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Many-body localization in large-N conformal mechanics

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Mainly based on collaboration with Pramod Padmanabhan

References:

- Nandkishore and Huse, arXiv:1404.0686
- Altman and Vonk, arXiv:1408.2834
- Imbre, Ros and Scardicchi, arXiv:1609.08076

1 Introduction

In this Talk, I discuss on recent topics in quantum statistical mechainics -

thermalization and localization in quantum many-body systems.

1.1 What is thermalization?

Let's consider a closed quantum system S. Time evolution of the system is given by

$$\rho(t) = e^{-iHt}\rho(0)e^{iHt}, \qquad H \text{ : Hamiltonian}$$

We can consider the same system in thermal equilibrium at temperature β^{-1}:

$$\rho^{(\mathrm{eq})}(\beta) = \frac{1}{Z(\beta)}e^{-\beta H}, \qquad Z(\beta) = \mathrm{Tr}\, e^{-\beta H}$$

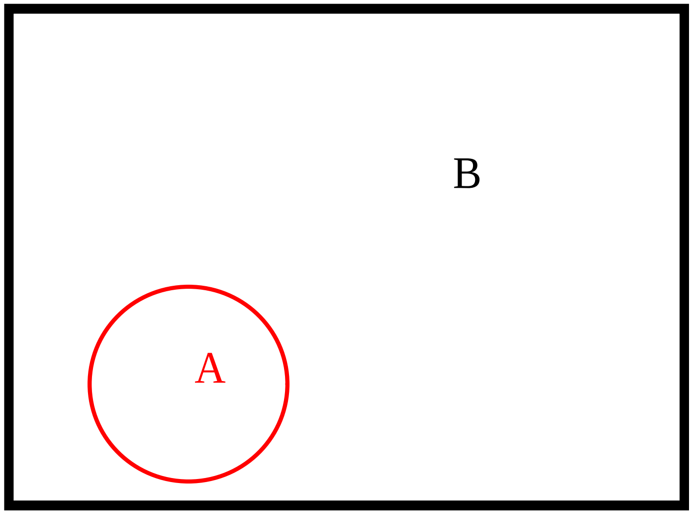

Figure 1: The closed system $\boldsymbol{S}$ is inside of the box. The subregion $\boldsymbol{A}$ is a region bounded by the red circle, and $\boldsymbol{B} = \boldsymbol{S} - \boldsymbol{A}$.

Pick a small subregion $\boldsymbol{A}$ in the system in real space.
$\boldsymbol{B} = \boldsymbol{S} - \boldsymbol{A}$ is regarded as a reservoir.

Reduced density matrix of $\boldsymbol{A}$:

$$\boldsymbol{\rho}_A(t) = \mathrm{Tr}_B\, \boldsymbol{\rho}(t).$$

Also,

$$\boldsymbol{\rho}_A^{(\mathrm{eq})}(\beta) = \mathrm{Tr}_B\, \boldsymbol{\rho}^{(\mathrm{eq})}(\beta).$$

Then, the system thermalizes for the temperature β^{-1} if

$$\rho_A(t) \to \rho_A^{(\mathrm{eq})}(\beta) \qquad \text{as } t \to \infty \text{ and } |S| \to \infty \text{ with } |A| \text{ fixed}$$

holds for all subsystems A.

Note: Thermalization does not imply $\rho(t) \to \rho^{(\mathrm{eq})}(\beta)$.

1.2 Eigenstate Thermalization Hypothesis (ETH)

Suppose $\rho(0)$ is a pure state of an energy eigenstate:

$$\rho(0) = |E_n\rangle\langle E_n|, \qquad H|E_n\rangle = E_n|E_n\rangle.$$

$\Longrightarrow$ ρ is time-independent: $\rho(t) = \rho(0)$.
$\Longrightarrow$ $\rho_A(t) = \rho_A(0)$ for any A.

In this case, we expect all the energy eigenstates are thermalized (ETH).
[Deutsch 1991, Srednicki 1994, Tasaki 1998,...]

If ETH holds,

- The temperature at the thermal equilibrium β_n^{-1} is determined by

$$E_n = \langle H\rangle_{\beta_n} \equiv \frac{1}{Z(\beta_n)}\mathrm{Tr}\left(H\, e^{-\beta_n H}\right).$$

- Entanglement entropy

$$S_A = -\mathrm{Tr}_A\left(\rho_A \ln \rho_A\right)$$

is equal to the equilibrium thermal entropy of A.
In particular, S_A obeys the volume law: $S_A \propto |A|$.

- Initial density matrix

$$\rho(0) = \sum_{n,m} \gamma_{n,m} \, |E_n\rangle\langle E_m|$$

evolves as

$$\rho(t) = \sum_{n,m} \gamma_{n,m} \, e^{-i(E_n - E_m)t} \, |E_n\rangle\langle E_m|$$

(Thermalization) = (Contribution from off-diagonals decays at late time)

For an operator X,

$$\langle X \rangle = \mathrm{Tr}\,[X \, \rho(t)] = \sum_{n,m} \gamma_{n,m} X_{m\,n} \, e^{-i(E_n - E_m)t}$$

When $\gamma_{n,m} X_{m\,n}$ slowly varies w.r.t. n and m, contribution form off-diagonals is strongly suppressed at late time due to rapidly oscillating behavior (dephasing).

Such "global" operators would be suitable observables treated in quantum statistical mechanics.

Local initial informations are 'hidden' in dephasing after long time.

However, ETH is a hypothesis.
Not true for one class of systems - localized systems.

1.3 Localized systems

◇ Anderson localization (single particle problem) [Anderson 1958]

$$H = -\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V_p(x) + V_q(x)$$

↑ periodic potential ↖ random noise

- For $V_q = 0$,
  Wave function $\psi(x)$ is oscillating (Bloch wave).

- When V_q is turned on,
  Wave function becomes localized:

$$\psi(x) \sim e^{-\mu_q |x|} \qquad (|x| \to \infty)$$

with a strictly positive constant μ_q.

◇ Many-body localization (MBL):

- localization with many-body interactions
- occurs for highly excited states

Typical system (quantum spin-$\frac{1}{2}$ chain)

$$\boldsymbol{H} = \sum_i \boldsymbol{h_i \sigma_i^z} + \sum_{\langle i,j \rangle} \boldsymbol{J_{ij}\, \vec{\sigma}_i \cdot \vec{\sigma}_j}$$

$\nearrow$ $\qquad\qquad\qquad\qquad$ $\nwarrow$

$-\boldsymbol{W} \lesssim$ (random magnetic fields) $\lesssim \boldsymbol{W}$ $\qquad$ short-range interations

All $\boldsymbol{J_{ij}}$ are almost same. $\boldsymbol{J_{ij} \sim J}$.

1. For all $\boldsymbol{J_{ij} = 0}$,
   eigenstates are product states $|\boldsymbol{\sigma_1^z}\rangle|\boldsymbol{\sigma_2^z}\rangle \cdots$.
   $\Longrightarrow$ the system is fully localized.

   Strictly local integrals of motion (LIOM): $\boldsymbol{\sigma_i^z}$ $(\boldsymbol{i = 1, 2, \cdots})$

   $[\boldsymbol{H},\ \boldsymbol{\sigma_i^z}] = 0$

2. For $J \neq 0$ and $J \ll W$,
$\Longrightarrow$ MBL takes place.

Quasi LIOM (with exponentially decaying tails) are constructed perturbatively and nonperturbatively

[Basko-Aleiner-Altshuler 2006, Imbre 2014]

$$
\begin{aligned}
H &= E_0 + \sum_i h'_i \tau_i^z + \sum_{i<j} J'_{ij}\, \tau_i^z \tau_j^z \\
&\quad + \sum_{n=1}^{\infty} \sum_{i<k_1<\cdots<k_n<j} K^{(n)}_{i\,\{k\}\,j}\, \tau_i^z \tau_{k_1}^z \cdots \tau_{k_n}^z \tau_j^z
\end{aligned}
$$

$\tau_i^z = e^{-iS} \sigma_i^z e^{iS}$: Unitary transf. of σ_i^z

Couplings J'_{ij}, $K^{(n)}_{i,\{k\},j}$ fall off exponentially as $|i-j| \to \infty$.

- Each of the terms in H is quasi LIOM.
- No spin flip operators (τ^x, τ^y) in $H \Longrightarrow$ No dissipation

1. First, consider the site $i = 2$.
   Unitary transformation (Schrieffer-Wolff transf.):
   $H \to H' = e^{iS}\, H\, e^{-iS}$
   S: sum of local operators is chosen so that off diagonals σ_2^x, σ_2^y can be eliminated
   $\Longrightarrow [\sigma_2^z,\ H'] = 0$:

   Writing $H = H_0 + V$ with $H_0 \equiv h_2 \sigma_2^z$, $V \equiv$ (rest),

$$S = \frac{-i}{2h_2}(P_+ V P_- - P_- V P_+) + \mathcal{O}(V^2), \qquad P_\pm = \frac{1}{2}(1_2 \pm \sigma_2^z).$$

2. Continue for the other sites. $\Longrightarrow H'(\sigma_i^z)$ (σ_i^x, σ_i^y: eliminated)

$$H = e^{-iS} H'(\sigma_i^z)\, e^{iS} = H'(\tau_i^z) \text{ with } \tau_i^z = e^{-iS} \sigma_i^z\, e^{iS}$$

3. For $J \gg W$,
$\Longrightarrow$ ETH true

Phase transition between MBL and ETH phases around $J \sim W$?
New type of phase transition between thermal equilibrium and out-of-equilibrium.

For applications, it is expected that localization is an intriguing phenomenon to protect the system from thermal decoherence
and to construct devices for quantum computation.

Some properties (known from spin systems)

Thermal phase	Many-body localizaton
ETH true	ETH false
Memory of local initial info. 'hidden'	Memory of some local initial info.preserved
Continuous spectrum	Discrete spectrum
Eigenstates with volume-law EE	Eigenstates with area-law EE
Power-law spreading of entanglement from non-entangled initial state $S_A \sim t^p$	Logarithmic spreading of entanglement from non-entangled initial state $S_A \sim \ln t$

Issues:

- Almost these analysis have been performed only for spin systems.
  Extension to other systems should be important to understand universal properties for localizations.

- While numerical evidence has been accumulated, analytic treatment is hard.
  Hard to investigate large volume systems in numerics. $\sim$ 20 sites

- Nonthermal phases other than Anderson localization phase and MBL phase?

Here, we construct an integrable model of many-body conformal quantum mechanics by using coproducts,
and analyze its thermal or localization properties.

Contents

Section 2: Many-body interacting model by using coproducts

Section 3: Eigenstates and eigenvalues

Section 4: Entanglement entropies

Section 5: Summary and discussions

2 Many-body interacting model by using coproducts

Based on the idea by [Ballesteros-Ragnisco 1998].

2.1 Conformal quantum mechanics

◇ $SL(2,R)$ (1d conformal) generators: $L_0,\ L_+,\ L_-$

$$[L_0,\ L_\pm] = \pm L_\pm, \qquad [L_+,\ L_-] = -2L_0$$

◇ Realization as 1 particle QM [De Alfaro-Fubini-Furlan 1976]

$$L_0 = \frac{1}{4}\left(p^2 + \frac{g}{x^2} + x^2\right), \leftarrow \text{ Hamiltonian}$$

$$L_\pm = \frac{1}{4}\left(-p^2 - \frac{g}{x^2} + x^2\right) \mp i\frac{1}{4}(xp + px).$$

For simplicity, we consider the case of $g = 0$ (harmonic oscillator).

2.2 Coproducts

$\boldsymbol{L_{a,\,i}}$ ($\boldsymbol{a = 0,\ \pm}$) : $\boldsymbol{L}$-operators of particle $\boldsymbol{i}$ (or at site $\boldsymbol{i}$)

$$\Delta^{(2)}(L_a) = L_a \otimes 1 + 1 \otimes L_a = L_{a,\,1} + L_{a,\,2},$$

$$\begin{aligned}
\Delta^{(3)}(L_a) &= (\mathbb{1} \otimes \Delta^{(2)}) \circ \Delta^{(2)}(L_a) \\
&= (\mathbb{1} \otimes \Delta^{(2)}) \circ (L_a \otimes 1 + 1 \otimes L_a) \\
&= L_a \otimes \Delta^{(2)}(1) + 1 \otimes \Delta^{(2)}(L_a) \\
&= L_a \otimes 1 \otimes 1 + 1 \otimes (L_a \otimes 1 + 1 \otimes L_a) \\
&= L_{a,\,1} + L_{a,\,2} + L_{a,\,3},
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
\Delta^{(k)}(L_a) &= (\overbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}^{k-2} \otimes \Delta^{(2)}) \circ \Delta^{(k-1)}(L_a) \\
&= L_{a,\,1} + \cdots + L_{a,\,k}
\end{aligned}$$

Note: The copropuct acts as homomorphism:

$$[\Delta^{(k)}(L_0),\ \Delta^{(k)}(L_\pm)] = \pm\Delta^{(k)}(L_\pm),$$
$$[\Delta^{(k)}(L_+),\ \Delta^{(k)}(L_-)] = -2\Delta^{(k)}(L_0).$$

For Casimir operator

$$C = L_0^2 - \frac{1}{2}\{L_+, L_-\} = L_0^2 - L_0 - L_+L_-,$$

$$\Delta^{(k)}(C) = \left(\Delta^{(k)}(L_0)\right)^2 - \Delta^{(k)}(L_0) - \Delta^{(k)}(L_+)\Delta^{(k)}(L_-).$$

$\Longrightarrow \Delta^{(k)}(C)$ commutes with $\Delta^{(k)}(L_a)$

$\Longrightarrow \Delta^{(k')}(C)$ commutes with $\Delta^{(k)}(L_a)$ when $k' \leq k$

2.3 N-particle interacting system

[PP-FS]

$$H_N = \Delta^{(N)}(L_0) + \sum_{k=2}^{N} \alpha_k \, \Delta^{(k)}(C)$$

$\nearrow$ N free harmonic oscillators $\nearrow$ coupling consts. $\nwarrow$ (nonlocal) interactions

Notes:

- $\Delta^{(N)}(L_0)$ and $\Delta^{(k)}(C)$ $(k = 2, \cdots, N)$ mutually commute.
  $\Longrightarrow$ N conserved quantities (integrable system)
  But, not local. (Nontrivial for MBL)

- In terms of x, p variables,

$$H_N = \sum_{i=1}^{N} \frac{1}{4}\left(p_i^2 + x_i^2\right) + \sum_{k=2}^{N} \alpha_k \left\{ \frac{1}{4} \sum_{k \geq i > j \geq 1} M_{ij}^2 + \frac{k(k-4)}{16} \right\}$$

with $M_{ij} \equiv x_i p_j - x_j p_i$ ("angular momenta").

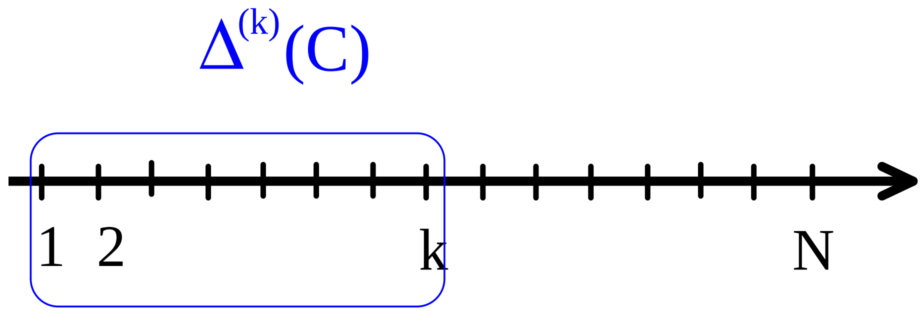

Figure 2: The operator $\Delta^{(k)}(C)$ has interactions between any two sites among $\{1, 2, \cdots, k\}$.

- Taking $\alpha_k \sim e^{-k/\xi}$ makes the interactions exponentially local w.r.t. the site k.

◇ Level 0 (Lowest weight states):

$L_{-,i}|s\rangle_N = 0$ for $i = 1, \cdots, N$

$\Longrightarrow$ $|s\rangle_N = |r_0^{(1)}, \cdots, r_0^{(N)}\rangle$ with $L_{0,i}|r_0^{(i)}\rangle = r_0^{(i)}|r_0^{(i)}\rangle$ ($r_0^{(i)} = \frac{1}{4}$ or $\frac{3}{4}$).

Energy: $E = R_N + \sum_{k=2}^{N} \alpha_k R_k (R_k - 1)$ $\qquad R_k \equiv r_0^{(1)} + \cdots + r_0^{(k)}$

$\Longrightarrow$ Fock space

$$\mathcal{F} = \bigoplus_{r_0^{(1)}, \cdots, r_0^{(N)}} \mathcal{F}_{(r_0^{(1)}, \cdots, r_0^{(N)})}$$

with

$$\mathcal{F}_{(r_0^{(1)}, \cdots, r_0^{(N)})} \equiv \left\{ L_{+,1}^{k_1} \cdots L_{+,N}^{k_N} |s\rangle_N \,;\, k_1, \cdots, k_N = 0, 1, 2, \cdots \right\}$$

↖ (level: $k_1 + \cdots + k_N$)

◇ Level p excited states: $\binom{p+N-1}{p}$ states

$$
\begin{aligned}
|v_{p,\,(m_1,n_1),\cdots,(m_q,n_q)}\rangle_N \;=\; & \left(\Delta^{(N)}(L_+)\right)^n \\
& \times F_{m_1}(\Delta^{(n_1)}(L_+),\; L_{+,\,n_1+1}) \\
& \times F_{m_2}(\Delta^{(n_2)}(L_+),\; L_{+,\,n_2+1}) \\
& \times \cdots \\
& \times F_{m_q}(\Delta^{(n_q)}(L_+),\; L_{+,\,n_q+1})\,|s\rangle_N,
\end{aligned}
$$

where $F_{m_k}(\Delta^{(n_k)}(L_+),\; L_{+,\,n_k+1})$ is a degree-m_k homogeneous polynomial of $\Delta^{(n_k)}(L_+)$ and $L_{+,\,n_k+1}$ (indep. of α_k's),
$p = n + \sum_{j=1}^{q} m_j$,
$N - 1 \geq n_1 > n_2 > \cdots > n_q \geq 1$.

Energy:

$$
\begin{aligned}
E_{p,\,(m_1,n_1),\cdots,(m_q,n_q)} \;=\; & R_N + p + \sum_{k=2}^{n_q} \alpha_k R_k (R_k - 1) \\
& + \sum_{\ell=2}^{q} \sum_{k=n_\ell}^{n_{\ell-1}} \alpha_k \left(R_k + \sum_{j=\ell}^{q} m_j \right) \left(R_k + \sum_{j=\ell}^{q} m_j - 1 \right) \\
& + \sum_{k=n_1}^{N} \alpha_k \left(R_k + \sum_{j=1}^{q} m_j \right) \left(R_k + \sum_{j=1}^{q} m_j - 1 \right)
\end{aligned}
$$

Remarks:

- For highly excited states ($p = n + \sum_{j=1}^{q} m_j$ large), huge degeneracy at free case - $\begin{pmatrix} p + N - 1 \\ p \end{pmatrix}$ degeneracy - is completely resolved by turning on α_k's.

- For $\alpha_k \sim e^{-k/\xi}$, the level splitting between states with different m_j's is $\mathcal{O}(e^{-N/\xi})$. Spectrum becomes continuous at large N.
  $\Longrightarrow$ Thermalization expected

  On the other hand, the splitting between states with different p's (m_j fixed) is $\mathcal{O}(1)$.
  $\Longrightarrow$ Nonthermal behavior (localization) expected

4 Entanglement entropies [PP-FS]

The total system $S = \{1, 2, \cdots, N\}$ is divided into

$$A = \{N - \nu + 1, \cdots, N\}, \qquad B = \{1, \cdots, N - \nu\}$$

with $\nu \ll N$.

◇ Entanglement entropy of highly excited states with $n_1 < N - \nu$ for large N:

- For $n (\equiv p - \sum_j m_j) \ll \hat{R}_N (\equiv R_N + \sum_j m_j)$,

$$S_A \sim \left(\sum_{i \in A} r_0^{(i)} \right) \times \frac{n}{\hat{R}_N} \ln \hat{R}_N$$

↗

Volume law like behavior! — seems to support thermal phase,

although S_A is tiny.

Entanglement spreading from low-level (noneigen)states $\sim t^2$ when $\alpha_k \sim e^{-k/\xi}$

- For $n \gg \hat{R}_N$,

$$S_A \sim \ln n$$

Indep. of ν! $\Longrightarrow$ Area law like — seems to support localization phase

5 Summary and discussions

◇ Summary:

- We briefly reviewed quantum thermalization and localizations.
- We constructed an integrable model with many-body interactions by using the coproducts, and discussed its thermalization and localization properties.
  - Seems more tractable compared with the coupled harmonic oscillators.

◇ Future directions:

- Figure out clearer physical picture for this model.
  - Complete the computation of the entanglement entropy, and investigate other indicators for MBL (entanglement growth, transport properties, ...).
  - Analyze "phase transition" between the thermal and localization phases around $n \sim \hat{R}_N$.

- Role of conformal symmetry?   Implication in AdS/CFT?
- Models of the coproducts based on different groups.

Thank you very much for your attention!