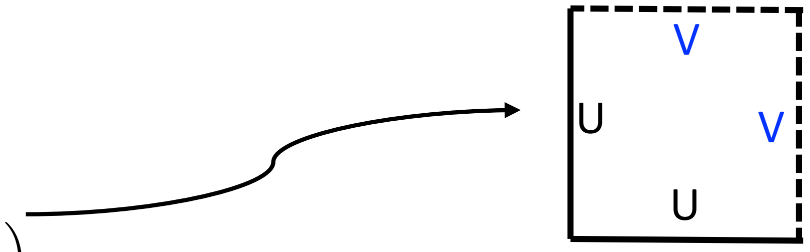
$V_\mu(x)$ : non-dynamical link variable

$$\mathcal{O}_b = \mathrm{tr}\Big(Q^2(U_1(x)V_2(x+\hat{1}))V_1(x+\hat{2})^\dagger U_2(x)^\dagger$$

$$- U_1(x)V_2(x+\hat{1})Q^2(V_1(x+\hat{2})^\dagger U_2(x)^\dagger)\Big)$$

$$\rightarrow \mathrm{tr}\left(\epsilon^{\mu\nu}A_\mu\mathcal{D}_\nu\phi + \partial_1 A_1 - \partial_2 A_2\right)$$

under $V_\mu = e^{iaB_\mu}, \quad B_\mu = 0$



# Future works

- There would be a possibility to avoid the No-Go theorem.
- Reformulation in terms of graph theory
- Including matter multiplets
- Embed into matrix?
- Quantum gravity?