Emergent Spacetime: Toward a background-independent formulation of quantum gravity

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In this talk

☞ I emphasize that noncommutative (NC) spacetime necessarily implies emergent spacetime if spacetime at microscopic scales should be viewed as NC.

☞ The emergent gravity from NC $U(1)$ gauge theory is the large N duality and the emergent spacetime picture admits a background-independent formulation of quantum gravity.

☞ Cosmic inflation in this picture corresponds to the dynamical emergence of spacetime.

☞ In order to understand NC spacetime correctly, we need to deactivate the thought patterns that we have installed in our brains and taken for granted for so many years.

What we have learned from quantum mechanics

Quantum mechanics is a prominent example of NC space whose coordinate generators satisfy the commutation relation

$$[x^i, p_j] = i\hbar\delta^i_j, \qquad (i, j = 1, \cdots, n). \qquad (1)$$

$$a_i \equiv \frac{x^i + ip_i}{\sqrt{2\hbar}}, \qquad a_i^\dagger \equiv \frac{x^i - ip_i}{\sqrt{2\hbar}}; \qquad [x^i, p_j] = i\hbar\delta^i_j \quad \Leftrightarrow \quad [a_i, a_j^\dagger] = \delta_{ij}.$$

A☞ NC phase space (1) introduces a separable Hilbert space $\mathcal{H} = \{|n\rangle |\ n = 0,1, \cdots, \infty\}$ and physical observables become operators in $\mathcal{A}_\hbar$ acting on the Hilbert space $\mathcal{H}$.

B☞ NC algebra $\mathcal{A}_\hbar$ admits a nontrivial inner automorphism $\mathfrak{A}_\hbar$

$$f(x + a) = U_a^\dagger f(x)\, U_a \qquad (2)$$

for $f(x) \in \mathcal{A}_\hbar$ and $U_a = e^{-i\frac{p\cdot a}{\hbar}} \in \mathfrak{A}_\hbar$. This means that every points in the NC phase space are unitarily equivalent. Thus the concept of classical (phase) space is doomed and a quantum algebra $(\mathcal{H}, \mathcal{A}_\hbar)$ plays a more fundamental role and the classical (phase) space is derived (emergent) from the quantum algebra $(\mathcal{H}, \mathcal{A}_\hbar)$.

What we have learned from quantum mechanics

C☞ The Hilbert space $\mathcal{H}$ has a countable basis which is orthonormal, $\langle n|m\rangle = \delta_{nm}$, and complete, $\sum_{n=0}^{\infty} |n\rangle\langle n| = I_{\mathcal{H}}$. Since a physical observable $f(x,p) \in \mathcal{A}_\hbar$ is a linear operator acting on the Hilbert space $\mathcal{H}$, it can be represented as a matrix in End $(\mathcal{H}) \equiv \mathcal{A}_N$:

$$\sum_{n,m=0}^{\infty} |n\rangle\langle n| f(x,p) |m\rangle\langle m| = \sum_{n,m=0}^{\infty} f_{nm} |n\rangle \langle m|.$$

Therefore the map $\mathcal{A}_\hbar \to \mathcal{A}_N$ is a Lie algebra homomorphism where N =dim $(\mathcal{H}) \to \infty$.

D☞ Infinitesimal generators of the inner automorphism $\mathfrak{A}_\hbar$ form an inner derivation $\mathfrak{D}_\hbar$. For example, $p_i = -i\hbar \frac{\partial}{\partial x^i}$ or $x^i = i\hbar \frac{\partial}{\partial p_i}$. Recall also that angular momentum operators in quantum mechanics can be represented by differential operators in $\Gamma(T\mathbb{S}^2)$.
In general, any dynamical variable in $\mathcal{A}_\hbar$ can be represented by a differential operator in $\mathfrak{D}_\hbar$ by the adjoint map

$$\mathcal{A}_\hbar \to \mathfrak{D}_\hbar : f(x,p) \mapsto ad_f = [f(x,p), \cdot\ ] \equiv \hat{V}_f. \quad (3)$$

The adjoint map $\mathcal{A}_\hbar \to \mathfrak{D}_\hbar$ is also a Lie algebra homomorphism:

$$\hat{V}_{[f,g]} = [\hat{V}_f, \hat{V}_g], \quad \text{for } f(x,p),\ \ g(x,p),\ \ [f,g](x,p) \in \mathcal{A}_\hbar. \quad (4)$$

Heuristic Example

Consider the $SU(2)$ Lie algebra: $[J_i, J_j] = i\varepsilon_{ijk} J_k \in \mathcal{A}_\hbar$, $i, j, k = 1,2,3.$

(1) Matrix representation: $\mathcal{A}_\hbar \to \mathcal{A}_N$, e.g. spin-1 representation $(J_i)_{jk} = -i\,\varepsilon_{ijk}$,

$$J_1 = -i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad J_2 = -i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J_3 = -i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

(2) Differential operator representation: $\mathcal{A}_\hbar \to \mathfrak{D}_\hbar$, e.g. $J_i = J_i^\alpha \frac{\partial}{\partial x^\alpha} \in \Gamma(T\mathbb{S}^2)$, $x^\alpha = (\theta, \phi)$,

$$J_1 = sin\phi \frac{\partial}{\partial\theta} + \cot\theta \cos\phi \frac{\partial}{\partial\phi}, \quad J_2 = -cos\phi \frac{\partial}{\partial\theta} + \cot\theta \sin\phi \frac{\partial}{\partial\phi}, \quad J_3 = -\frac{\partial}{\partial\phi},$$

$$g^{\alpha\beta} = \sum_{i=1}^{3} J_i^\alpha J_i^\beta = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix} \quad \Rightarrow \quad g = d\theta^2 + \sin^2\theta\, d\phi^2,$$

So we can derive the underlying geometry from the $SU(2)$ Lie algebra using the duality chain : $\mathcal{A}_\hbar \to \mathcal{A}_N \to \mathfrak{D}_\hbar$.

NC spacetime as α'-quantization

Lesson from quantum mechanics:

Quantum mechanics is the more fundamental description of nature.
Classical world is a coarse graining description of quantum world, so emerges from the quantum mechanics in a specific limit $\hbar \to 0$.

Suppose that nature admits a *new* physical constant α' whose physical dimension is of l_s^2.
It is important to notice that a new physical constant such as $\hbar$ and α' introduces a deformation of some structure in a physical theory. For example, $\hbar$ deforms the algebraic structure of particle phase space from commutative to NC space.

An educated reasoning motivated by the fact that $[\alpha'] =$ (length)×(length) leads to a natural speculation that α′ brings about the deformation of the algebraic structure of spacetime itself such that

$$xy - yx = 0 \quad \Longrightarrow \quad xy - yx = i\,\alpha'. \qquad (5)$$

Since the mathematical structure of the NC space (5) is essentially the same as quantum mechanics, we will consider the deformation (5) as α'-quantization.

NC spacetime as α'-quantization

Lesson from quantum mechanics:

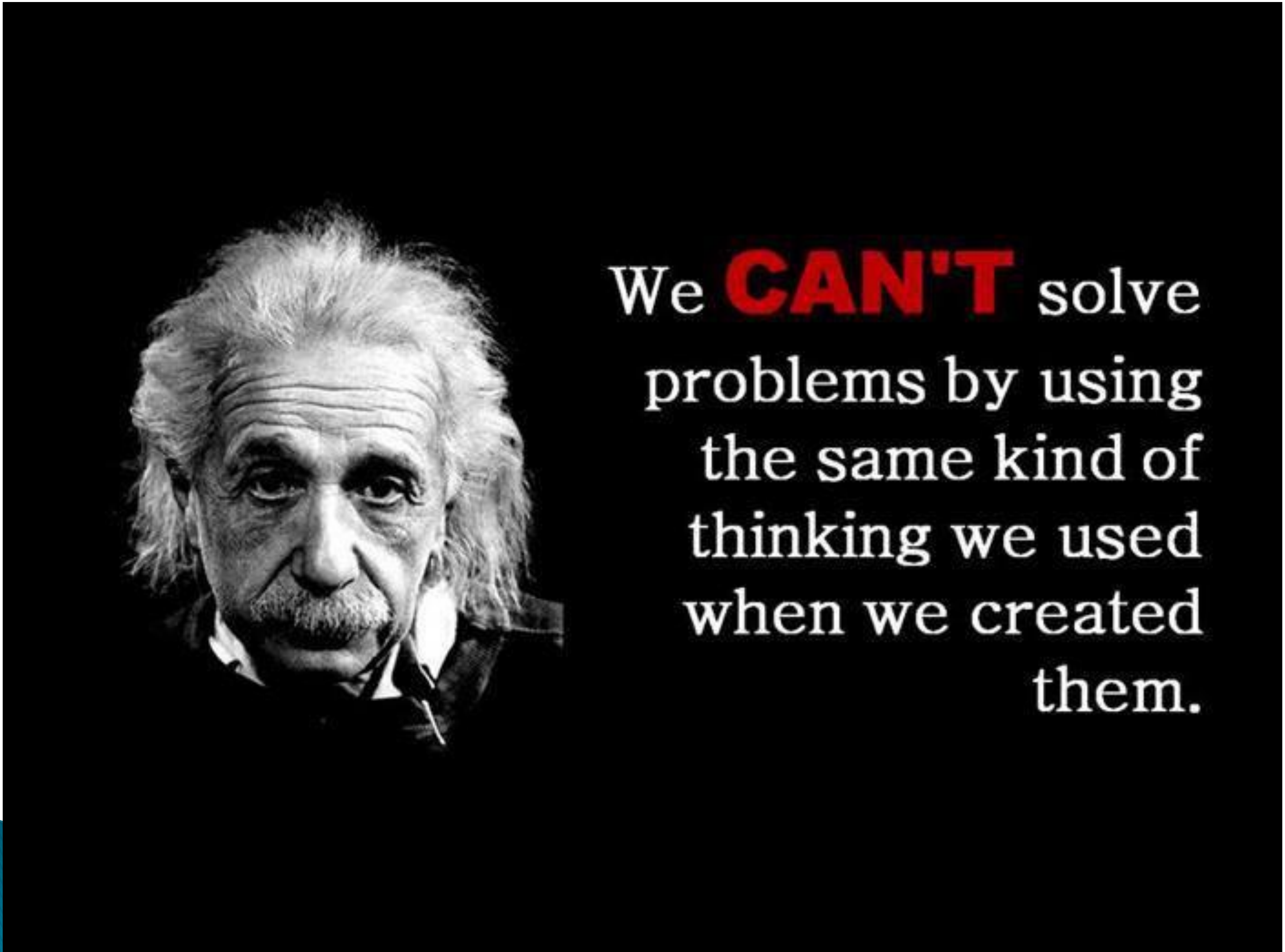
NC spacetime is the more fundamental description of nature.
Classical spacetime is a coarse graining description of quantum geometry, so emerges from the NC spacetime in a specific limit $\alpha' \to 0$.

Suppose that a $U(1)$ gauge theory is defined on $\mathbb{R}^{1,d-1} \times \mathbb{R}^{2n}_\theta$ where $\mathbb{R}^{2n}_\theta$ is the NC space whose coordinate generators obey the commutation relation

$$[y^a, y^b] = i\theta^{ab}, \qquad (a, b = 1, \cdots, 2n) \tag{6}$$

where $(\theta)^{ab} = \alpha'(I_n \otimes i\sigma^2)$ is a $2n \times 2n$ symplectic matrix and the NC space (5) corresponds to the $n = 1$ case.

Let us denote the NC $\star$-algebra generated by $\mathbb{R}^{2n}_\theta$ by $\mathcal{A}_\theta \cong \mathcal{A}_\hbar$ and the NC $\star$-algebra on $\mathbb{R}^{1,d-1} \times \mathbb{R}^{2n}_\theta$ by $\mathcal{A}^d_\theta \equiv \mathcal{A}_\theta\big(C^\infty(\mathbb{R}^{1,d-1})\big) = C^\infty(\mathbb{R}^{1,d-1}) \otimes \mathcal{A}_\theta$.
Note that the $U(1)$ gauge theory on $\mathbb{R}^{1,d-1} \times \mathbb{R}^{2n}_\theta$ takes values in $\mathcal{A}^d_\theta$.


We CAN'T solve
problems by using
the same kind of
thinking we used
when we created
them.

Lesson from quantum mechanics

History is a mirror to the future.
If we do not learn from the mistakes of history, we are doomed to repeat them.

George Santayana (1863-1952)

Since $\mathcal{A}_\theta \cong \mathcal{A}_\hbar$, let us apply the propositions (A~D) in quantum mechanics to the $U(1)$ gauge theory on $\mathbb{R}^{1,d-1} \times \mathbb{R}^{2n}_\theta$ which takes values in $\mathcal{A}^d_\theta$.

$\mathbb{A}$☞ NC space (6) introduces a separable Hilbert space $\mathcal{H} = \{|n\rangle|\ n = 0,1,\cdots,\infty\}$ and dynamical variables become operators in $\mathcal{A}^d_\theta$ acting on the Hilbert space $\mathcal{H}$.

$\mathbb{B}$☞ NC algebra $\mathcal{A}^d_\theta$ admits a nontrivial inner automorphism $\mathfrak{A}^d_\theta$

$$f(x, y + d) = U_d^\dagger\, f(x, y)\, U_d \qquad (7)$$

for $f(x, y) \in \mathcal{A}^d_\theta$ and $U_d = e^{ip\cdot d} \in \mathfrak{A}^d_\theta$ and $p_a = B_{ab}\, y^b$ with $B = \theta^{-1}$.
This means that every points in the NC space $\mathbb{R}^{2n}_\theta$ are unitarily equivalent. Thus the concept of classical space(time) is doomed and the space(time) is replaced by a quantum algebra $(\mathcal{H},\ \mathcal{A}^d_\theta)$ and the classical spacetime is derived (emergent) from the quantum algebra $(\mathcal{H},\ \mathcal{A}^d_\theta)$ in a specific limit.

Lesson from quantum mechanics

In the presence of NC $U(1)$ gauge fields which appear in the form of background-independent variables $\phi_A = \left(iD_\mu, \phi_a\right)$ where $D_\mu = \partial_\mu - iA_\mu(x,y)$ and $\phi_a = p_a + A_a\,(x,y)$, one can covariantize the *inner* automorphism with $U_d = e^{i\phi_A d^A} \in \mathfrak{A}_\theta^d$ by introducing open Wilson lines. (See, Ishibashi et. al.; Das & Rey; Gross et. al.)

ℂ☞ The map $\mathcal{A}_\theta^d \to \mathcal{A}_N^d \equiv \mathcal{A}_N\left(C^\infty\left(\mathbb{R}^{1,d-1}\right)\right) = C^\infty\left(\mathbb{R}^{1,d-1}\right) \otimes \mathcal{A}_N$ is a Lie algebra homomorphism where N =dim $(\mathcal{H}) \to \infty$:

$$\sum_{n,m=0}^{\infty} |n\rangle\langle n| f(x,y)\, |m\rangle\langle m| = \sum_{n,m=0}^{\infty} f_{nm}(x)|n\rangle \langle m| \qquad (8)$$

for $f(x,y) \in \mathcal{A}_\theta^d$ and $[f(x)]_{nm} \in \mathcal{A}_N^d$.

𝔻☞ For any dynamical variable, e.g. $\phi_A(x,y) \in \mathcal{A}_\theta^d$, we can associate a differential operator, the so-called polyvector fields in $\mathcal{D}_\theta^d$, by the adjoint map

$$\mathcal{A}_\theta^d \to \mathcal{D}_\theta^d: \phi_A(x,y) \mapsto ad_{\phi_A} = [\phi_A(x,y), \,\cdot\,] \equiv \hat{V}_A. \qquad (9)$$

The adjoint map $\mathcal{A}_\theta^d \to \mathcal{D}_\theta^d$ is also a Lie algebra homomorphism.

Physical consequence from ($\mathbb{A} \cup \mathbb{C}$)

Using the matrix representation (8), the $D = (d + 2n)$-dimensional NC $U(1)$ gauge theory on $\mathbb{R}^{1,d-1} \times \mathbb{R}_\theta^{2n}$ is exactly mapped to the d-dimensional $U(N \to \infty)$ Yang-Mills theory on $\mathbb{R}^{1,d-1}$:

$$S = -\frac{1}{G_{YM}^2} \int d^D Y \frac{1}{4} (\widehat{F}_{AB} - B_{AB})^2 \quad (10)$$

$$= -\frac{1}{g_{YM}^2} \int d^d x \mathrm{Tr} \Big(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \phi_a D^\mu \phi_a - \frac{1}{4} [\phi_a, \phi_b]^2 \Big) \quad (11)$$

where $G_{YM}^2 = (2\pi)^n |Pf\theta| \, g_{YM}^2$ and $B_{AB} = \begin{pmatrix} 0 & 0 \\ 0 & B_{ab} \end{pmatrix}$.

We emphasize that the equivalence between the D-dimensional NC $U(1)$ gauge theory (10) and d-dimensional $U(N \to \infty)$ Yang-Mills theory (11) is not a dimensional reduction but an exact mathematical identity although they are defined in different dimensions with different gauge groups.

What is a physical consequence of this mathematical identity?

Physical consequence from (𝔹 ∪ 𝔻)

In a large-distance limit, i.e. $|\theta| \to 0$, one can expand the NC vector fields $\widehat{V}_A$ in Eq. (9) using the explicit form of the Moyal $\star$-product. The result takes the form

$$\widehat{V}_A = V_A^M(x,y)\frac{\partial}{\partial X^M} + \sum_{p=2}^{\infty} V_A^{a_1\cdots a_p}(x,y)\frac{\partial}{\partial y^{a_1}}\cdots\frac{\partial}{\partial y^{a_p}} \in \mathfrak{D}^d, \qquad (12)$$

where $X^M = (x^\mu, y^a)$ are local coordinates on a D-dimensional emergent Lorentzian manifold $\mathcal{M}$. In general, the module of derivations $\mathcal{D}_\theta^d$ is a direct sum of the submodules of horizontal and inner derivations:

$$\mathcal{D}_\theta^d = \mathrm{Hor}\,(\mathcal{A}_\theta^d) \oplus \mathcal{D}(\mathcal{A}_\theta^d), \qquad (13)$$

where horizontal derivation is locally generated by a vector field $k^\mu(x,y)\frac{\partial}{\partial x^\mu} \in \mathrm{Hor}\,(\mathcal{A}_\theta^d)$.

Thus the Taylor expansion of NC vector fields in $\mathcal{D}_\theta^d$ generates an infinite tower of the so-called polyvector fields. Note that the leading term gives rise to the ordinary vector fields that will be identified with a frame basis associated to the tangent bundle $T\mathcal{M}$ of an emergent manifold $\mathcal{M}$.

Physical consequence from (𝔸~𝔻)

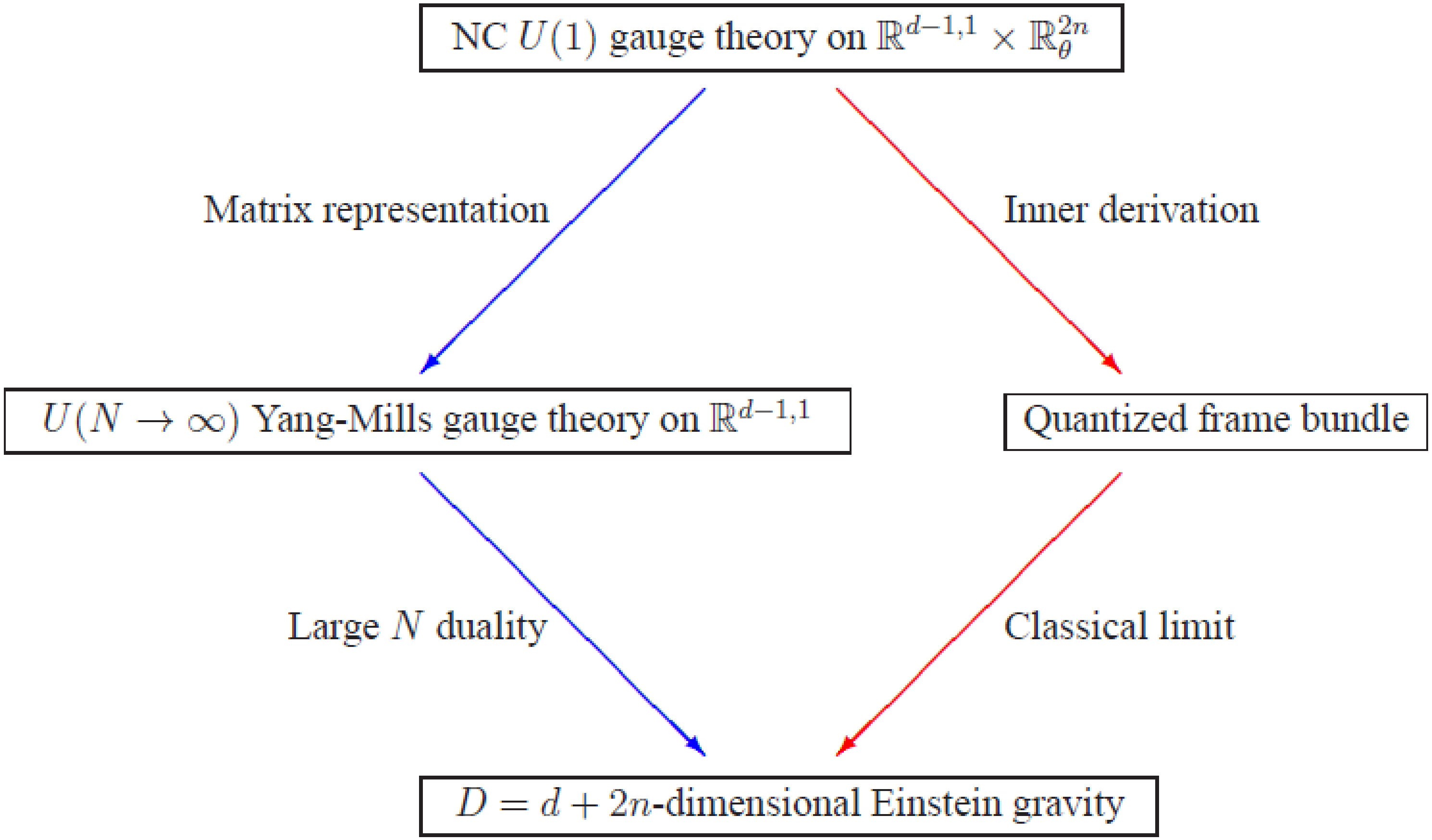


Figure 1: Flowchart for emergent gravity

Large N duality

The large N duality is still a conjecture, not proven yet. But we can use the emergent gravity from NC $U(1)$ gauge theory to verify the conjectural large N duality by realizing the equivalence between the actions (10) and (11) in a reverse way.

It is based on the observation that there are two different kinds of vacua in Coulomb branch if we consider the $N \to \infty$ limit and the NC space (6) arises as a vacuum solution of the d-dimensional $U(N \to \infty)$ Yang-Mills theory (11) in the Coulomb branch.

The conventional choice of vacuum in the Coulomb branch of $U(N)$ Yang-Mills theory is given by

$$[\phi_a, \phi_b]|_{\text{vac}} = 0 \qquad \Rightarrow \qquad \langle \phi_a \rangle_{\text{vac}} = \text{diag}\big((\alpha_a)_1, (\alpha_a)_2, \cdots, (\alpha_a)_N\big) \qquad (14)$$

for $a = 1, \cdots, 2n$. In this case the $U(N)$ gauge symmetry is broken to $U(1)^N$.
If we consider the $N \to \infty$ limit, the large N limit opens a new phase of the Coulomb branch given by

$$[\phi_a, \phi_b]|_{\text{vac}} = -iB_{ab} \qquad \Rightarrow \qquad \langle \phi_a \rangle_{\text{vac}} = p_a \equiv B_{ab} y^b \qquad (15)$$

where the vacuum moduli y^a satisfy the Moyal-Heisenberg algebra (6).

Large N duality

The vacuum (15) will be called the NC Coulomb branch.

Note that the Moyal-Heisenberg vacuum (15) saves the NC nature of matrices while the conventional vacuum (14) dismisses the property.

Suppose that fluctuations around the vacuum (15) take the form

$$D_\mu = \partial_\mu - iA_\mu(x,y), \qquad \phi_a = p_a + A_a\,(x,y) \in \mathcal{A}^d_\theta. \tag{15}$$

The above adjoint scalar fields now obey the deformed algebra given by

$$[\phi_a, \phi_b] = -i\,(B_{ab} - F_{ab}), \qquad F_{ab} = \partial_a A_b - \partial_b A_a - i[A_a, A_b],$$

with the definition $\partial_a \equiv -i\, ad_{p_a} = -i[p_a,\ \cdot\,]$. Plugging the fluctuations (15) into the d-dimensional $U(N \to \infty)$ Yang-Mills theory (11), we finally get the $D = (d + 2n)$-dimensional NC $U(1)$ gauge theory (10) . Thus we arrive at the reversed version of the equivalence:

$$\begin{aligned} S &= -\frac{1}{g_{YM}^2} \int d^d x \mathrm{Tr}\Big(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi_a D^\mu\phi_a - \frac{1}{4}[\phi_a, \phi_b]^2\Big) \\ &= -\frac{1}{G_{YM}^2} \int d^D Y \frac{1}{4}(\widehat{F}_{AB} - B_{AB})^2, \end{aligned} \tag{16}$$

Flowchart for large N duality

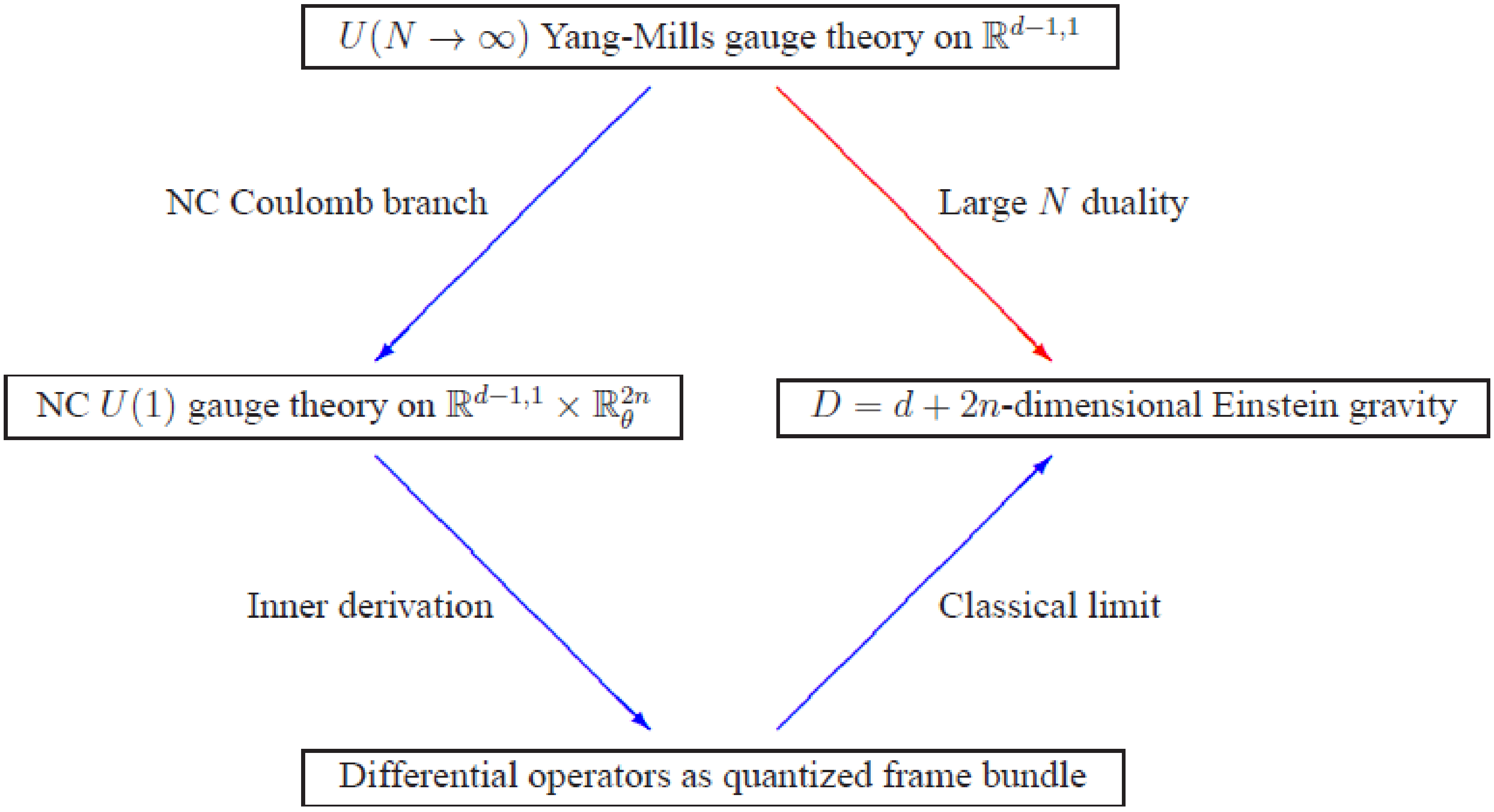


Figure 2: Flowchart for large N duality

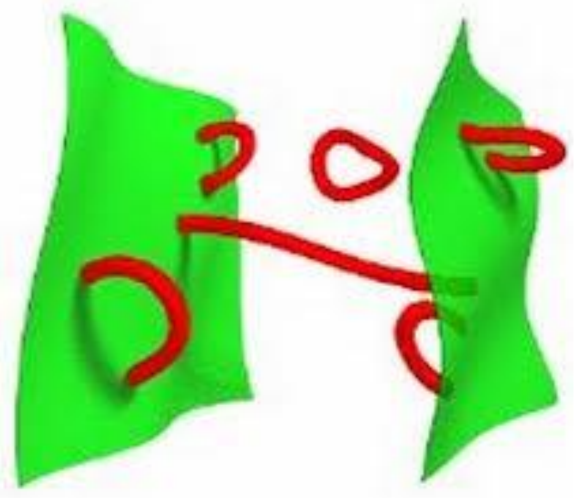
Equivalence principle for electromagnetic force

If general relativity emerges from a symplectic or NC $U(1)$ gauge theory, it is necessary to realize the equivalence principle and general covariance from the $U(1)$ gauge theory. Now I will try to elucidate why *NC spacetime* requires us to take a radical departure from the 20th century physics.

Consider a general open string action defined by

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma |dX|^2 + \int_\Sigma B + \int_{\partial\Sigma} A, \qquad (17)$$

where $X: \Sigma \to M,\ B(\Sigma) = X^* B(M)$ and $A(\partial\Sigma) = X^* A(M)$.



The string action (17) respects two local gauge symmetries:

(I) Diff(M)-symmetry: $X \to X' = \phi(X) \in$ Diff(M),

(II) Λ-symmetry: $(B, A) \to (B - d\Lambda,\ A + \Lambda)$,
where the gauge parameter Λ is a one-form in $\Gamma(T^* M)$.

The Λ-symmetry is present only when $B \neq 0$. When $B = 0$, the symmetry is reduced to $A \to A + d\lambda$, which is the ordinary $U(1)$ gauge symmetry.

Equivalence principle for electromagnetic force

Suppose that $B \in \Gamma(\Lambda^2 T^* M)$ is a symplectic structure on M, i.e., a nondegenerate, closed two-form. The symplectic two-form defines a bundle isomorphism $B: TM \to T^*M$ by $X \mapsto A = \iota_X B$. Then $F = dA = \mathcal{L}_X B$ where $\mathcal{L}_X = d\iota_X + \iota_X d$ is the Lie derivative with respect to $X \in \Gamma(TM)$.

$B + F \in \Gamma(\Lambda^2 T^* M)$ is a gauge invariant quantity under the Λ-symmetry, but we have the relation $B + F = B + \mathcal{L}_X B = (1 + \mathcal{L}_X)B \approx e^{\mathcal{L}_X} B$. Note that a vector field is an infinitesimal generator of local coordinate transformations, in other words, a Lie algebra generator of Diff(M). Thus there always exists a diffeomorphism $\phi \in$ Diff(M) such that $\phi^*(B + F) = B$ where $\phi^* = (1 + \mathcal{L}_X)^{-1} \approx e^{-\mathcal{L}_X}$. Actually this statement is precisely the Darboux theorem or the Moser lemma in symplectic geometry.

Therefore the Λ-symmetry can be identified with a coordinate transformation generated by a vector field $X \in \Gamma(TM)$. As a result, the electromagnetic force can always be eliminated by a local coordinate transformation as far as $U(1)$ gauge theory is defined on a spacetime with symplectic structure, in other words, a microscopic spacetime becomes NC.

Emergent geometry and quantum entanglement

DBI action: Diff(M) $\bigoplus$ Λ-symmetry

$$g_c + \kappa(B + F) \rightarrow g_c + \kappa(B + \mathcal{L}_X B) \rightarrow \phi^*(g_c + \kappa(1 + \mathcal{L}_X)B) \rightarrow G + \kappa B$$

where $G \equiv \phi^*(g_c)$ & $\phi^* = (1 + \mathcal{L}_X)^{-1} \approx e^{-\mathcal{L}_X}$: *exponential map*

$$\frac{1}{g_s}\int d^{p+1}x\sqrt{\det(g_c + \kappa(B + F))} = \frac{1}{g_s}\int d^{p+1}y\sqrt{\det(G + \kappa B)} \qquad (18)$$

$$= \frac{1}{G_s}\int d^{p+1}y\sqrt{\det\left(g_o + \kappa\left(\hat{F} - B\right)\right)}$$

where $G_{\mu\nu}(y) = \frac{\partial x^a}{\partial y^\mu}\frac{\partial x^b}{\partial y^\nu} g_{ab}(x)$ and $\phi: y^a \mapsto x^a(y) = y^a + \theta^{ab}\hat{A}_b(y)$.

We remark that NC gauge fields $\hat{A}_b(y)$ are locally defined by quantizing the Poisson algebra on a local Darboux chart, so the quantum algebra $(\mathcal{H}, \mathcal{A}_\theta)$ defining the NC gauge theory is locally defined. Therefore, we need to glue these local data on Darboux charts to yield global vector fields which will eventually be identified with gravitational fields, i.e., vielbeins.

Question: What is the operation of the gluing from the Hilbert space point of view?
Answer?: According to Raamsdonk, it may be a quantum entanglement for locally defined Hilbert spaces on overlapping Darboux patches.

Emergent spacetime from large N duality

Let us start with a one-dimensional matrix model, a.k.a. BFSS matrix model, with a bunch of $N \times N$ Hermitian matrices, $\{\phi_A \in \mathcal{A}_N^1 | A = 1, \cdots, d\}$, whose action is given by

$$S = \frac{1}{g^2} \int dt\, Tr(-\frac{1}{2}(D_0\phi_A)^2 + \frac{1}{4}[\phi_A, \phi_B]^2) \qquad (19)$$

where $D_0\phi_A = \frac{\partial \phi_A}{\partial t} - i[A_0, \phi_A]$. Consider a translation invariant vacuum defined by

$$\langle A_0 \rangle_{vac} = \mathcal{E}\, \mathbb{1}_{N\times N}, \quad \langle \phi_a \rangle_{vac} = p_a = B_{ab}\, y^b \in \mathcal{A}_N^1, \qquad (a, b = 1, \cdots, 2n), \quad (20)$$
$$\langle \phi_i \rangle_{vac} = \text{diag}\big((\phi_i)_1, \cdots, (\phi_i)_N\big) \equiv \alpha_i \in \mathcal{A}_N^1, \quad (i = 2n+1, \cdots, d),$$

where $[y^a, y^b] = i\theta^{ab}\mathbb{1}_{N\times N}$ and $[\alpha_i, \alpha_j] = 0$. The above vacuum is a consistent solution of the theory (19). Introduce fluctuations around the vacuum

$$D_0 = \frac{\partial}{\partial t} - iA_0(t, y), \quad \phi_a = p_a + A_a(t, y), \quad \phi_i = \alpha_i + \varphi_i(t, y).$$

The action (19) for these fluctuations is given by

$$S = \frac{1}{g_{YM}^2} \int d^{2n+1}X \left(-\frac{1}{4}\mathcal{F}_{MN}\mathcal{F}^{MN} - \frac{1}{2}(D_M\varphi_i)^2 + \frac{1}{4}[\varphi_i, \varphi_j]^2\right), \qquad (21)$$

where $X^M = (t, y^a)$, $A_M = (A_0,\ A_a)$, $M = 0,1, \cdots, 2n$ and $\mathcal{F}_{MN} = B_{MN} - F_{MN}$.

Emergent spacetime from large N duality

Therefore the fluctuations around the NC vacuum are described by the $(2n+1)$-dimensional NC $U(1)$ gauge theory with $(d-2n)$ adjoint scalar fields.
In this case, the emergent geometry are determined by applying the flowchart in Fig. 2 to the action (21) and takes the form of a $(2n+1)$-dimensional Lorentzian manifold embedded in $\mathbb{R}^{1,d}$. But, let us set $\varphi_i(t,y)=0$ for simplicity.

Then the time-dependent vector fields $\hat{V}_A(t) \in \mathfrak{D}^1_\theta$ take the following form

$$\widehat{V}_0(t) = \frac{\partial}{\partial t} + A_0^\mu(t,y)\frac{\partial}{\partial y^\mu} + \sum_{p=2}^{\infty} A_0^{\mu_1\cdots\mu_p}(t,y)\frac{\partial}{\partial y^{\mu_1}}\cdots\frac{\partial}{\partial y^{\mu_p}},$$

$$\widehat{V}_a(t) = V_a^\mu(t,y)\frac{\partial}{\partial y^\mu} + \sum_{p=2}^{\infty} V_a^{\mu_1\cdots\mu_p}(t,y)\frac{\partial}{\partial y^{\mu_1}}\cdots\frac{\partial}{\partial y^{\mu_p}}.$$

Let us truncate the above polyvector fields to ordinary vector fields given by

$$\mathcal{X}(M) = \left\{ V_A = V_A^M(t,y)\frac{\partial}{\partial X^M} \,|A, M = 0,1,\cdots,2n \right\}.$$

Emergent spacetime from large N duality

The orthonormal vielbeins on TM are obtained by the prescription

$$(V_0, V_a) = (E_0, \lambda E_a) \in \Gamma(TM) \quad \text{or} \quad (e^0, e^a) = (v^0, \lambda v^a) \in \Gamma(T^*M). \qquad (22)$$

The conformal factor $\lambda \in C^\infty(M)$ is determined by the volume-preserving condition

$$\mathcal{L}_{V_A} \upsilon_t = (\nabla \cdot V_A + (2-2n) V_A \ln \lambda) = 0 \;\; \text{with} \;\; \upsilon_t = \lambda^2 dt \wedge v^1 \wedge \cdots \wedge v^{2n}.$$

If the structure equation of vector fields $V_A \in \Gamma(TM)$ is defined by $[V_A, V_B] = -g_{AB}{}^C \, V_C$, the volume-preserving condition can be written as

$$g_{BA}{}^B = V_A \ln \lambda^2. \qquad (23)$$

In the end, the Lorentzian metric on a $(2n+1)$-dimensional emergent spacetime manifold is given by

$$\begin{aligned} ds^2 &= \mathcal{G}_{MN}(X) dX^M \otimes dX^N = \eta_{AB} e^A \otimes e^B \\ &= -v^0 \otimes v^0 + \lambda^2 v^a \otimes v^a = -dt^2 + \lambda^2 v^a_\mu v^a_\nu (dy^\mu - \mathbf{A}^\mu)(dy^\nu - \mathbf{A}^\nu) \end{aligned} \qquad (24)$$

where $\boldsymbol{A}^\mu := A_0^\mu(t, y) dt$.

Origin of flat Minkowski spacetime

The large N duality in Fig. 2 says that the gravitational variables such as vielbeins in general relativity arise from the commutative limit of NC $U(1)$ gauge fields. Then one may ask where the flat Minkowski spacetime comes from. Let us look at the metric (24) to identify the origin of the flat Minkowski spacetime.

It turns out that the flat Minkowski spacetime is originated from the NC vacuum (20) since in this case $\langle V_A \rangle_{\text{vac}} = \delta_A^M \partial_M$, so $\lambda = 1$ according to (23). Thus the inverse vielbeins and the metric for the vacuum geometry are given by

$$E_A^{(0)} = V_A^{(0)} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial y^a}\right), \qquad ds^2 = -dt^2 + d\vec{y} \cdot d\vec{y}. \qquad (25)$$

We emphasize that the NC Coulomb vacuum (20) is responsible for the emergence of the Minkowski spacetime (25), but this vacuum has a nontrivial vacuum energy density caused by the condensate (20). We can calculate it using the action (21):

$$\rho_{vac} = \frac{1}{4g_{YM}^2} |B_{ab}|^2. \qquad (26)$$

A striking fact is that the flat spacetime is originated from the uniform vacuum energy (26) known as the cosmological constant in general relativity. This is a tangible difference from Einstein gravity, in which $T_{\mu\nu} = 0$ in flat spacetime.

Cosmic inflation from time-dependent matrices

However, since we have started with the matrix model (19) in which any spacetime structure has not been assumed in advance, the spacetime was not existent at the beginning but simply emergent from the vacuum condensate (20) . Therefore the Planck energy condensation into vacuum must be regarded as a dynamical process.

It is not difficult to show that the dynamical process for the vacuum condensate is described by the time-dependent vacuum matrices given by

$$\langle \phi_a(t) \rangle_{vac} = p_a e^{\frac{\kappa t}{2}}, \quad \langle A_0(t) \rangle_{vac} = \frac{\kappa}{2} \int_0^1 d\sigma \frac{dy^a(\sigma)}{d\sigma}\, p_a(\sigma), \tag{27}$$

where the open Wilson line is defined along a path parameterized by the curve $y^a(\sigma) = y_0^a + \zeta^a(\sigma)$ with $0 \leq \sigma \leq 1$ and $y^a(\sigma = 0) \equiv y_0^a,\ y^a(\sigma = 1) \equiv y^a$.
Using the formula

$$\frac{\partial}{\partial y^a} \int_0^1 d\sigma \frac{dy^b(\sigma)}{d\sigma} K\big(y(\sigma)\big) = \delta_a^b K(y)$$

for some differentiable function $K(y)$, it is easy to check that the time-dependent matrices in (27) satisfy the equations of motion, $D_0^2 \phi_a + [\phi_b, [\phi_a, \phi_b]] = 0,$ as well as the Gauss constraint, $[\phi_a, D_0 \phi_a] = 0$. The constant $H = (n-1)\kappa$ will be identified with the inflationary Hubble constant.

Cosmic inflation from time-dependent matrices

The $(2n+1)$-dimensional basis for the time-dependent vacuum (27) can easily be calculated using the map (9):

$$V_0 = \frac{\partial}{\partial t} - \frac{\kappa}{2} y^a \frac{\partial}{\partial y^a}, \quad V_a = e^{\frac{\kappa t}{2}} \frac{\partial}{\partial y^a} \quad \Longrightarrow \quad e^0 = dt, \ \ e^a = e^{Ht}\, dy_t^a \qquad (28)$$

where $y_t^a = e^{\frac{\kappa t}{2}} y^a$ and $\lambda = e^{n\kappa t}$. Finally, the time-dependent metric for the inflating background (27) is given by

$$ds^2 = -dt^2 + e^{2Ht}\, d\vec{y}_t \cdot d\vec{y}_t. \qquad (29)$$

Note that the conformal vector field $Z \equiv \frac{1}{2} y^a \frac{\partial}{\partial y^a}$ in Eq. (28) is known as the Liouville vector field in locally conformal symplectic manifolds and generated by the open Wilson line (27). It obeys $\mathcal{L}_Z B = \kappa B$, so $B(t) = e^{\kappa t} B$. This explains why the volume of spacetime phase space whose symplectic two-form is B exponentially expands.

Since $\phi_a(t=0)$ are operators acting on a Hilbert space, this means that the inflationary vacuum (27) creates a spacetime of the Planck size and evolves to the inflation epoch unlike the traditional inflationary models that describe just the exponential expansion of a preexisting spacetime. This picture is similar to the birth of inflationary universes (Hawking, Moss, Vilenkin) in which the universe is spontaneously created by quantum tunneling from nothing into a de Sitter space.

Emergent spacetime, Quantum gravity and Cosmic inflation

The cosmic inflation arises as a time-dependent solution of a background-independent theory describing the dynamical process of Planck energy condensate in vacuum without introducing any inflaton field as well as an *ad hoc* inflation potential.

The large N duality in Fig. 2 also implies that cosmic inflation triggered by the Planck energy condensate into vacuum must be a single event.

Thus the emergent spacetime is a completely new paradigm so that the multiverse debate in physics circles has to seriously take it into account.

An underlying idea must be clear although it has been dormant so far.
In order to understand NC spactime correctly, we need to deactivate the thought patterns that we have installed in our brains and taken for granted for so many years. However, if it is understood correctly, its impact must be huge as we have described in this talk. Therefore I hope an underlying idea of this talk has not been hindered by too restricted concepts and traditional prejudices.