# APS index theorem for domain-wall fermion

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# 1. Introduction

# a. What is APS index theorem?

Atiyah, Patodi and Singer “Spectral asymmetry and Riemanian geometry”, Math. Proc. Cambridge Philos. Soc. 77, 43

- Generalization of Atiyah-Singer Index theorem on a manifold with boundary.

- Key notion in bulk-edge correspondence for insulators in symmetry protected topological phase.

# Atiyah-Singer index theorem

Index theorem for massless Dirac op.
on even-dim manifold without boundary.

$$\text{Index of } D_{4D} = \frac{1}{32\pi^2}\int d^4x\epsilon_{\mu\nu\rho\sigma}\text{tr}\left[F^{\mu\nu}F^{\rho\sigma}\right]$$

Can be derived by Fujikawa's method

$$\lim_{\Lambda\to\infty} \text{Tr}\gamma_5 e^{D_{\rm 4D}^2/\Lambda^2}$$

Simple 1-loop calculation
Insertion of complete set of plane-wave states

# APS index theorem

Index theorem for massless Dirac op.
on even-dim manifold with boundary.


$$\text{Index of } D_{4D} = \frac{1}{32\pi^2}\int_{x_4>0} d^4x\,\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}\left[F^{\mu\nu}F^{\rho\sigma}\right] - \frac{\eta(iD_{\mathrm{3D}})}{2}$$

$$\eta(iD_{\mathrm{3D}}) = \sum_{\lambda}^{\mathrm{reg}} \mathrm{sgn}(\lambda) \qquad \lambda : \text{ eigenvalue of } iD_{\mathrm{3D}}$$

# Proof of APS index theorem

- Quite technical
- Proof is given only for the case

$$F_{4i} = 0 \text{ near the boundary}$$

- Non-local boundary condition is imposed

$$D^{\rm 4D} = \gamma_4(\partial_4 + H), \quad H = \gamma_4\gamma_i D^i \qquad A_4 = 0 \text{ gauge}$$

With non-local boundary condition (APS b.c.)

$$\frac{1}{2}\left(H + |H|\right)\psi\Big|_{x_4=0} = 0$$

# b. Symmetry protected topological phase

- Insulator with T-symmetry can have nontrivial topology (SPT phase)

Partition function after integrating massive fermion

$$Z_{\text{bulk}} = |Z_{\text{bulk}}| \exp\left( i\pi \frac{1}{32\pi} \int d^4x \epsilon_{\mu\nu\rho\sigma} \text{tr}\left[F^{\mu\nu} F^{\rho\sigma}\right] \right)$$

↓

$$Z'_{\text{bulk}} = |Z_{\text{bulk}}| \exp\left( -i\pi \frac{1}{32\pi^2} \int d^4x \epsilon_{\mu\nu\rho\sigma} \text{tr}\left[F^{\mu\nu} F^{\rho\sigma}\right] \right)$$

$$= |Z_{\text{bulk}}| \exp\left( i\pi \frac{1}{32\pi^2} \int d^4x \epsilon_{\mu\nu\rho\sigma} \text{tr}\left[F^{\mu\nu} F^{\rho\sigma}\right] \right)$$

$$= Z_{\text{bulk}}$$

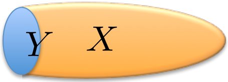
Atiyah-Singer index = integer !

T-invariance holds, but nontrivially,
owing to Atiyah-Singer index theorem.

# What happens if there is a boundary?

$$Z_{\text{bulk}} \approx |Z_{\text{bulk}}| e^{i\pi P}$$

$$P = \frac{1}{32\pi^2} \int_{x>0} d^4x \epsilon_{\mu\nu\rho\sigma} \text{tr}\left[F^{\mu\nu} F^{\rho\sigma}\right]$$

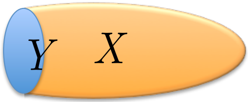



P can no longer be integer → Violation of T symmetry!

Bulk-edge correspondence: Massless edge modes can appear at the boundary of insulator in SPT phase

# Bulk-edge correspondence: additional massless edge mode can appear on Y

Low energy 3-dim effective action on Y




$$S_{\text{eff}} = \int_Y d^3x \bar{\psi} D^{3d} \psi + \cdots$$

Additional T-anomaly in $\text{Det}(D^{3d})$ from T-violating regularization

In 3-dim, (Pauli-Villars regulator) mass term always breaks T-symmetry

$$\text{Det}(D^{3d}) = \prod_i \frac{\lambda_i}{\lambda_i + i\Lambda} = |\text{Det}(D^{3d})| \exp\left(-i\frac{\pi}{2}\eta(D^{3d})\right)$$

$\lambda_i$ : eigenvalue of $D^{3d}$

$$\eta(D^{3d}) = \lim_{s \to 0} \sum_k \text{sign}(\lambda_k)|\lambda_k|^{-s} \ : \eta\text{-invariant}$$

# Argument based on low energy effective theory

Witten "Fermion path integrals and topological phases", Rev. Mod. Phys. 88 (2016)

c. f. Metlitski 15, Seiberg-Witten 16, Tachikawa-Yonekura 16, Freed-Hopkins 16, Yonekura 16

Full theory is invariant under T symmetry

→Anomaly cancelled between the bulk & boundary ?

Does this really happen? → YES! (Witten)

$$Z_{\rm bulk} = |Z_{\rm bulk}| \exp\left( i\pi \frac{1}{32} \int_{x_4>0} d^4x \epsilon_{\mu\nu\rho\sigma} {\rm tr}\left[F^{\mu\nu}F^{\rho\sigma}\right]\right)$$

$$Z_{\rm edge} = |Z_{\rm edge}| \exp\left( -i\pi \frac{\eta(iD_{\rm 3D})}{2}\right)$$

→ $$Z_{\rm bulk} Z_{\rm edge} = |Z_{\rm bulk} Z_{\rm edge}| (-1)^{\mathcal{J}}$$

$$\mathcal{J} = \frac{1}{32\pi^2} \int_{x_4>0} d^4x \epsilon_{\mu\nu\rho\sigma} {\rm tr}\left[F^{\mu\nu}F^{\rho\sigma}\right] - \frac{\eta(iD_{\rm 3D})}{2}$$

$$\mathcal{J} = \text{APS index} = \text{integer}$$

Recovers T symmetry!

This is the reason for the stable existence of edge modes!

## Unsatisfactory points in Witten's argument:

- **Reasonable argument but not derivation**

  Imposing theoretical consistency :

  "symmetry (microscopic) = symmetry (low energy)"

  Similar to anomaly matching condition

- **Why APS index ?**

  - massless 4d Dirac op. does not appear in DW fermion
  - appearance of APS index in T-anomaly looks just accidental
  - APS index theorem do not deal with localized edge modes
  - Non-local b.c. in APS ←??→ local b.c. in DW

  → Physics of "APS index" and "DW fermion" are quite different

- **Simple and Direct Derivation like Fujikawa's method?**

# C. Our goal

We directly compute the index for domain-wall fermion.

If successful,

- Microscopic derivation of the determinant phase of the Domain-wall fermion in a "physicist-friendly" way, similar to the Fujikawa method.
- Better understanding of anomaly descent equations.

Our main result:

✓ index for domain-wall fermion = APS index

✓ T-anomaly cancellation using Fujikawa's method.

# 2. Index for domain-wall Dirac operator

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## a. Determinant phase of Domain-wall Dirac op.

Domain-wall fermion in 4-dim with 3-dim boundary

$$\det\left(\frac{D+M\epsilon(x_4)}{D-M}\right)$$

Domain-wall with kink mass

Pauli-Villars regulator

Defining

$$H_{\mathrm{DW}} = \gamma_5\left(D + M\epsilon(x_4)\right)$$
$$H_{\mathrm{PV}} = \gamma_5\left(D - M\right)$$

Phase can be evaluated as

$$\det\left(\frac{D+M\epsilon(x_4)}{D-M}\right)$$
$$= \left|\det\left(\frac{D+M\epsilon(x_4)}{D-M}\right)\right| \exp\left(\frac{i\pi}{2}\left(\eta(H_{\mathrm{DW}}) - \eta(H_{\mathrm{PV}})\right)\right)$$

4-dim Hamiltoinan $H_{\rm DW}, H_{\rm PV}$ : Hermitian

Det. should be real if properly regularized.

$$\frac{1}{2}(\eta(H_{\rm DW}) - \eta(H_{\rm PV})) = \text{integer} \equiv \mathcal{J}^{\rm DW}$$

We propose to define this quantity $\mathcal{J}_{\rm DW}$ as "the index of Domain-wall Dirac op."

# Computation of eta-invariant

$$\eta(H) = \lim_{s\to 0} \sum_k \frac{\mathrm{sign}(\frac{\lambda_k}{M})}{|\frac{\lambda}{M}|^s} = \lim_{s\to 0} \mathrm{Tr}\left[ \frac{\frac{H}{M}}{\sqrt{\frac{H^2}{M^2}}^{1+s}} \right]$$

$$= \lim_{s\to 0} \frac{1}{\Gamma(\frac{1+s}{2})} \int_0^\infty dt\ t^{\frac{s-1}{2}} \mathrm{Tr}\left[ \frac{H}{M} \exp\left( -t\frac{H^2}{M^2} \right) \right]$$

# b. Computation of $\eta(H_{\mathrm{PV}})$

$$\frac{H_{\mathrm{PV}}}{M} = -\gamma_5 + \gamma_5 \frac{D^{4d}}{M}, \quad \frac{H_{\mathrm{PV}}^2}{M^2} = 1 + \frac{(D^{4d})^2}{M^2}$$

$$\eta(H_{\mathrm{PV}}) = \lim_{s \to 0} \int_0^\infty dt \frac{t^{\frac{s-1}{2}}}{\Gamma(\frac{1+s}{2})} e^{-t} \mathrm{Tr}\left[\left(-\gamma_5 + \gamma_5 \frac{D^{4d}}{M}\right) \exp\left(-t \frac{(D^{4d})^2}{M^2}\right)\right]$$

Does not contributue

Calculation of trace using plane wave gives

$$\eta(H_{\mathrm{PV}}) = -\frac{1}{32\pi^2} \int_{-\infty}^{\infty} dx_4 \int d^3x \epsilon^{\mu\nu\alpha\beta} \mathrm{tr}\left[F_{\mu\nu} F_{\alpha\beta}\right] + O(\frac{1}{M^2})$$

Essentially the same calculation as Fujikawa's method.

# c. Computation of $\eta(H_{\rm DW})$

DW fermion: feels $\gamma_4$ -dependent potential at the origin

$$\frac{H_{\rm DW}}{M} = \gamma_5 \epsilon(x_4) + \gamma_5 \frac{D^{4d}}{M}, \quad \frac{H_{\rm DW}^2}{M^2} = 1 + \frac{(D^{4d})^2 - 2M\gamma_4\delta(x_4)}{M^2}$$

Complete set of plane wave states should be modified!

Taking the basis $\gamma_4 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$
and solving eigenstates for the free part of $H^2_{\rm DW}$

$$H^2_{\rm DW} = -\partial_4^2 - \sum_{i=1}^{3} \partial_i^2 + M^2 - 2M\gamma_4\delta(x_4)$$

Complete set of states

$$\varphi_o^{\omega,\vec{k}}(x_4) = \frac{1}{\sqrt{4\pi}}\left(e^{i\omega x_4} - e^{-i\omega x_4}\right) e^{i\vec{k}\cdot\vec{x}} \begin{pmatrix} \phi \\ 0 \end{pmatrix},$$

$$\varphi_e^{\omega,\vec{k}}(x_4) = \frac{1}{\sqrt{4\pi(\omega^2+M^2)}}\left((i\omega - M)e^{i\omega|x_4|} + (i\omega + M)e^{-i\omega|x_4|}\right) e^{i\vec{k}\cdot\vec{x}} \begin{pmatrix} \phi \\ 0 \end{pmatrix},$$

$$\varphi_e^{\rm edge}(x_4) = \sqrt{M}e^{-M|x_4|} \begin{pmatrix} \phi \\ 0 \end{pmatrix},$$

$$\varphi_{-,o}^{\omega}(x_4) = \frac{1}{\sqrt{4\pi}}\left(e^{i\omega x_4} - e^{-i\omega x_4}\right) e^{i\vec{k}\cdot\vec{x}} \begin{pmatrix} 0 \\ \chi \end{pmatrix},$$

$$\varphi_{-,e}^{\omega}(x_4) = \frac{1}{\sqrt{4\pi(\omega^2+M^2)}}\left((i\omega + M)e^{i\omega|x_4|} + (i\omega - M)e^{-i\omega|x_4|}\right) e^{i\vec{k}\cdot\vec{x}} \begin{pmatrix} 0 \\ \chi \end{pmatrix},$$

$$\eta(H_{\rm DW}) = \lim_{s\to 0}\int_0^\infty dt \frac{t^{\frac{s-1}{2}}}{\Gamma(\frac{1+s}{2})} e^{-t} {\rm Tr}\left[\left(\gamma_5\epsilon(x_4) + \gamma_5\frac{D^{4d}}{M}\right)\exp\left(-t\frac{H_{\rm DW}^2}{M^2}\right)\right]$$

(1) (2)

Also contributue due to $\gamma_4$ -dep. complete set of states

Calculation of (1) using complete set of plane wave states gives

$$\eta^{(1)}(H_{\rm DW}) = \frac{1}{32\pi^2}\int_{-\infty}^{\infty} dx_4 \int d^3x \epsilon(x_4)\epsilon^{\mu\nu\alpha\beta}{\rm tr}\left[F_{\mu\nu}F_{\alpha\beta}\right] + O(\frac{1}{M^2})$$

Essentially the same calculation as Fujikawa's method.

## Calculation of (2) using complete set of states.

by differentiating & integrating over gauge field

( integer part can be dropped )

After technical calculation

using gaussian integral and error functions

$$\eta^{(2)}(H_{\mathrm{DW}}) = 2\left(-\frac{\eta(D^{3d})}{2} + \mathrm{mod}(\mathrm{integer})\right)$$

$$\eta(H_{\mathrm{DW}}) = \eta^{(1)}(H_{\mathrm{DW}}) + \eta^{(2)}(H_{\mathrm{DW}})$$
$$= \frac{1}{32\pi^2}\int d^4x\epsilon(x_4)\epsilon^{\mu\nu\alpha\beta}\mathrm{tr}[F_{\mu\nu}F_{\alpha\beta}] + 2\left(-\frac{\eta(D^{3d})}{2} + \mathrm{mod}(\mathrm{integer})\right)$$

# d. Final results

Combining PV and DW contributions, we obtain

$$\mathcal{J}^{\mathrm{DW}} \equiv \frac{1}{2}\left(\eta(H_{\mathrm{DW}}) - \eta(H_{\mathrm{PV}})\right)$$
$$= \frac{1}{32\pi^2}\int_{x_4>0} d^4x \mathrm{tr}[\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}] - \frac{\eta(D^{3d})}{2} (\mathrm{mod\ integer})$$

This agrees with the result by APS index theorem!

Direct macroscopic derivation of
Domain-Wall fermion determinant phase,

No fictitious massless Dirac op. needed as a mathematica tool.

We also confirm that domain-wall index

$$\mathcal{J} = \frac{\eta(H_{\rm DW})}{2} - \frac{\eta(H_{\rm PV})}{2}$$

Is idependent of mass, $\dfrac{\partial \mathcal{J}}{\partial M} = 0$

And stable against change of gauge fields

$$\frac{\delta \mathcal{J}}{\delta A_\mu(x)} = 0.$$

# 3. Summary

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- We have succeeded in direct derivation of the determinant phase for Domain-wall fermion using Fujikawa's method.
- We reformulated APS index theorem with domain-wall Dirac op.
- T-anomaly cancellation was understood from microscopic theory.

$$\mathfrak{I}(D^{4\mathrm{D}})|_{\text{APS boundary}} = \eta(H^{4\mathrm{D}}(m))|_{\text{SO(3) symmetirc boundary}}$$
$$= \int_{x_4>0} d^4xF\wedge F - \frac{\eta(iD^{3\mathrm{D}})}{2}$$

# What is next?

1. Generalization to Shamir-type domain-wall fermions

2. Generalization to odd dimensions

3. Non-perturbative formulation of APS index theorem on a lattice

4. Application to 6D formulation of lattice chiral gauge theory.