Exact Solution of Noncommutative Φ^3 Model

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Discrete Approaches to the Dynamics of Fields and Space-Time
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Talk plan

1. History
2. The Origin from N.C. field theory
3. Set up
4. Ward-Takahashi Id. & Schwinger-Dyson Eqs.
5. Large (N, L) limit & Solutions
6. Which kind of Q.F.T.
7. Axiomatic construction of Φ^3 field theory

§1 History

- 90s' Matrix model

◎ 2D gravity $\Leftrightarrow$ random matrix

Brezin -Kazakov, Gross-Migdal, etc.

Kontsevich model (Witten conjecture)

$$Z[J] = \int \mathcal{D}\Phi \, \exp\left(-\mathrm{tr}\left(-J\Phi + \frac{N}{2}\beta\Phi^2 + \frac{N}{3}\alpha\Phi^3\right)\right)$$

Φ : Hermite. matrix

Makeenko - Semenoff solve this.

▶ 2000's

N.C. field Theory $\Rightarrow$ Matrix model.

Grosse-Steinacker

Φ^3 model (Kontsevich model) Renormalizable

Grosse-Wulkenhaar

Φ^4 models in 2, 4 dim are Renormalizable

Φ^4 model is solvable

(SD-eq is recursivly determined.)

Contents

- Φ^3 models (Kontsevich model 2, 4, 6 dim) are solved exactly at large N limit (Every N-pt function is given explicitly.)

- Approch to the Axiomatic Construction of the Φ^3 Quantum Field Theory

§2. ~The Origin from N.C. field Theory~

$\mathbb{R}^2_\theta$: Moyal plane $\quad [A,B] := AB - BA$

$$[x^1, x^2] = i\underbrace{\theta}_{\text{N.C. parameter}} \Leftrightarrow [z, \bar{z}] = 2\theta$$

- Annihilation $\left(a := \frac{z}{\sqrt{2\theta}} \right)$ Creation $\left(a^\dagger := \frac{\bar{z}}{\sqrt{2\theta}} \right)$ op.

$$\Rightarrow [a, a^\dagger] = 1, \quad [a,a] = [a^\dagger, a^\dagger] = 0$$

- $\frac{\partial}{\partial z} = -\frac{1}{\sqrt{2\theta}}[a^\dagger, \ ], \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{\sqrt{2\theta}}[a, \ ]$

- Fock sp. $[a, a^\dagger] = 1$, $[a, a] = [a^\dagger, a^\dagger] = 0$

$|0\rangle$: $a|0\rangle = 0$, $|n\rangle := \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle$

Number op. $N := a^\dagger a$ $\quad N|n\rangle = n|n\rangle$

$\langle n|$ = dual of $|n\rangle$

$\langle n|m\rangle = \delta_{nm}$

- Scalar field $\phi = \sum \phi_{nm} |m\rangle\langle n|$

$$\int d^2x \rightarrow \theta^2 \, \mathrm{Tr}$$

Action

$$S_1 = \int d^4x \, -\phi \left(\frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \right) \phi$$

$$= \frac{\theta^2}{2\theta} \mathrm{Tr}\, \phi \, [a^\dagger, [a, \phi]] \qquad N = a^\dagger a$$

$$= \theta \, \mathrm{Tr} (\phi N \phi - \underline{\theta a^\dagger \phi a \phi})$$

Removing this term by a counter Lagrangian

Renormalizable model is obtained.

$$S_m = \theta \, \mathrm{Tr} \frac{\mu^2}{2} \phi^2$$

μ: const. (mass)

Interaction

$$S_{int} = \Theta \frac{\lambda}{3} \mathrm{Tr}\, \phi^3$$

λ : const

coupling const

$$S = S_1 + S_m + S_{int} + \text{tadpole}$$

$$= \Theta\, \mathrm{Tr} \left(\phi E \phi - A\phi + \frac{\lambda}{3} \phi^3 \right)$$

where $E_{nm} = \left(\frac{1}{2}\mu^2 + n\right)\delta_{nm}$

A : const

§3 ~Setup~ (2-dim case for simplicity)

Hermitian Matrix $\Phi = \Phi^\dagger \in M_N(\mathbb{C})$

Action $S = L\,\mathrm{Tr}(E\Phi^2 - A\Phi) + V(\Phi)$

$$V(\Phi) = L\frac{\lambda}{3}\mathrm{Tr}\,\Phi^3$$

$$E = (E_m \delta_{mn}) = \begin{pmatrix} E_1 & & 0 \\ 0 & E_2 & \\ & & \ddots \end{pmatrix}$$

$$E_m = \mu^2\left(\frac{1}{2} + e\left(\frac{m}{\mu^2 L}\right)\right),$$

θ

A, λ, L, μ : const. $e(0)=0$, C^∞-fun

$$S = L\left(\sum_{n,m}^{N} \frac{1}{2}\Phi_{nm}\Phi_{mn}H_{nm} - A\sum_{m=0}^{N}\Phi_{mm} + \frac{\lambda}{3}\sum_{k,l,m}^{N}\Phi_{kl}\Phi_{lm}\Phi_{mk}\right)$$

$$H_{mn} := E_m + E_n = \mu^2\left(1 + e\left(\frac{m}{\mu^2 L}\right) + e\left(\frac{n}{\mu^2 L}\right)\right)$$

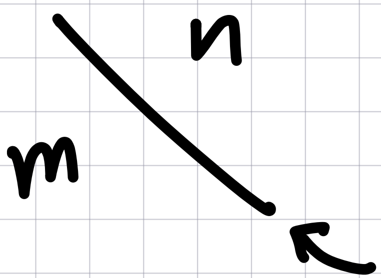
$$Z[J] := \int \mathcal{D}\Phi \, e^{-S + L\,\mathrm{Tr}(J\Phi)}$$

$$= K \exp\left(-V\left(\frac{1}{L}\frac{\partial}{\partial J}\right)\right) Z_{free}[J]$$

$$Z_{free} = e^{\sum \frac{L}{2}(\delta_{nm}A + J_{nm})H_{nm}^{-1}(\delta_{nm}A + J_{mn})}$$

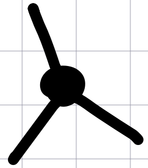
Remark) Correspondence with Graphs.

propergator:



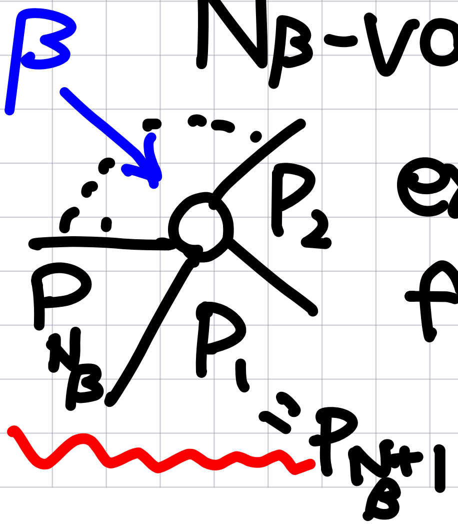
$\sim \dfrac{1}{H_{mn}}$

Black vertex:



$\sim \lambda \, \mathrm{tr}\, \Phi^3$

white vertex:



N_β-valence external field $\sim \mathbb{J}_{p_1 \cdots p_{N_\beta}} = \prod_{j=1}^{N_\beta} J_{p_j p_{j+1}}$

$(N_\beta + 1 \equiv 1)$

$$\boxed{\log\frac{Z[J]}{Z[0]} = \sum_{B=1}^{\infty}\sum_{1\le N_1\le\cdots\le N_B}\sum_{\forall P_i^j=0}^{N} L^{2-B}\,\frac{\overbrace{G_{|P_1^1\cdots P_{N_1}^1|\cdots|P_1^B\cdots P_{N_B}^B|}}^{(N_1+\cdots+N_B)\text{-point fun}}}{S(N_1,\cdots,N_B)}\prod_{\beta=1}^{B}\frac{J_{P_1^\beta\cdots P_{N_\beta}^\beta}}{N_\beta}}$$

$= \text{Generating fun of connected graphs}$

(Green fun.)

$$\left(\begin{array}{l} S(N_1,\cdots,N_B) := \prod_{i=1}^{s}\nu_i! \quad \text{statistical factor} \\ \text{for } (N_1,\cdots,N_B) = (\underbrace{N_1',\cdots,N_1'}_{\nu_1},\cdots,\underbrace{N_s',\cdots,N_s'}_{\nu_s}) \end{array} \right)$$

L^{2-B} : We choose this factor to obtain all Npt function as finite at Large (L,N) lim

§4. ~W.T. Id & SD Eqs.~

~Ward Takahashi like Id.~

$$\Phi \to \Phi' = \Phi + [u, \Phi] \quad \Leftarrow Z[J] \text{ is inv.}$$

WT-Id.

$$\boxed{\sum_m \frac{\partial^2 Z[J]}{\partial J_{am} \partial J_{mb}} = \sum_m \frac{L}{E_a - E_b}\left(J_{ma} \frac{\partial}{\partial J_{mb}} - J_{bm} \frac{\partial}{\partial J_{am}} \right) Z[J]}$$

2pt functions are reduced to 1pt functions by the WT-Id.

~ Schwinger-Dyson Eqs. ~

▷ 1pt function

$$G_{|a|} := \frac{1}{L} \frac{\partial \log Z[J]}{\partial J_{aa}} \Big|_{J=0}$$

$$= H_{aa}^{-1} \left(A - \lambda G_{|a|}^2 - \frac{\lambda}{L} \sum_{m=0}^{N} G_{|am|} - \frac{\lambda}{L^2} G_{|a|a|} \right)$$

— ①

▷ 2pt fun.

$$G_{|ab|} := \frac{1}{L} \frac{\partial^2 \log Z[J]}{\partial J_{ab} \partial J_{ba}} \Big|_{J=0}$$

Using W-T id.
2pt → 1pt

$$= H_{ab}^{-1} \left(1 + \lambda \frac{G_{|a|} - G_{|b|}}{E_a - E_b} \right)$$

— ②

▷ Renormalization Condition

$$G_{|0|} = 0 \iff A = \frac{\lambda}{L}\sum_{m=0}^{N} G_{0m} + \frac{\lambda}{L^2} G_{|0|0|}$$

⇓ Remove A in ① by using this condition

$$G_{|a|} = H^{-1}_{aa}\Big\{-\lambda G^2_{|a|} - \frac{\lambda}{L}\sum_m \left(H^{-1}_{am} - H^{-1}_{om}\right) - \frac{\lambda}{L^2}\left(G_{|a|0|} - G_{|0|0|}\right)$$
$$- \frac{\lambda^2}{L}\sum_m \left(H^{-1}_{am}\frac{(G_{|a|} - G_{|m|})}{E_a - E_m} - H^{-1}_{om}\frac{G_{|m|}}{E_m - E_o}\right)\Big\} \quad \cdots ①'$$

⇓ using $\frac{W_{|a|}}{2\lambda} := G_{|a|} + \frac{H_{aa}}{2\lambda} = G_{|a|} + \frac{E_a}{\lambda}$

①' ② are simplified.

Schwinger-Dyson Eqs. for 1pt, 2pt fun.

- $$W_{|a|}^2 = 4E_a^2 - \frac{4\lambda^2}{L^2}\left(G_{|a|a|} - G_{|0|0|}\right) - \frac{2\lambda^2}{L}\sum_{m=0}^{N}\left(\frac{W_{|a|}-W_{|m|}}{E_a^2-E_m^2} - \frac{W_{|m|}-W_{|0|}}{E_m^2-E_0^2}\right)$$

- $$G_{|ab|} = \frac{1}{2}\,\frac{W_{|a|}-W_{|b|}}{E_a^2-E_m^2}$$

§5 ~Large (N, L)-lim & Solutions~

matrix size & N.C. parameter

$$N, L \longrightarrow \infty \quad \text{with fixing } \frac{N}{L} = \mu^2\Lambda^2$$

$$\boxed{\begin{aligned} \lim \frac{1}{L}\sum_{m=0}^{N} f\left(\frac{m}{L}\right) &= \mu^2\Lambda^2\int_0^1 f(\mu^2\Lambda^2 x)\,dx \\ &= \mu^2\int_0^{\Lambda^2} f(\mu^2 x)\,dx \end{aligned}}$$

$$\left.\begin{aligned} \mu^2 W(x) &= \lim W_{|L\mu^2 x|} \\ G(x) &= \lim G_{|L\mu^2 x|} \quad \text{etc.} \end{aligned}\right\} \text{Using these expression}$$

$$X := (2e(x)+1)^2$$

Schwinger-Dyson eq for 1pt fun.

$$W^2(X) + \int_1^{\Box} dY\, \rho(Y) \frac{W(X)-W(Y)}{X-Y} = X + \overbrace{\int_1^{\Box} dY\, \rho(Y) \frac{W(1)-W(Y)}{1-Y}}^{\text{const.}}$$

where $\rho(Y) = \dfrac{2\tilde{\lambda}^2}{\sqrt{Y}\, e'\!\left(e^{-1}\!\left(\frac{\sqrt{Y}-1}{2}\right)\right)}$, $\Box = \left(1+2e(\tilde{\lambda}^2)\right)^2$

$$\tilde{\lambda} = \frac{\lambda u}{2}$$

Makeenko-Semenoff solved similar type

Solution

$$W(X) := \sqrt{X+c} + \frac{1}{2}\int_1^{\Box} dZ\, \frac{\rho(Z)}{\left(\sqrt{X+c}+\sqrt{Z+c}\right)\sqrt{Z+c}}$$

ex). N.C. scalar ϕ^3 field theory

$$e(x) = x\,,\quad X = (2x+1)^2\,,\quad \rho(Y) = \frac{2\tilde{\lambda}^2}{\sqrt{Y}}$$

$$\Downarrow$$

$$W(X) = \sqrt{X+c} + \frac{2\tilde{\lambda}^2}{\sqrt{X}} \log\left(\frac{(\sqrt{X}+1)(\sqrt{X+c}+\sqrt{X})}{\sqrt{X+c}\,X + \sqrt{X+c}} \right)$$

$$\Updownarrow$$

$$\boxed{\begin{aligned} G(x) &= \frac{\sqrt{(2x+1)^2+c} - (2x+1)}{2\tilde{\lambda}} \\ &+ \frac{\tilde{\lambda}}{2x+1} \log\left(\frac{(2x+2)\left(\sqrt{(2x+1)^2+c} + 2x+1\right)}{(2x+1)\sqrt{1+c} + \sqrt{(2x+1)^2+c}} \right) \end{aligned}}$$

~ $(N_1+\cdots+N_B)$-pt function ~

$$G|a^1_1\cdots a^1_{N_1}|\cdots|a^B_1\cdots a^B_{N_B}| = L^{B-2}\frac{\partial^{N_1}}{\partial J_{a^1_1\cdots a^1_{N_1}}}\cdots\frac{\partial^{N_B}}{\partial J_{a^B_1\cdots a^B_{N_B}}}\log\frac{Z[J]}{Z[0]}$$

where $$\frac{\partial^N}{\partial J_{a_1\cdots a_N}} = \frac{\partial}{\partial J_{a_1a_2}}\frac{\partial}{\partial J_{a_2a_3}}\cdots\frac{\partial}{\partial J_{a_{N-1}a_N}}\frac{\partial}{\partial J_{a_Na_1}}$$

Feynman graph

$N, L \to \infty$

(Using W-T Id.)

$$G(x^1_1,\cdots,x^1_{N_1}|\cdots|x^B_1,\cdots,x^B_{N_B})$$

$$=\tilde{\lambda}^{N_1+\cdots+N_B-B}\sum_{k_1=1}^{N_1}\cdots\sum_{k_B=1}^{N_B}G(x^1_{k_1}|\cdots|x^B_{k_B})\prod_{\beta=1}^{B}\prod_{\substack{l_\beta=1\\ l_\beta\neq k_\beta}}^{N_\beta}\frac{4}{x^\beta_{k_\beta}-x^\beta_{l_\beta}}$$

If we obtain this, then every $(N_1+\cdots+N_B)$-pt function is solved!!

$$G|a^1|\cdots|a^B| = L^{B-2} \frac{\partial}{\partial J_{a_1 a_1}} \cdots \frac{\partial}{\partial J_{a_B a_B}} \log \frac{Z[J]}{Z[0]} \Big|_{J=0}$$

SD-eq. for this
$L, N \to \infty$ lim

Similar process as previous discussions

S-D eq

$$W(x^1) G(x^1 | X^{\{2,\cdots,B\}}) + \frac{1}{2} \int_1^{\infty} dT \rho(T) \frac{G(x^1 | X^{\{2,\cdots,B\}}) - G(T | X^{\{2,\cdots,B\}})}{(x - T)}$$

$$= -\tilde{\lambda} \sum_{\beta=2}^{B} G(x^1, x^\beta, x^\beta | X^{\{2,\cdots,B\}}) - \tilde{\lambda} \sum_{\substack{J \subset \{2,\cdots,B\} \\ 1 \le |J| \le B-2}} G(x^1 | X^J) G(x^1 | X^{\{2,\cdots,B\} \setminus J})$$

where $G(X | Y^J) = G(X | Y^{j_1} | Y^{j_2} | \cdots | Y^{j_p})$ for $J = \{j_1, \cdots, j_p\}$

▷ Solution for (1+1)-pt fun.

S-D eq.

$$W(x)G(x|Y) = -\tilde{\lambda}\, G(x,Y,Y) - \frac{1}{2}\int_1^\infty dz\,\rho(z)\,\frac{G(x|Y)-G(z|Y)}{x-z}$$

Solution

$$\boxed{G(x|Y) = \frac{4\tilde{\lambda}^2}{\sqrt{x+c}\sqrt{Y+c}\left(\sqrt{x+c}+\sqrt{Y+c}\right)^2}}$$

▷ Solution for $B \geq 3$

$$G(x^1|\cdots|x^B) = (-2\tilde{\lambda})^{3B-4}\left(\frac{d}{dt}\right)^{B-3}\left.\left(\frac{\left(\frac{1}{\sqrt{x^1+c-2t}}\right)^3\cdots\left(\frac{1}{\sqrt{x^B+c-2t}}\right)^3}{\left(1-\int_1^\infty dT\,\rho(T)\frac{1}{\sqrt{T+c}\sqrt{T+c-2t}\left(\sqrt{T+c}+\sqrt{T+c-2t}\right)}\right)^{B-2}}\right)\right|_{t=0}$$

Every N-pt function is solved exactly!

Comments

- For 4-dim, 6-dim cases

2-dim action

$$S = L\,\mathrm{Tr}\left(E\Phi^2 - A\Phi\right) + L\frac{\lambda}{3}\mathrm{Tr}\,\Phi^3$$

4, 6-dim action

$$S = V\,\mathrm{Tr}\left(Z E\Phi^2 + (\kappa + \nu E + \xi E^2)\Phi + \frac{\lambda Z^{\frac{3}{2}}}{3}\Phi^3\right)$$

for renormalization

~ 2, 4, 6 - dim ~

$$\boxed{\text{Every } (N_1+\cdots+N_B)\text{-pt fun } G(x^1_1\cdots x^1_{N_1}|\cdots|x^B_1\cdots x^B_{N_B}) \text{ is given explicitly by solving S-D eq.}}$$

$V, N \to \infty$

$$\log \frac{Z[J]}{Z[0]} =: \sum_{B=1}^{\infty} \sum_{1\le N_1 \le \cdots \le N_B}^{\infty} \sum_{\forall p^j_i = 0}^{N} V^{2-B} \frac{G(|p^1_1\cdots p^1_{N_1}|\cdots|p^B_1\cdots p^B_{N_B}|)}{S(N_1,\cdots,N_B)} \prod_{\beta=1}^{B} \frac{J_{p^\beta_1}\cdots J_{p^\beta_{N_\beta}}}{N_\beta}$$

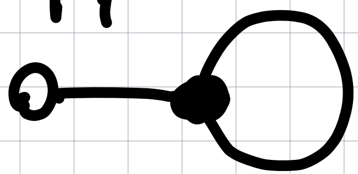
This Φ^3_d Q.F.T. is completely solved !

$d = 2, 4, 6$

§6. ~ Which kind of Quantum Field Theory ~

@ 2-dim case

ex. Γ_1



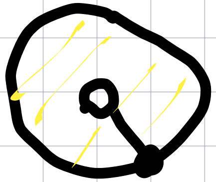
Γ_i : planar graph on S^2

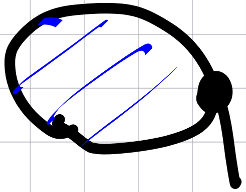
$\circ$: white vertex $\sim \mathbb{J}_{P_1\cdots P_N} = J_{P_1P_2} J_{P_2P_3} \cdots J_{P_NP_1}$

↑ puncture, external vertex,

$\bullet$: black vertex ← internal vertex

Face : A number of "$\circ$" touching a face is 1 or 0.

 : Face with "o" : "external" $\leftarrow x$

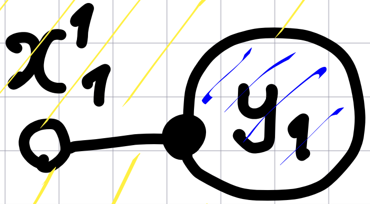
 : Face without "o" : "internal" $\leftarrow y$

x^{i}_{j}

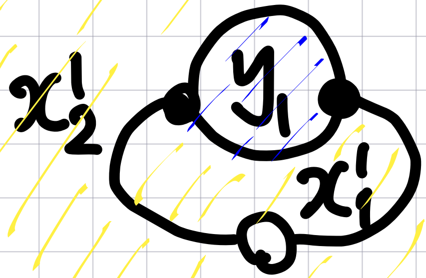
i ← label for "o" $(1 \sim B)$

j ← j-th face in N_i faces touching i-th "o" $(1 \sim N_i)$

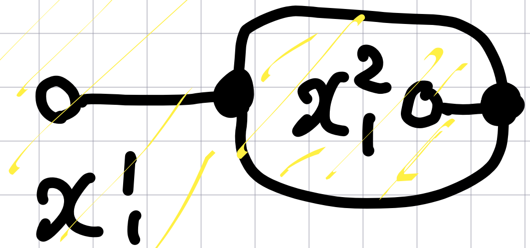
ex). Γ_1 :



Γ_2 :



Γ_3 :

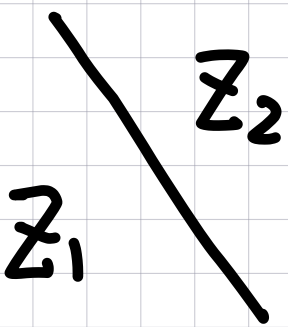


(⇐ Dual graph of a triangulation of S^2 with B-puncture)

Feynman rules

● : 3-point interaction $\longleftrightarrow (-\tilde{\lambda})$

○ : external vertex $\longleftrightarrow 1$



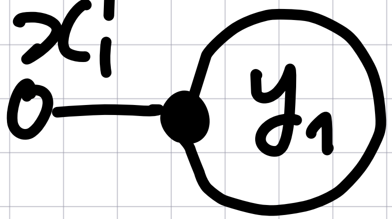
: border line between z_1-face and z_2-face

$$\longleftrightarrow \frac{1}{z_1+z_2+1}$$

y_i : internal face variables

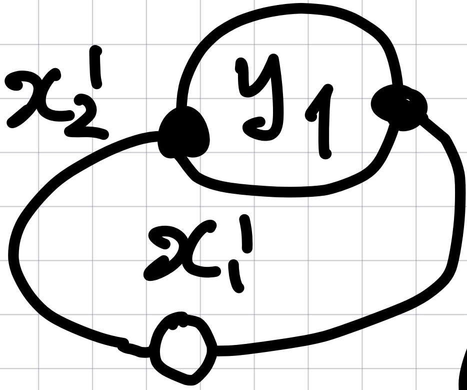
$$\longleftrightarrow \int_0^{\Lambda^2} dy_i$$

Γ_1 :



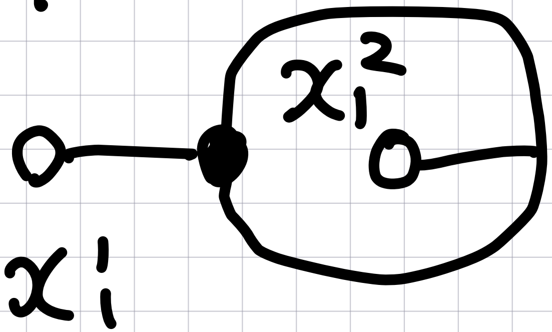
$$\tilde{G}^{\Lambda}_{\Gamma_1}(x_1^1) = \frac{(-\tilde{\lambda})}{2x_1^1+1}\int_0^{\Lambda^2}\frac{dy_1}{x_1^1+y_1+1}$$

Γ_2 :



$$\tilde{G}^{\Lambda}_{\Gamma_2}(x_1^1, x_2^1) = \frac{(-\tilde{\lambda})^2}{(x_1^1+x_2^1+1)^2}\int_0^{\Lambda^2}\frac{dy_1}{(x_1^1+y_1+1)(x_2^1+y_1+1)}$$

Γ_3 :



$$\tilde{G}^{\Lambda}_{\Gamma_3}(x_1^1 | x_1^2) = \frac{(-\tilde{\lambda})^2}{(2x_1^1+1)(2x_1^2+1)(x_1^1+x_1^2+1)^2}$$

§7. Axiomatic Construction of Φ^3 F.T.

▷ The process of our calculation

$$\text{Action } S(\Phi) \to \log\frac{Z[J]}{Z[0]} \xrightarrow{N\to\infty} (\text{renormalization})$$

$$\to \boxed{\text{S-D eqs.} \to N \text{ pt. function}}$$

If we look at only $\boxed{\text{SD}\to\text{N}_{\text{pt}}}$ process, there is no difficulty to describe the field theory as a mathematical theory.

- A standard approach to construct an Axiomatic field theory on a Minkowski sp.

@ Euclidean field theory

$\updownarrow$ Wick rotation

Minkowski field theory

Osterwalder - Schrader Axiom

O-S Axioms require the following.

- $\boxed{\text{Euclidean Inv.}}$: translation & rotation inv. in $\mathbb{R}^d$ O.K.

- $\boxed{\text{Symmetry}}$ O.K.

$$S_c(\xi_1, \cdots, \xi_N) = S_c(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(N)})$$

- $\boxed{\text{Reflection positivity}}$

$$\theta x = (-x^1, x^2, \cdots, x^d)$$

When $\theta f(x_1, \cdots, x_n) = f(\theta x_1, \cdots, \theta x_n)$

$\Downarrow$

$$\sum_{n,m=0}^{N} S_c^{(n+m)}\left(\theta\bar{\Phi}(x_1),\cdots,\theta\bar{\Phi}(x_n),\Phi(y_1),\cdots,\Phi(y_m)\right)\geq 0$$

← n+m pt. Schwinger function

$\Downarrow$ Especially 2pt function

D=2. No! ⇒ Not field theory on the Minkowski sp.

D=4, 6 O.K.

⇒ We have to check $N\geq 3$ pt function farthermore!!

Conclusion

Thank you for your attention!!

- Φ^3_D models (Kontsevich model 2, 4, 6 dim) are solved exactly at a large N limit.

  Every N-pt function is given explicitly.

- If SD eqs $\Longleftrightarrow$ Def. of Q.F.T,

  Φ^3 QFT is defined on Euclidean $\mathbb{R}^2, \mathbb{R}^4, \mathbb{R}^6$.

  $\Downarrow$ Reflection positive.

  Minkowski

  2pt fun. $\mathbb{R}^4, \mathbb{R}^6$ O.K.!

  npt fun. $(n \geq 3)$??

- Mass spectrum $\begin{cases} \text{isolated} & M^2 \sim \mu^2 \\ \text{scattering} & M^2 \gtrsim 2\mu^2 \end{cases}$

Appendix

~ Osterwalder-Schrader Axioms ~

Schwinger fun. for Moyal plane case

$$S_c(\mu\xi_1,\cdots,\mu\xi_N)$$

$$=\lim_{L,N\to\infty}\sum_{N_1+\cdots+N_B=N}\sum_{q_1^1,\cdots,q_{N_B}^B=0}^{N}\frac{G|q_1^1\cdots q_N^1|\cdots|q_1^B\cdots q_{N_B}^B|}{8\pi\mu^{2(2-B-N)}S(N_1,\cdots,N_B)}$$

$$\times\sum_{\sigma\in S_N}\prod_{\beta=1}^{B}\frac{|q_1^\beta\rangle\langle q_2^\beta|(\xi_{\sigma(s_\beta+1)})\cdots|q_{N_\beta}^\beta\rangle\langle q_1^\beta|(\xi_{\sigma(s_\beta+N_\beta)})}{L\mu^2N_\beta}$$

$|n\rangle\langle m|$: Laguerre polynomial. ⇐ Fock rep.

$s_\beta = N_1+\cdots+N_{\beta-1}$