An Alternative Lattice Field Theory Formulation Inspired by Lattice Supersymmetry

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Noboru Kawamoto

Hokkaido University

In collaboration with

A. D'Adda and J. Saito

If we require exact supersymmetry on the lattice
to what formulation we are forced to reach ?

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Non-local lattice field theory formulation equivalent
to the corresponding continuum theory.
Locality is recovered in the continuum limit.

No chiral fermion problem

Symmetries of the corresponding
continuum theory including SUSY are exactly kept
however
gauge invariance is exact only in the limit

Major difficulties for lattice SUSY

Let's consider the simplest lattice SUSY algebra:

$$Q^2 = H = i\partial \to i\hat{\partial} \qquad \hat{\partial}F(x) = \frac{1}{2a}\{F(x+a) - F(x-a)\}$$

$$(a : \text{lattice constant})$$

(1) Difference operator does not satisfy Leibniz rule.

(2) Species doublers of lattice chiral fermion copies appear:
unbalance of d.o.f. between bosons and fermions

Major difficulties for lattice SUSY

(1) Difference operator does not satisfy Leibniz rule.

$$\hat{\partial}(F(x)G(x)) = \frac{1}{2a}(F(x+a)G(x+a) - F(x-a)G(x-a))$$

$$= \frac{1}{2a}(F(x+a) - F(x-a))G(x-a) + F(x+a)\frac{1}{2a}(G(x+a) - G(x-a))$$

$$= \hat{\partial}F(x)G(x-a) + F(x+a)\hat{\partial}G(x)$$

$$= \hat{\partial}F(x)G(x+a) + F(x-a)\hat{\partial}G(x)$$

cancelation

breakdown of Leibniz rule

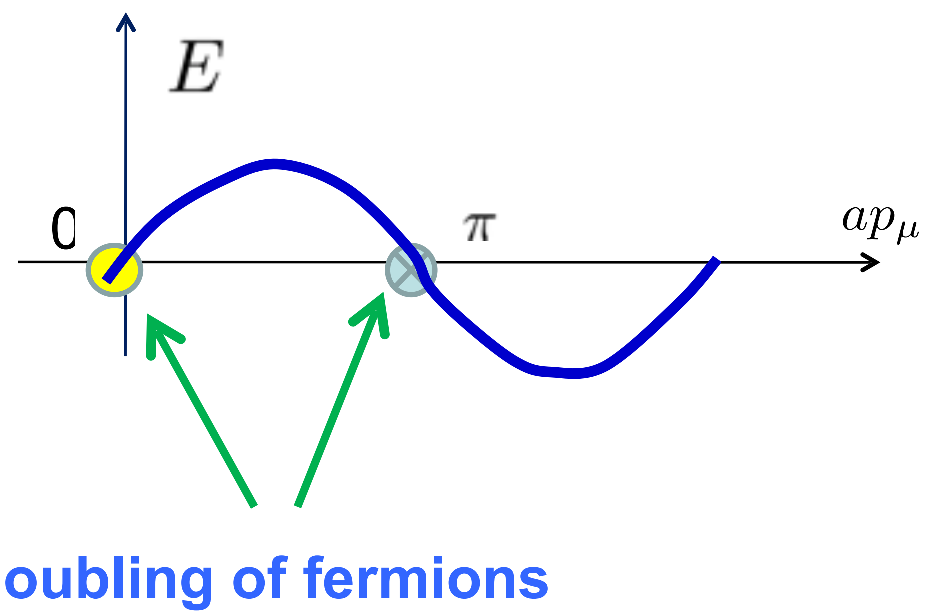
$$Q^2 = H = i\partial \rightarrow i\hat{\partial}$$

(2) Species doublers of lattice chiral fermion copies appear: unbalance of d.o.f. between bosons and fermions

Massless fermion ⟹ species doublers

$$\bar{\psi}(x) i\gamma_\mu \{\psi(x+a\hat{\mu}) - \psi(x-a\hat{\mu})\}/2a$$

$$\frac{1}{\gamma^\mu \sin a p_\mu}, \qquad a p_\mu = 0, \pi$$



pole in the propagator with a replacement:

$$(p_0, p_1) = (iE, p)$$

For simplicity in Einstein relation:

Continuum: $\frac{dE}{dp} = \frac{p}{\sqrt{p^2+m^2}} \xrightarrow{m \to 0} \frac{p}{|p|} = \pm 1 \ (\text{helicity})$

How do we solve these problems (1) and (2) at the same time ?

We claim that these two problems are related !

- Introduce half lattice structure: $a \to \frac{a}{2}$
    - What is the half lattice translation ?
- Change the identification of conserved continuum momentum not by the lattice momentum itself but another properly chosen function:

$$p \to \Delta(p)$$

$$\delta(p_1 + p_2 + \cdots) \to \delta(\Delta(p_1) + \Delta(p_2) + \cdots)$$

- Is the continuum mom. same as lattice mom. p ?
- If not necessary how do we choose the $\Delta(p)$?
- What is the change in the coordinate ?

Why half lattice structure ?

Let's consider the simplest lattice SUSY algebra:

$$Q^2 = H = i\partial \to i\hat{\partial} \to \frac{\sin ap}{a} \quad ?$$

symmetric difference operator ⟵ Hermiticity

$$\hat{\partial}\Psi(x) = (\Psi(x+a) - \Psi(x-a))/2a \qquad \frac{\sin ap}{ia}\Psi(p)$$

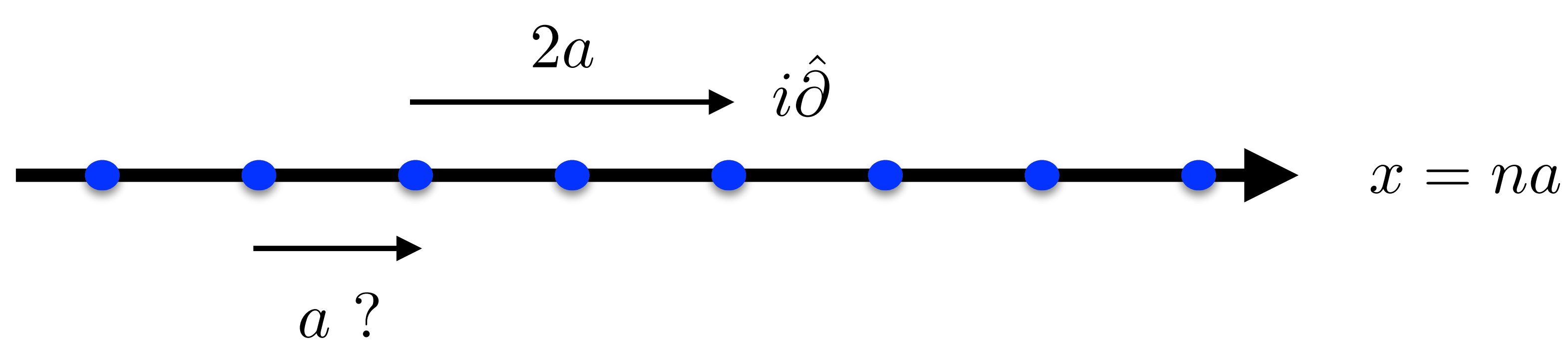
$2a$ step ?!

$(a : \text{lattice constant})$

$$\neq (\Psi(x+a) - \Psi(x))/a \qquad \frac{e^{iap}-1}{a}\Psi(p)$$

$\hat{\partial}$ defines lattice translation generator $2a$ step ?!

We consider that physical states are translational invariant.



$\Psi = C = \text{constant}$ $\quad\quad$ $\Psi' = C'(-1)^n$

species doubler

in mom. space

Translation on the lattice should be identified as $2a$ lattice shift !

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Introduce half lattice and identify lattice shift a as translation !

- **What is the half lattice shift then ?**

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half lattice $\frac{a}{2}$ shift = SUSY transformation

$$\{Q,Q\} = i\Delta(p)$$

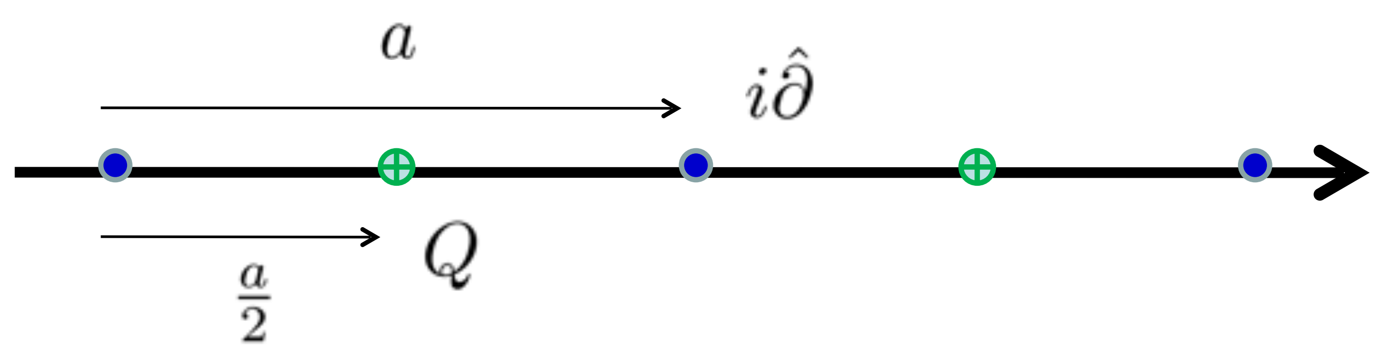
$$\frac{a}{2} + \frac{a}{2} = a$$

$$\hat{p} = \Delta_s(p) = \frac{2}{a}\sin\frac{ap}{2} \quad \left(-\frac{\pi}{a} \leq p \leq \frac{3\pi}{a}\right)$$

$$Q^2 = \hat{P} = \Delta(p) = \frac{2}{a}\sin\frac{ap}{2}$$

$Q^2 = \hat{P} = i\hat{\partial}$ → translation generator of a

↓ half translation generator $\frac{a}{2}$



Brillioun zone $\frac{4\pi}{a}$

$x = n\frac{a}{2} \leftrightarrow \frac{2x}{a} = n \in Z$

$-\frac{\pi}{a} \le p < \frac{3\pi}{a}$

SUSY transformation from dimensional analyses

$$Q^2 = i\partial$$

$$[Q^2] = L^{-1} \longrightarrow [Q] = L^{-\frac{1}{2}}$$

1-dim. $L = \psi\partial\psi + \partial\phi\partial\phi, \qquad [S] = [\int dx L] = L^0$

$$[\psi] = L^0, \qquad [\phi] = L^{\frac{1}{2}}$$

$$Q\phi \sim \psi, \qquad Q\psi \sim \partial\phi$$

How can we realize this structure naturally on a lattice ?

Lattice superfield: SUSY transformation:

$$\Phi(x) = \varphi(x) + \tfrac{1}{2}\boxed{(-1)^{\frac{2x}{a}}}\psi(x) \qquad (x = \frac{na}{2})$$

alternating sign key of the lattice SUSY

$$\delta\Phi(x) = a^{-\frac{1}{2}}\alpha(-1)^{\frac{2x}{a}}\left(\Phi(x+\tfrac{a}{2}) - \Phi(x)\right)$$

hermiticity

$$\delta\varphi(x) = \tfrac{i\alpha}{2}\left[\psi(x+\tfrac{a}{4}) + \psi(x-\tfrac{a}{4})\right] \rightarrow i\alpha\psi(x)$$

$$\delta\psi(x) = 2a^{-1}\alpha\left[\varphi(x+\tfrac{a}{4}) - \varphi(x-\tfrac{a}{4})\right] \rightarrow \alpha\frac{\partial\varphi(x)}{\partial x}$$

D=1 N=2 Lattice SUSY

$$e^{i\frac{2\pi}{a}x}$$

$$\frac{1}{2}\sum_{x=\frac{na}{2}+\frac{a}{4}} e^{ipx}(-1)^{\frac{2x}{a}}\psi(x) = \psi\left(p+\frac{2\pi}{a}\right) = -\psi\left(p-\frac{2\pi}{a}\right)$$

alternating sign → species doubler

$$\delta_1\Phi(p) = i\cos\frac{ap}{4}\alpha\Psi(p)$$

$$\delta_1\Psi(p) = -4i\sin\frac{ap}{4}\alpha\Phi(p)$$

$$\Psi(p) \to -i\Psi\left(\frac{2\pi}{a}-p\right)$$

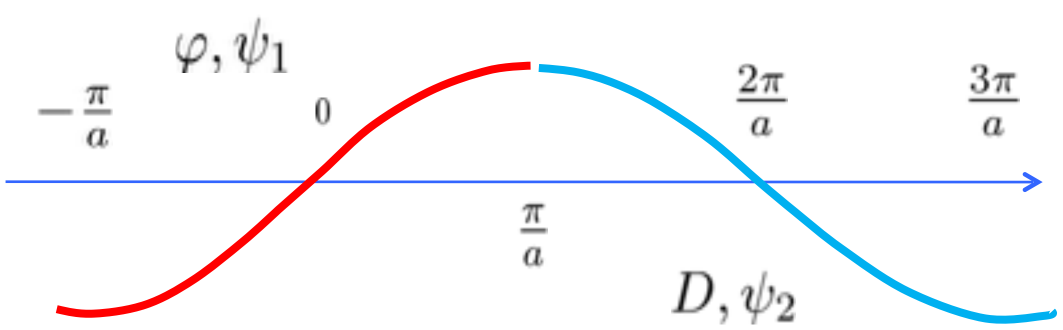
$$\delta_2\Phi(p) = \cos\frac{ap}{4}\alpha\Psi\left(\frac{2\pi}{a}-p\right)$$

$$\delta_2\Psi\left(\frac{2\pi}{a}-p\right) = 4\sin\frac{ap}{4}\alpha\Phi(p)$$

N=2 lattice SUSY algebra

$$\delta_1 = \alpha Q_1, \qquad \delta_2 = \alpha Q_2$$

$$Q_1^2 = Q_2^2 = 2\sin\frac{ap}{2}, \qquad \{Q_1, Q_2\} = 0$$



We propose a solution to (1).

Dondi & Nicolai

$$\{Q, Q\} = i\Delta(p) \qquad \hat{p} = \Delta_s(p) = \frac{2}{a}\sin\frac{ap}{2} \quad \left(-\frac{\pi}{a} \leq p \leq \frac{3\pi}{a}\right)$$

Natural identification of the conserved continuum momentum $\hat{p}$ is $\Delta_s(p)$!

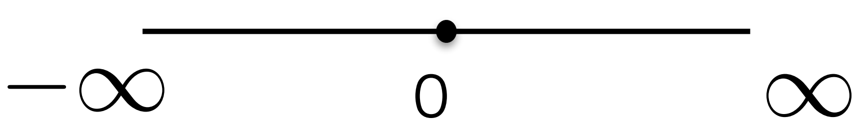
(A) Identification of species doublers as super partners

(In one dimensional example)

- Should the conserved continuum mom. $\hat{p}$ be the same as the lattice mom. p in finite a ?

$\hat{p} = p$ only in the $a \to 0$

$$-\infty < \hat{p} < \infty \quad \xleftarrow{a \to 0} \quad -\frac{\pi}{a} \le p \le \frac{\pi}{a} \quad \left(\text{diff. op.} = \frac{\sin ap}{a}\right)$$



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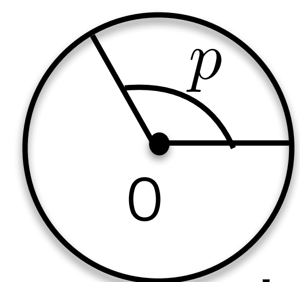


lattice:

angular variable

topologically different

It is very unnatural to identify: $\hat{p} = p$

natural one is: $\hat{p} = \Delta(p) = \frac{2}{a} \sin \frac{ap}{2}$ topologically same

generator of the SUSY algebra

(1) Solution to the exact lattice Leibniz rule

In order that lattice Leibniz rule is satisfied exactly as in the continuum case the lattice counter part of the conserved momentum should be modified !

$$F(p) \cdot G(p) = \int dp_1 dp_2 \delta(p - p_1 - p_2) F(p_1) G(p_2)$$

$$F(p) \star G(p) = \int dp_1 dp_2 \delta(\hat{p} - \hat{p}_1 - \hat{p}_2) F(p_1) G(p_2)$$

$$\hat{p} = \frac{2}{a} \sin \frac{ap}{2}$$

Momentum: $\delta(p - p_1 - p_2) \longrightarrow \delta(\hat{p} - \hat{p}_1 - \hat{p}_2)$

$$\hat{p}(F \star G)(p) = \int dp_1 dp_2 \; \hat{p}(F(p_1)G(p_2))\delta(\hat{p} - \hat{p}_1 - \hat{p}_2)$$

$$= \int dp_1 dp_2 [(\hat{p}_1 F(p_1))G(p_2) + F(p_1)(\hat{p}_2 G(p_2))]\delta(\hat{p} - \hat{p}_1 - \hat{p}_2)$$

Coordinate: $\phi(x) \cdot \phi(x) \longrightarrow \phi(x) \star \phi(x)$

$$\hat{\partial}(F(x) \star G(x)) = \hat{\partial} F(x) \star G(x) + F(x) \star \hat{\partial} G(x)$$

$$F = \tilde{\Phi}_1, \quad G = \tilde{\Phi}_2$$

$$\Delta(p)(\tilde{\Phi}_1 \star \tilde{\Phi}_2)(p) = \int dp_1 \int dp_2 \Delta(p) \tilde{\Phi}_1(p_1) \tilde{\Phi}_2(p_2) \delta(\Delta(p) - \Delta(p_1) - \Delta(p_2))$$

$$= \int dp_1 \int dp_2 [(\Delta(p_1)\tilde{\Phi}_1(p_1))\tilde{\Phi}_2(p_2) + \tilde{\Phi}_1(p_1)(\Delta(p_2)\tilde{\Phi}_2(p_2))] \delta(\Delta(p) - \Delta(p_1) - \Delta(p_2))$$

return to coordinate representation

$$\int dp e^{-ip\frac{na}{2}} \int dp_1 dp_2 [(\hat{p}_1 \tilde{\Phi}_1(p_1))\tilde{\Phi}_2(p_2) + \tilde{\Phi}_1(p_1)(\hat{p}_2 \tilde{\Phi}_2(p_2))] \delta(\hat{p} - \hat{p}_1 - \hat{p}_2)$$

$$= (\{-i\hat{\partial}\Phi_1\} \star \Phi_2 + \Phi_1 \star \{-i\hat{\partial}\Phi_2\})(\tfrac{na}{2})$$

$$(i\hat{\partial})(\Phi_1 \star \Phi_2)(\tfrac{na}{2}) = (\{i\hat{\partial}\Phi_1\} \star \Phi_2 + \Phi_1 \star \{i\hat{\partial}\Phi_2\})(\tfrac{na}{2})$$

Exact lattice Leibniz rule on $*$ product

Explicit expression of $*$ product

$$(\Phi_1 \star \Phi_2)(\tfrac{na}{2}) =$$

$$\int dp e^{-ip\frac{na}{2}} (\tilde{\Phi}_1 \star \tilde{\Phi}_2)(p) = \int dp e^{-ip\frac{na}{2}} \int dp_1 dp_2 \tilde{\Phi}_1(p_1) \tilde{\Phi}_2(p_2) \delta(\Delta(p) - \Delta(p_1) - \Delta(p_2))$$

$$= \int dp e^{-ip\frac{na}{2}} \int dp_1 dp_2 \sum_{n_1} e^{ip\frac{n_1 a}{2}} \Phi(\tfrac{n_1 a}{2}) \sum_{n_2} e^{ip\frac{n_2 a}{2}} \Phi(\tfrac{n_2 a}{2}) \int d\lambda e^{i\lambda(\Delta(p) - \Delta(p_1) - \Delta(p_2))}$$

$$= \int d\lambda \int dp e^{-i(p\frac{na}{2} - \lambda\Delta(p))} \sum_{n_1} \int dp_1 e^{i(p_1\frac{n_1 a}{2} - \lambda\Delta(p_1))} \Phi(\tfrac{n_1 a}{2}) \sum_{n_2} \int dp_2 e^{i(p_2\frac{n_2 a}{2} - \lambda\Delta(p_2))} \Phi(\tfrac{n_2 a}{2})$$

$$= \sum_{n_1, n_2} K(n, n_1, n_2)\ \Phi_1(\tfrac{n_1 a}{2})\ \Phi_2(\tfrac{n_2 a}{2})$$ non-local product

$$K(n, n_1, n_2) = \int_{-\infty}^{+\infty} d\lambda J_\Delta\left(\lambda, \tfrac{na}{2}\right) J_\Delta\left(\lambda, \tfrac{n_1 a}{2}\right) J_\Delta\left(\lambda, \tfrac{n_2 a}{2}\right)$$

$$J_\Delta\left(\lambda, \tfrac{na}{2}\right) = \frac{a}{2} \int_0^{\frac{4\pi}{a}} \frac{dp}{2\pi} e^{-i\left(n\frac{ap}{2} - \lambda\frac{a\Delta(p)}{2}\right)} \qquad \left(\Delta(p) = \hat{p} = \frac{2}{a}\sin\frac{ap}{2}\right)$$

For $\Delta(p) = \sin\frac{ap}{2}$:Bessel function J_Δ

(1) Lattice Leibniz rule is exactly recovered on the $\star$-product.

(2) We can kill the extra d.o.f. of species doublers by imposing:

(B): $\Phi_A(p) = \Phi_A\left(\frac{2\pi}{a} - p\right)$ (for all A)

In Wess-Zumino model this is chiral condition !

This is possible also for fermions since species doublers chirality is same as the particle chirality.

No chiral fermion

$$\hat{p} = \Delta(p) = \frac{2}{a}\sin\frac{ap}{2}$$

$$\frac{d\hat{p}}{dp} = \frac{d\Delta(p)}{dp}\Big|_{p=0} = \frac{d\Delta(\frac{2\pi}{a} - p)}{dp}\Big|_{p=\frac{2\pi}{a}} = \text{helicity}$$

We need: $\Delta\left(\frac{2\pi}{a} - p\right) = \Delta(p)$

In d-dimension this condition can be given as:

$$\Phi_A(p_1, \cdots, \frac{2\pi}{a} - p_j, \cdots, p_d) = \Phi_A(p_1, \cdots, p_j, \cdots, p_d)$$

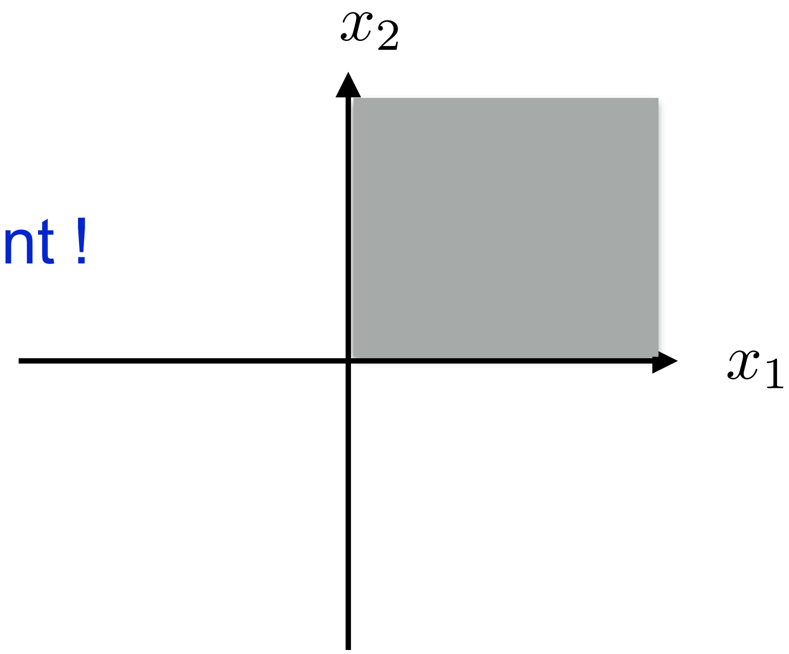
In the coordinate representation:

$$\Phi_A(\frac{n_1 a}{2}, \cdots, -\frac{n_j a}{2}, \cdots, \frac{n_d a}{2}) = (-1)^{n_j} \Phi_A(\frac{n_1 a}{2}, \cdots, \frac{n_j a}{2}, \cdots, \frac{n_d a}{2})$$

This amounts to consider only in the first quadrant of coordinate space !

Since we have introduced half lattice structure,

truncation of half d.o.f. for each dimension is consistent !



However

- **Breakdown of associativity**

$$(\phi_1 \star (\phi_2 \star \phi_3))(p) \neq ((\phi_1 \star \phi_2) \star \phi_3)(p)$$

$$\hat{p}_i = \frac{2}{a} \sin \frac{ap}{2}$$

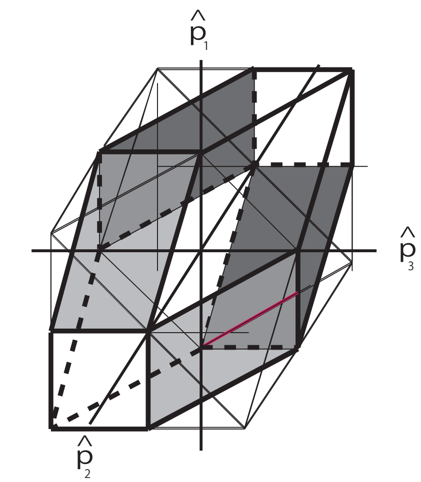
$$(\phi_1 \star (\phi_2 \star \phi_3))(p) = \int dp_1 dq \phi_1(p_1) \delta(\hat{p} - \hat{p}_1 - \hat{q})$$
$$\times (\int dp_2 dp_3 \phi_2(p_2) \phi_3(p_3) \delta(\hat{q} - \hat{p}_2 - \hat{p}_3))$$

$$((\phi_1 \star \phi_2) \star \phi_3)(p) = \int dp_3 dq (\int dp_1 dp_2 \phi_1(p_1) \phi_2(p_2) \delta(\hat{q} - \hat{p}_1 - \hat{p}_2))$$
$$\times \phi_3(p_3) \delta(\hat{p} - \hat{q} - \hat{p}_3)$$

Integration region

(A) $|\hat{p}_i| < \frac{2}{a}, \quad |\hat{p}_2 + \hat{p}_3| < \frac{2}{a}, \quad |\hat{p}_1 + \hat{p}_2 + \hat{p}_3|, < \frac{2}{a}$

(B) $|\hat{p}_i| < \frac{2}{a}, \quad |\hat{p}_1 + \hat{p}_2| < \frac{2}{a}, \quad |\hat{p}_1 + \hat{p}_2 + \hat{p}_3|, < \frac{2}{a}$



Since ϕ_1, ϕ_2 and ϕ_3 are different fields, associativity is broken

But we can recover the associativity if we find a derivative operator $\Delta(p)$ which satisfies:

1) $\Delta(-p) = -\Delta(p), \quad -\infty < \Delta < \infty$

2) $\frac{a}{2}\Delta(p) = \frac{ap}{2} + O\left((\frac{ap}{2})^3\right)$

3) $\Delta(p)$ has to cover twice the whole real axis as p goes through the $\frac{4\pi}{a}$ period.

$$\boxed{\Delta(p) = \Delta(\frac{2\pi}{a} - p)} \qquad \lim_{p \to \pm\frac{\pi}{a}} \Delta(p) = \pm\infty$$

Associativity is recovered and thus all the difficulties can be solved at the same time !

$$-\infty < \hat{p}_i = \Delta_G(p_i) < \infty, \qquad -\infty < \Delta_G(p_i) + \Delta_G(p_j) < \infty$$

Second step proposal

$$\frac{a}{2}\Delta_G(p) = \sin\frac{ap}{2} + \frac{1}{3}\left(\sin\frac{ap}{2}\right)^3 + \frac{1}{5}\left(\sin\frac{ap}{2}\right)^5 + \frac{1}{7}\left(\sin\frac{ap}{2}\right)^7 + \cdots$$

$$= 2\left[\sin\frac{ap}{2} - \frac{1}{3}\sin\frac{3ap}{2} + \frac{1}{5}\sin\frac{5ap}{2} - \frac{1}{7}\sin\frac{7ap}{2} + \cdots\right]$$

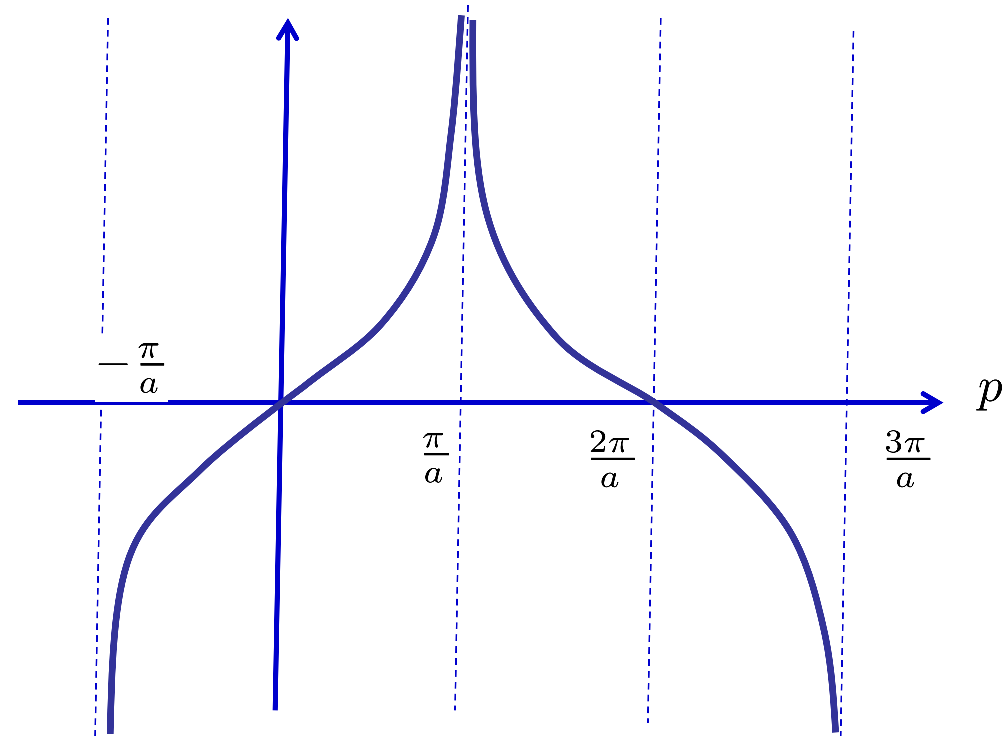
$$\hat{p} = \Delta_G(p) = \frac{1}{a}\log\frac{1+\sin\frac{ap}{2}}{1-\sin\frac{ap}{2}}$$

Gudermannian function

$$\Delta_G\Phi(x) = \frac{2}{a}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}\left[\Phi\left(x+\frac{(2k-1)a}{2}\right) - \Phi\left(x-\frac{(2k-1)a}{2}\right)\right]$$

nonlocal dif. op.

$$\Delta_G(p) = \frac{1}{a}\log\frac{1+\sin\frac{ap}{2}}{1-\sin\frac{ap}{2}}$$



associativity OK

$$-\infty < \hat{p}_i = \Delta_G(p_i) < \infty$$

$$-\infty < \Delta_G(p_i) + \Delta_G(p_j) < \infty$$

$$\Delta\left(\frac{2\pi}{a} - p\right) = \Delta(p)$$

$\Delta(p)$ has to cover twice the whole real axis as p goes through the $\frac{4\pi}{a}$ period

We can find continuum equivalent lattice field theory formulation !

How come this is possible ?

(1) Difference operator satisfies exact Leibniz rule on $\star$-product

(2) No chiral fermion doublers: truncation of the doublers d.o.f.

(3) Associativity is satisfied for the $\star$-product.

Sketch of the derivation

Conventional product in mom. representation: **convolution**

$$\hat{p}_\mu = \Delta(p_\mu) \qquad : \Delta\left(\pm\frac{\pi}{a}\right) = \pm\infty$$

$$\widetilde{\Phi_1 \star \Phi_2}(\hat{p}) = \frac{2}{\pi}\int_{\Delta(-\frac{\pi}{a})}^{\Delta(\frac{\pi}{a})} d\hat{p}_1 d\hat{p}_2 \tilde{\Phi}_1(\hat{p}_1)\tilde{\Phi}_2(\hat{p}_2)\delta\left(\hat{p} - \hat{p}_1 - \hat{p}_2\right)$$

$$f(p)\frac{1}{f(p)}\frac{d\hat{p}}{dp}\widetilde{\Phi_1 \star \Phi_2}(\hat{p}) = \frac{d\hat{p}}{dp}\frac{2}{\pi}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dp_1 dp_2 f(p_1)\frac{1}{f(p_1)}\frac{d\hat{p}_1}{dp_1}\tilde{\Phi}_1(\hat{p}_1) f(p_2)\frac{1}{f(p_2)}\frac{d\hat{p}_2}{dp_2}\tilde{\Phi}_2(\hat{p}_2)\delta\left(\hat{p} - \hat{p}_1 - \hat{p}_2\right)$$

Non-local star-product on the lattice with arbitrary f(p)

$$f(p)\ \widetilde{\varphi_1 \star \varphi_2}(p) = \frac{2}{\pi}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dp_1 dp_2 \frac{d\Delta(p)}{dp} f(p_1)\tilde{\varphi}_1(p_1)\ f(p_2)\tilde{\varphi}_2(p_2)\delta\left(\Delta(p) - \Delta(p_1) - \Delta(p_2)\right)$$

continuum mom. $\hat{p}$ as a function of lattice mom. p

$$\hat{p} = \Delta(p) \qquad \tilde{\varphi}(p) = \frac{1}{f(p)}\frac{d\Delta(p)}{dp}\tilde{\Phi}(\Delta(p)) \qquad -\frac{\pi}{a} \le p \le \frac{\pi}{a}$$

equivalent to the continuum product: **convolution**

$$f(p)\ \widetilde{\varphi_1\star\varphi_2}(p)=\frac{2}{\pi}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}dp_1dp_2\frac{d\Delta(p)}{dp}f(p_1)\tilde{\varphi}_1(p_1)\ f(p_2)\tilde{\varphi}_2(p_2)\delta\left(\Delta(p)-\Delta(p_1)-\Delta(p_2)\right)$$

substitute: $f(p)=\sqrt{\frac{d\Delta_G}{dp}}=\frac{1}{\sqrt{\cos\frac{ap}{2}}}$

$$\sqrt{\cos\frac{ap}{2}}\ \widetilde{\varphi_1\star\varphi_2}(p)=\frac{2}{\pi}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}\frac{dp_1}{\sqrt{\cos\frac{ap_1}{2}}}\frac{dp_2}{\sqrt{\cos\frac{ap_2}{2}}}\tilde{\varphi}_1(p_1)\ \tilde{\varphi}_2(p_2)\int d\xi\ e^{i\xi(\Delta_G(p)-\Delta_G(p_1)-\Delta_G(p_2))}$$

$$\tilde{\varphi}(p)=\frac{a}{2}\sum_n\varphi_0(x_n)e^{-i\frac{na}{2}p},\qquad -\frac{\pi}{a}\le p\le\frac{\pi}{a}$$

Coordinate representation of star product: non-local

$$(\varphi_1\star\varphi_2)^{(0)}(x_n)=\frac{a^2}{4}\sum_{n_1,n_2}K_{n,n_1,n_2}\varphi_1^{(0)}(x_{n_1})\varphi_2^{(0)}(x_{n_2})$$

$$K_{n,n_1,n_2}=\int_{-\infty}^{+\infty}d\xi\ J_{\Delta_G}^{(0)}(\xi,x_n)\ J_{\Delta_G}^{(0)}(\xi,x_{n_1})\ J_{\Delta_G}^{(0)}(\xi,x_{n_2})$$

transformation function from lattice to continuum:

$$J_{\Delta_G}^{(0)}(\xi,x_n)=\frac{1}{2\pi}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}\frac{dp}{\sqrt{\cos\frac{ap}{2}}}\ e^{-ix_np+i\xi\Delta_G(p)}$$

lattice to continuum

$$\Phi(\xi) = \frac{a}{2}\sum_n J^{(0)}_{\Delta_G}(\xi, x_n)\varphi_0(x_n)$$

continuum to lattice

$$\varphi_0(x_n) = \int_{-\infty}^{\infty} d\xi \ \bar{J}^{(0)}_{\Delta_G}(x_n, \xi)\ \Phi(\xi)$$

Invertible ! if $f(p) = \sqrt{\frac{d\Delta_G}{dp}} = \frac{1}{\sqrt{\cos\frac{ap}{2}}}$

transformation function

$$e^{ipx} \leftrightarrow \quad \bar{J}^{(0)}_{\Delta_G,}(\eta, \xi) = J^{(0)}_{\Delta_G,}(\xi, \eta) = \frac{N}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\cos(\theta)}}\ e^{iN\left(\xi \mathrm{gd}^{-1}(\theta) - \eta\theta\right)}$$

$$\lim_{N\to\infty}\int_{-\infty}^{+\infty} d\xi\ J^{(0)}_{\Delta_G}(\xi, \eta)\chi(\xi) = \chi(\eta)$$

$$(N = \frac{2}{a},\ \theta = \frac{ap}{2},\ \mathrm{gd}^{-1}(\theta) = \frac{a}{2}\Delta_G(\theta))$$

$$\frac{Na}{2} \to \eta \quad (a \to 0, N \to \infty)$$

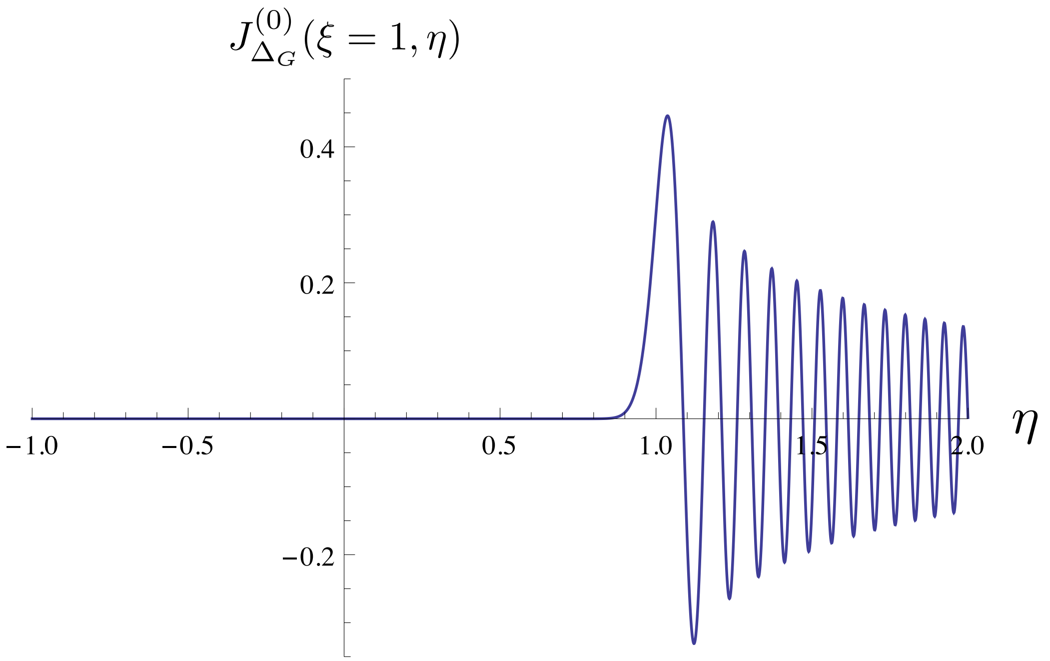
continuum limit coordinate

locality assured

$$\lim_{N\to\infty} J^{(0)}_{\Delta_G}(\xi, \eta) = \delta(\xi - \eta)$$

transformation function leads to delta func.

$$\lim_{N\to\infty} J^{(0)}_{\Delta_G}(\xi,\eta) = \delta(\xi-\eta)$$



the relation between the lattice and cont. wave func.

$$\tilde{\varphi}(p) = \frac{1}{f(p)}\frac{d\Delta(p)}{dp}\tilde{\Phi}(\Delta(p)) \qquad -\frac{\pi}{a} \le p \le \frac{\pi}{a} \qquad f(p) = \sqrt{\frac{d\Delta_G}{dp}} = \frac{1}{\sqrt{\cos\frac{ap}{2}}}$$

$$\tilde{\varphi_\mu}(p_\mu) = \sqrt{\frac{d\Delta_G(p_\mu)}{dp_\mu}}\tilde{\Phi}_\mu(\hat{p}_\mu) = \frac{1}{\sqrt{\cos\frac{ap_\mu}{2}}}\tilde{\Phi}_\mu(\hat{p}_\mu)$$

$$\left(\tilde{\varphi}_A(\frac{2\pi}{a}-p) = \tilde{\varphi}_A(p),\ \varphi(-\frac{na}{2}) = (-1)^n\ \varphi(\frac{na}{2})\right)$$

doublers identified !

We can construct continuum equivalent lattice field theory as follows:

(1) Write down the momentum representation of continuum action.

(2) Replace the continuum derivative operator by $\Delta_G(p)$

(3) Replace the continuum wave function by the lattice wave function as:

$$\tilde{\varphi}_j(p_j) = \sqrt{\frac{\Delta_G(p_j)}{dp_j}}\,\tilde{\Phi}_j(\hat{p}_j) = \frac{1}{\sqrt{\cos\frac{ap_j}{2}}}\,\tilde{\Phi}_j(\hat{p}_j)$$

(4) Then we obtain continuum equivalent non-local lattice action.

Continuum theory

||

Non-local lattice field theory

but local in the limit

so far no regularization yet !

How do we regularize the non-local lattice theory ?

Possible regularization >> cut off theory

$$\Delta_G^{(z)}(p) = \frac{2}{a}\mathrm{gd}^{-1}(x, \hat{z}) \qquad (\hat{z} = \frac{2z}{1+z^2})$$

$$\mathrm{gd}^{-1}(x, \hat{z}) = \frac{1}{2\hat{z}} \log \frac{1+\hat{z}\sin x}{1-\hat{z}\sin x} \qquad \left(\mathrm{gd}^{-1}(x) = \frac{1}{2}\log\frac{1+\sin x}{1-\sin x}\right)$$

extrapolation between local and non-local difference op.

$$\hat{z} \to 0 : \quad \sin x : \text{symmetric difference operator}(x = \frac{ap}{2})$$

$$\frac{1}{a}\mathrm{gd}^{-1}(\frac{ap}{2}, 0) = \frac{1}{a}\sin\frac{ap}{2} \to \left(\varphi(na + \frac{a}{2}) - \varphi(na - \frac{a}{2})\right)/a$$

$$\hat{z} = 1 : \quad \text{Inverse Gudermannian} \qquad \frac{1}{a}\mathrm{gd}^{-1}(\frac{ap}{2}, 1) \to \Delta_G\varphi(x)$$

$$|\hat{p}| = |\Delta_G^{(z)}(p)| \le \frac{1}{a\hat{z}}\log\frac{1+\hat{z}}{1-\hat{z}} = \hat{p}^{(\mathrm{cutoff})}$$

For a given momentum two parameters: a and $\hat{z}$

Blocking transformation (block spin) from continuum to lattice:

$$\varphi_A(n) = \int dx f(nl - x)\Phi_A(x) \longleftrightarrow \varphi_0(x_n) = \int_{-\infty}^{\infty} d\xi \ \bar{J}^{(0)}_{\Delta_G}(x_n, \xi) \ \Phi(\xi)$$

$$e^{-S_\Delta(\tilde{\varphi})} = \int \mathcal{D}\tilde{\Phi}_A \prod_\mu \prod_{p_\mu = -\frac{\pi}{a}}^{\frac{3\pi}{a}} \prod_A \delta\left(\tilde{\Phi}_A(\Delta(p)) - \frac{F(p)}{\left|\prod_{\mu=1}^d \frac{d\Delta(p_\mu)}{dp_\mu}\right|}\tilde{\varphi}_A(p)\right) e^{-S_{cl}(\tilde{\Phi})}$$

$$= \prod_{\mu=1}^d \sqrt{\cos\frac{ap_\mu}{2}} \quad \text{for } \Delta = \Delta_G$$

Regularization

$$\tilde{\varphi}_j(p_j) = \sqrt{\frac{\Delta_G(p_j)}{dp_j}}\tilde{\Phi}_j(\hat{p}_j) = \frac{1}{\sqrt{\cos\frac{ap_j}{2}}}\tilde{\Phi}_j(\hat{p}_j)$$

Example 1

Regularization for Wess-Zumino model (4-dim.)

Coordinate representation: continuum

$$S = \int d^4x \left(i\Psi^\dagger \bar{\sigma}^\mu \partial_\mu \Psi - \partial^\mu \Phi^\star \partial_\mu \Phi \right) \qquad (\bar{\sigma}^\mu) = (\sigma^0, -\sigma^i)$$

$$\delta_\epsilon \Phi = \epsilon \Psi, \qquad \delta_\epsilon \Phi^\star = \epsilon^\dagger \Psi^\dagger$$

$$\delta_\epsilon \Psi_\alpha = -i(\sigma^\mu \epsilon^\dagger)_\alpha \partial_\mu \Phi, \qquad \delta_\epsilon \Psi^\dagger_{\dot\alpha} = i(\epsilon\sigma^\mu)_{\dot\alpha} \partial_\mu \Phi^\star$$

Momentum representation:

$$S = \frac{1}{(2\pi)^4} \int d^4\hat{p}_1 d^4\hat{p}_2 \ \delta^4 \left(\hat{p}_1 + \hat{p}_2\right) \left[-\tilde{\Psi}^\dagger(\hat{p}_1)\bar{\sigma}_\mu \hat{p}_2^\mu \tilde{\Psi}(\hat{p}_2) + \hat{p}_{1\mu}\tilde{\Phi}(\hat{p}_1)\hat{p}_2^\mu \tilde{\Phi}^\dagger(\hat{p}_2) \right]$$

Regularized lattice action:

$$S^{(z)} = \frac{1}{\pi^4} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} d^4p_1 \ d^4p_2 \ \prod_\mu \delta\left(p_{1\mu} + p_{2\mu}\right) \left[-\tilde{\psi}^\dagger(p_1)\bar{\sigma}^\mu \Delta_G^{(z)}(p_{2\mu})\tilde{\psi}(p_2) + \Delta_G^{(z)}(p_{1\mu})\tilde{\varphi}(p_1)\Delta_G^{(z)}(p_2^\mu)\tilde{\varphi}^\dagger(p_2) \right]$$

The corresponding cut off continuum:

$$S^{(z)} = \frac{1}{(2\pi)^4} \int_{|\hat{p}_\mu| \le \hat{p}^{(\mathrm{cutoff})}} d^4\hat{p}_1 d^4\hat{p}_2 \ \delta^4 \left(\hat{p}_1 + \hat{p}_2\right) \left[-\tilde{\Psi}^\dagger(\hat{p}_1)\bar{\sigma}_\mu \hat{p}_2^\mu \tilde{\Psi}(\hat{p}_2) + \hat{p}_{1\mu}\tilde{\Phi}(\hat{p}_1)\hat{p}_2^\mu \tilde{\Phi}^\dagger(\hat{p}_2) \right]$$

SUSY exact even with interactions !

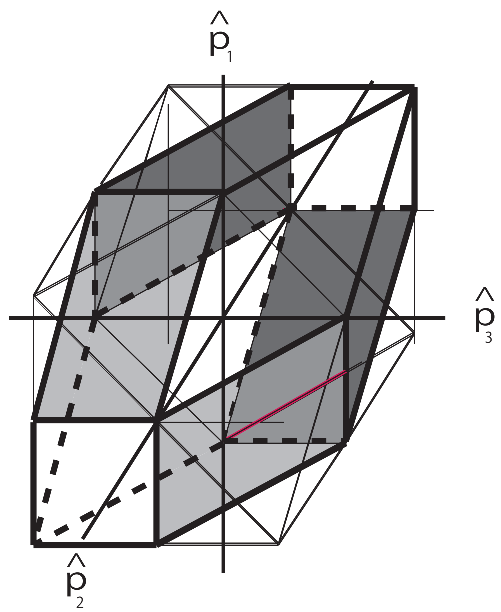
vacuum energy can be exactly vanishing !

Associativity is again broken when regularized !

$$\begin{aligned}
&(\Phi_1 \star (\Phi_2 \star \Phi_3))(p_{123}) \\
&= \int dp_{23} \int dp_1 \int dp_2 \int dp_3 \tilde{\Phi}_1(p_1)\tilde{\Phi}_2(p_2)\tilde{\Phi}_3(p_3)\delta(\hat{p}_{23}-\hat{p}_2-\hat{p}_3)\delta(\hat{p}_{123}-\hat{p}_1-\hat{p}_{23}) \\
&\neq \int dp_{12} \int dp_1 \int dp_2 \int dp_3 \tilde{\Phi}_1(p_1)\tilde{\Phi}_2(p_2)\tilde{\Phi}_3(p_3)\delta(\hat{p}_{12}-\hat{p}_1-\hat{p}_2)\delta(\hat{p}_{123}-\hat{p}_3-\hat{p}_{12}) \\
&= ((\Phi_1 \star \Phi_2) \star \Phi_3)(p_{123})
\end{aligned}$$

$$dp_1 dp_2 dp_3 \delta(\hat{p}_{23}-\hat{p}_2-\hat{p}_3)\delta(\hat{p}_{123}-\hat{p}_1-\hat{p}_{23}) \neq dp_1 dp_2 dp_3 \delta(\hat{p}_{12}-\hat{p}_1-\hat{p}_2)\delta(\hat{p}_{123}-\hat{p}_3-\hat{p}_{12})$$

$$-\hat{p}^{(\mathrm{cutoff})} \leq \hat{p}_i \leq \hat{p}^{(\mathrm{cutoff})}, \quad |\hat{p}_i + \hat{p}_j| \leq \hat{p}^{(\mathrm{cutoff})} = \frac{1}{a\hat{z}} \log \frac{1+\hat{z}}{1-\hat{z}}$$



Gauge invariance is not exact in the regularized formulation !

$$\Phi^\dagger(x)\star\Phi(x)\to(\Phi^\dagger(x)\star e^{-i\alpha(x)})\star(e^{i\alpha(x)}\star\Phi(x))$$
$$\neq\Phi^\dagger(x)\star(e^{-i\alpha(x)}\star e^{i\alpha(x)})\star\Phi(x)=\Phi^\dagger(x)\star\Phi(x)$$

Gauge invariance is only exact at the limit : $\hat{z}=z=1$

Example 2

Φ^4 -Theory (cut off formulation)

$$S^{(z)} = \int_{-\hat{p}^{(\mathrm{cutoff})}}^{\hat{p}^{(\mathrm{cutoff})}} d^4\hat{p}_1 d^4\hat{p}_2 \; \delta^{(4)}(\hat{p}_1+\hat{p}_2)\left[-\hat{p}_1^{\mu}\tilde{\Phi}(\hat{p}_1)\hat{p}_{2\mu}\tilde{\Phi}(\hat{p}_2)+m_0^2\tilde{\Phi}(\hat{p}_1)\tilde{\Phi}(\hat{p}_2)\right]$$
$$+ \quad \lambda_0 \int_{-\hat{p}^{(\mathrm{cutoff})}}^{\hat{p}^{(\mathrm{cutoff})}} \prod_{i=1}^{4} d^4\hat{p}_i \; \delta^{(4)}(\hat{p}_1+\hat{p}_2+\hat{p}_3+\hat{p}_4)\; \tilde{\Phi}(\hat{p}_1)\tilde{\Phi}(\hat{p}_2)\tilde{\Phi}(\hat{p}_3)\tilde{\Phi}(\hat{p}_4)$$

replace cotinuum mom. by regularized lattice mom.

$$\hat{p}_{i\mu} = \Delta_G^{(z)}(p_{i\mu})$$

replace continuum wave func. by lattice wave func.

$$\tilde{\Phi}(\Delta_G^{(z)}(p_\mu)) = \frac{2\tilde{\varphi}(p)}{\prod_\mu \sqrt{\frac{d\Delta_G^{(z)}(p_\mu)}{dp_\mu}}}$$

regularized momentum derivative:

$$\frac{d\Delta_G^{(z)}(p_\mu)}{dp_\mu} = \frac{\cos\frac{ap_\mu}{2}}{1-z^2\sin^2\frac{ap_\mu}{2}} \xrightarrow{z\to 1} \frac{1}{\cos\frac{ap_\mu}{2}}$$

from continuum to lattice:

$$d^4\hat{p}_1\delta^{(4)}(\hat{p}_1+\cdots)\tilde{\Phi}(\hat{p}_1) \to d^4p_1\delta(p_1+\cdots)\prod_{\mu=1}^{4}\sqrt{\frac{\cos\frac{ap_{1\mu}}{2}}{1-z^2\sin^2\frac{ap_{1\mu}}{2}}}\tilde{\varphi}(p_1)$$

A more general regularization with α

$$(z=1,\ \alpha \to \frac{1}{2})$$

$$d^4\hat{p}_1 \delta^{(4)}(\hat{p}_1+\cdots)\tilde{\Phi}(\hat{p}_1) \to d^4 p_1 \delta(p_1+\cdots)\left(\prod_{\mu=1}^{4} \frac{d\Delta_G^{(z)}(p_{i\mu})}{dp_{i\mu}}\right)^{\alpha} \tilde{\varphi}(p_1)$$

Lattice:

$$S^{\alpha} = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} d^4p_1 d^4p_2 \prod_{\mu=1}^{4}\delta(p_{1\mu}+p_{2\mu})\left[-\Delta_G(p_{1\mu})\Delta_G(p_{2\mu}) + m_0^2\left(\prod_{\mu}\cos\frac{ap_{1\mu}}{2}\right)^{1-2\alpha}\right]\tilde{\varphi}(p_1)\tilde{\varphi}(p_2)$$

$$+\lambda_0\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}\prod_{i=1}^{4}d^4p_i\prod_{\mu=1}^{4}\delta\left(\sum_{i=1}^{4}\Delta_G(p_{i\mu})\right)\left(\prod_{i,\mu=1}^{4}\cos\frac{ap_{i\mu}}{2}\right)^{-\alpha}\tilde{\varphi}(p_1)\tilde{\varphi}(p_2)\tilde{\varphi}(p_3)\tilde{\varphi}(p_4)$$

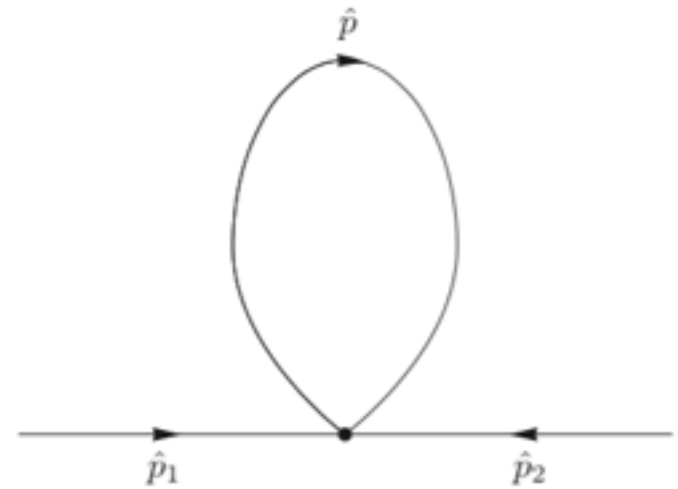
↓

cut off for $\hat{p}_{1\mu} > 1/\left(\frac{1}{2}-\alpha\right)$

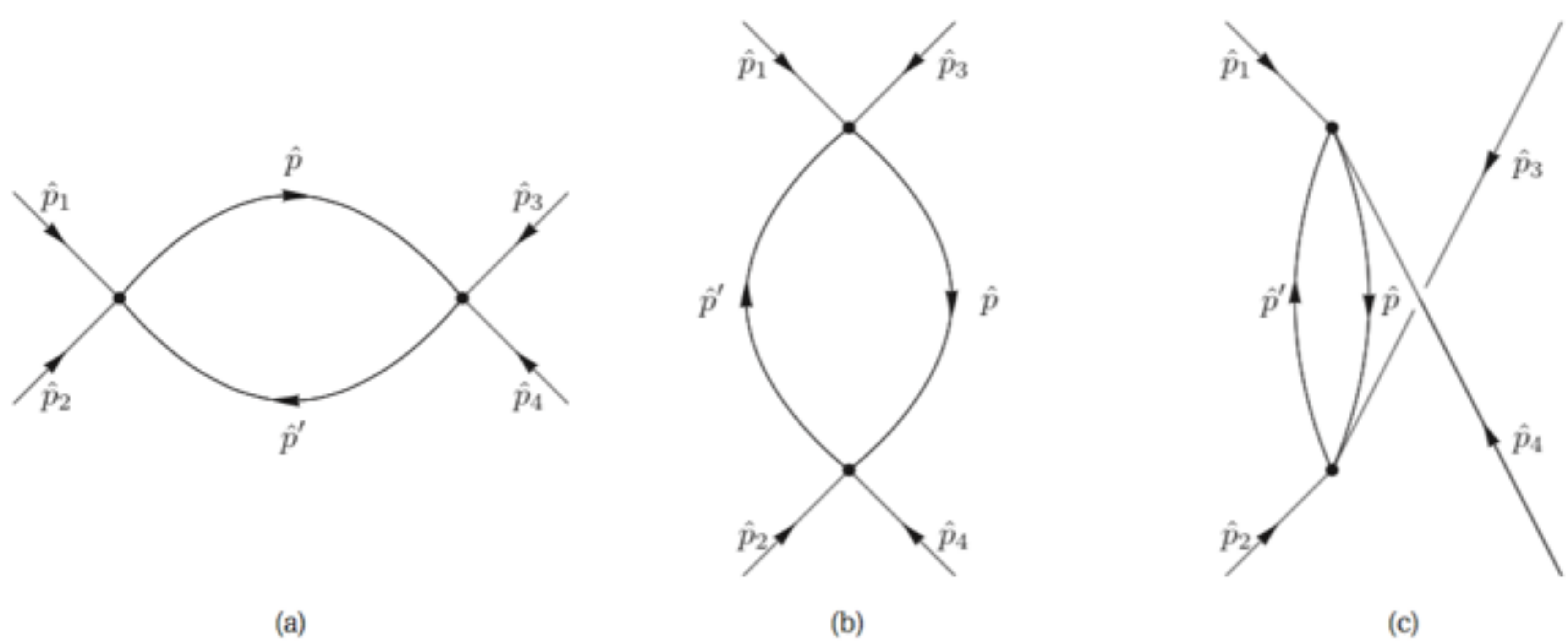
Regularized continuum:

$$S^{(\alpha)} = \int_{-\infty}^{\infty} d^4\hat{p}_1 d^4\hat{p}_2\ \delta^{(4)}(\hat{p}_1+\hat{p}_2)\left[-\hat{p}_1^{\mu}\hat{p}_{2\mu} + \frac{m_0^2}{\left(\prod_{\mu}\cosh\frac{a\hat{p}_{1\mu}}{2}\right)^{1-2\alpha}}\right]\tilde{\Phi}(\hat{p}_1)\tilde{\Phi}(\hat{p}_2)$$

$$+\ \lambda_0\int_{-\infty}^{\infty}\prod_{i=1}^{4}d^4\hat{p}_i\ \delta^{(4)}(\hat{p}_1+\hat{p}_2+\hat{p}_3+\hat{p}_4)\ \frac{\tilde{\Phi}(\hat{p}_1)\tilde{\Phi}(\hat{p}_2)\tilde{\Phi}(\hat{p}_3)\tilde{\Phi}(\hat{p}_4)}{\left(\prod_{\mu}\prod_{i=1}^{4}\cosh\frac{a\hat{p}_{i\mu}}{2}\right)^{1/2-\alpha}}$$



One-loop correction to the propagator.



One-loop corrections to the four point vertex.

$$D^{(\alpha)}(\hat{p}_1, \hat{p}_2) = \frac{\prod_\mu \delta\left(\hat{p}_{1\mu} + \hat{p}_{2\mu}\right)}{\sum_\mu \hat{p}_1^\mu \hat{p}_{1\mu} + \left(\prod_\mu \cosh \frac{a\hat{p}_{1\mu}}{2}\right)^{2\alpha-1} m_0^2},$$

mass renormalization

$$D^{(\alpha)}_{1\text{loop}}(\hat{p}_1, \hat{p}_2) = \frac{\prod_\mu \delta\left(\hat{p}_{1\mu} + \hat{p}_{2\mu}\right)}{\sum_\mu \hat{p}_1^\mu \hat{p}_{1\mu} + \left(\prod_\mu \cosh \frac{a\hat{p}_{1\mu}}{2}\right)^{2\alpha-1} (m_0^2 + \lambda_0 I_\alpha)}$$

$$I_\alpha = b_2 \left(\frac{1}{a(1/2-\alpha)}\right)^2 + b_0 \; m_0^2 \log(1/2-\alpha) + \text{regular terms}$$

similarly coupling constant renormalization

$$\lambda_0 \longrightarrow \lambda_0 + \lambda_0^2 \left(I_\alpha(\hat{p}_1 + \hat{p}_2) + I_\alpha(\hat{p}_1 + \hat{p}_3) + I_\alpha(\hat{p}_3 + \hat{p}_2)\right)$$

$$I_\alpha(\hat{p}) = c_0 \log(1/2-\alpha) + \text{regular terms}$$

Conclusion and Discussions

Continuum equivalent lattice field theory is defined.
It defines non-local field theory but locality is recovered in the continuum limit

Symmetries of continuum theory are kept exactly; Poincare symmetry, supersymmetry, $\cdots$ and no chiral fermion problems.

In the regularized version associativity is broken and thus gauge invariance is recovered only in the limit.

Finite box regularization: $\delta\left(\Delta_G(p_{12}) - \Delta_G(p_1) - \Delta_G(p_2)\right)$

Perfect action for regularized version !