Contravariant Gravity

-A Gravity on Poisson Manifolds-

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mainly based on

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Quantum Theory of Gravity

has been a long-standing problem in theoretical physics

(Un)fortunately, we still have no rigorous formulation of it, although many approaches have been pursued:

- Loop Quantum Gravity
- Discretization Methods
- Noncommutative Geometry
- String Theory
- etc. …

} SCOPE OF TODAY'S TALK

Minimal Length and Noncommutativity

Naïve Heuristic Argument:

A string, as an extended object, cannot be specified where it is no more precisely than its length

Similar to quantum mechanics,
where an uncertainty on phase space of particle motion
can be realized by replacing $\{x, p\}_{\mathrm{PB}} = 1 \longrightarrow [\hat{x}, \hat{p}] = i$,
the uncertainty on spatial position of string
can be (effectively) realized by noncommutative coordinates:

$$[\hat{x}^i, \hat{x}^j] = i\Theta^{ij}$$

Gravity on Noncommutative Spaces

would be interesting to investigate

as an effective theory / in itself

from the viewpoint of

string theory / noncommutative geometry

But it seems a bit far to tackle: We instead consider

Gravity on Poisson Manifolds

Noncomm. Spaces and Poisson Manifolds

Noncommutativity of spatial coordinates $[\hat{x}^i, \hat{x}^j] = i\Theta^{ij}$ is approximated by a Poisson bracket:

$$[\hat{x}^i, \hat{x}^j] = i\Theta^{ij} \longrightarrow \{x^i, x^j\} = \theta^{ij}$$

Poisson manifold is a manifold equipped with a Poisson bracket of functions

$$\{f, g\} = \theta^{ij}(\partial_i f)(\partial_j g)$$

Differential

A Poisson bracket of functions $\{f, g\} = \theta^{ij}(\partial_i f)(\partial_j g)$ induces a natural "differential":

$$D^i f := \{x^i, f\} = \theta^{ij}\partial_j f$$

- which is referred to as Hamiltonian vector field in math literature
- whose noncommutative counterpart is $\mathcal{D}_i \Phi = i[X^i, \Phi]$

We shall take this derivative as a guiding object to develop differential geometry and gravity

Differential Geometry with Poisson Bracket

Consider "contravariant extension" of $D^i f := \{x^i, f\}$:

Contravariant derivative

$$\bar{\nabla}^i \alpha_j = D^i \alpha_j + \bar{\Gamma}^{ik}_j \alpha_k$$

$$D^i = \theta^{ij} \partial_j$$

$$[D^i, D^j] \neq 0$$

Imposing both conditions

- -Metricity $\bar{\nabla}^i G_{jk} = 0$
- -Torsion-free $\bar{\Gamma}^{ij}_k - \bar{\Gamma}^{ji}_k - \partial_k \theta^{ij} = 0$

the connection is uniquely specified in the following form

$$\bar{\Gamma}^{ij}_k = \frac{1}{2} G_{mk} (\theta^{il} \partial_l G^{jm} + \theta^{jl} \partial_l G^{im} - \theta^{ml} \partial_l G^{ij} + G^{lj} \partial_l \theta^{mi} + G^{li} \partial_l \theta^{mj} + G^{lm} \partial_l \theta^{ij})$$

What's the difference from the usual one?

While we defined $D^i f := \{x^i, f\} = \theta^{ij}\partial_j f$,

is there any difference between

- contravariant derivative: $\bar{\nabla}^i \alpha_j := D^i \alpha_j + \bar{\Gamma}^{ik}_j \alpha_k$

$$= \theta^{ik}\partial_k \alpha_j + \bar{\Gamma}^{ik}_j \alpha_k$$

and

- covariantization of D: $\bar{D}^i \alpha_j := \theta^{ik}\nabla_k \alpha_j$

$$= \theta^{ik}\partial_k \alpha_j + \theta^{ik}\Gamma^l_{kj}\alpha_l$$

?

Contorsion Tensor

$\bar{\Gamma}^{ij}_{k}[G,\theta]$ and $\Gamma^{k}_{ij}[G]$ are definitely different from each other as their arguments indicate, more specifically

$$\bar{\Gamma}^{ij}_{k} = \Gamma^{j}_{kl}\theta^{li} + K^{ij}_{k}$$

where K^{ij}_{k} is a contorsion tensor given by

$$K^{ij}_{k} = \frac{1}{2}G_{kl}(\nabla^{l}\theta^{ij} - \nabla^{i}\theta^{jl} + \nabla^{j}\theta^{li})$$

which is understood as an obstruction of covariantly constant θ^{ij}

Some Mathematical Backgrounds I

Behind the construction of the connection,
the Lie algebroid $(T^*M, [\cdot\,,\,\cdot]_\theta)$, Lie algebra of 1-forms,
(*cf.* Watamura-san's talk) plays a role,
instead of usual Lie algebra of vector fields $(TM, [\cdot\,,\,\cdot]_{\mathrm{Lie}})$

In other words, we take a "Lie derivative" $\bar{\mathcal{L}}_\xi \eta = [\xi, \eta]_\theta$
as a generator of symmetry, instead of $\mathcal{L}_X Y = [X, Y]_{\mathrm{Lie}}$
which generates a usual diffeomorphism

Some Mathematical Backgrounds II

We then introduce an affine connection on 1-forms as follows:

$$\bar{\nabla}_{f\xi}(g\eta) = f(\bar{\mathcal{L}}_\xi g)\eta + fg\bar{\nabla}_\xi \eta$$

on local coordinates

$$\bar{\nabla}_{dx^i} dx^j = \bar{\Gamma}^{ij}_k dx^k \qquad\qquad cf. \quad \nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$$

The conditions which specify the connection read

- $\bar{\mathcal{L}}_\xi G(\eta, \zeta) = G(\bar{\nabla}_\xi \eta, \zeta) + G(\eta, \bar{\nabla}_\xi \zeta)$
- $T(\xi, \eta) = \bar{\nabla}_\xi \eta - \bar{\nabla}_\eta \xi - [\xi, \eta]_\theta = 0$

Curvature Tensor

From the affine connection, we straightforwardly define the curvature tensor:

$$\bar{R}(\xi,\eta,\zeta) = \bar{\nabla}_\xi \bar{\nabla}_\eta \zeta - \bar{\nabla}_\eta \bar{\nabla}_\xi \zeta - \bar{\nabla}_{[\xi,\eta]_\theta} \zeta$$

On a local patch the curvature tensor takes the form

$$\bar{R}_l^{kij} = \theta^{im} \partial_m \bar{\Gamma}_l^{jk} + \bar{\Gamma}_m^{jk} \bar{\Gamma}_l^{im} - (i \leftrightarrow j) - \bar{\Gamma}_l^{nk} \partial_n \theta^{ij}$$

Ricci tensor/scalar reads

$$\bar{R}^{kj} := \bar{R}_l^{klj}; \quad \bar{R} := G_{ij} \bar{R}^{ij}$$

$$[D^i, D^j] \neq 0$$

in other words

$$[dx^i, dx^j]_\theta \neq 0$$

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Bianchi Identities

Curvature tensor satisfies the Bianchi identities

$$\bar{R}_l^{kij} + \bar{R}_l^{ijk} + \bar{R}_l^{jki} = 0$$

$$\bar{\nabla}^k \bar{R}_l^{mij} + \bar{\nabla}^i \bar{R}_l^{mjk} + \bar{\nabla}^j \bar{R}_l^{mki} = 0$$

$$\bar{R}^{mkij} = -\bar{R}^{kmij}$$

implying

"Einstein Tensor"

$$\bar{\mathcal{G}}^{ij} = \bar{R}^{ij} - \frac{1}{2} G^{ij} \bar{R}$$

is divergence free

$$\bar{\nabla}^k \bar{\mathcal{G}}_{ki} = 0$$

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Invariant Measure

In order to ensure that we can do partial integration, we introduce an invariant measure $e^{\phi}\sqrt{G}$

such that $e^{\phi}\sqrt{G}\bar{\nabla}^{i}\xi_i = \partial_j(e^{\phi}\sqrt{G}\xi_i\theta^{ij})$

$$cf. \quad \sqrt{G}\nabla_i V^i = \partial_i(\sqrt{G}V^i)$$

which imposes a condition on ϕ : $D^i\phi = \{x^i, \phi\} = 2\bar{\Gamma}_j^{ji}$

When θ^{ij} is invertible, ϕ is explicitly given by

$$\phi = -\log\left(\det(\theta^{ij})\det(G_{ij})\right) \Rightarrow e^{\phi}\sqrt{G} = 1/(\theta\sqrt{G})$$

Einstein-Hilbert-type Action

Once we fixed an invariant measure,

we can define an Einstein-Hilbert-type action

$$S_{\mathrm{EH}} = \int d^n x \sqrt{G} e^{\phi} \bar{R}$$

and cosmological term

$$S_{\mathrm{cosm}} = \int d^n x \sqrt{G} e^{\phi} \Lambda$$

Field Equations

Taking a variation with respect to G_{ij} (and $\phi = \phi[G]$) we find field equations

$$\bar{R}^{ij} - \frac{1}{2} G^{ij} \bar{R} = \Lambda G^{ij}$$

As mentioned above, this equation is divergence-free:

$$\bar{\nabla}_i \left(\bar{R}^{ij} - \frac{1}{2} G^{ij} \bar{R} \right) = 0$$

thanks to the Bianchi identities

Solutions to E.O.M.: Example I

Complex Plane

is the most simple and trivial example solution:

Both the metric G_{ij} and Poisson tensor θ^{ij} are spatially constant, then

$$\bar{\Gamma}^{ij}_{k} \equiv 0$$

$$\bar{\Gamma}^{ij}_{k} = \frac{1}{2} G_{mk} \left(\theta^{il} \partial_l G^{jm} + \theta^{jl} \partial_l G^{im} - \theta^{ml} \partial_l G^{ij} + G^{lj} \partial_l \theta^{mi} + G^{li} \partial_l \theta^{mj} + G^{lm} \partial_l \theta^{ij} \right)$$

and thus the curvature becomes also trivial

Solutions to E.O.M.: Example II

Complex Projective Space

For instance, take the Fubini-Study metric and the symplectic form of $\mathbb{C}P^2$, we find

$$\bar{R}^{ij} = 3G^{ij}$$

which solves the field equation with a suitable cosmological constant

Solutions to E.O.M.: Example III

Eguchi-Hanson Space

whose Kähler potential is given by

$$\phi(z_1, \bar{z}_1, z_2, \bar{z}_2) = \log \left(\frac{\rho^2 e^{\sqrt{\rho^4 + a^4}/a^2}}{a^2 + \sqrt{\rho^4 + a^4}} \right)$$

with

$$\rho^2 = |z_1|^2 + |z_2|^2$$

giving

$$\bar{R}^{ij} = 0$$

i.e. it is still Ricci-flat in contravariant sense as well

Linearization

As there exist solutions to field equation,
it's worth considering field expansion around sol. $(\hat{G}, \hat{\theta})$

Linear approximation: $G^{ij} = \hat{G}^{ij} + \underline{\epsilon h}^{ij}$ $\qquad \theta^{ij} = \hat{\theta}^{ij} + \epsilon \underline{B}^{ij}$

$$\underline{(\tilde{\partial}^k \tilde{\partial}_i h^{ji} + \tilde{\partial}^j \tilde{\partial}_i h^{ki} - \tilde{\partial}^i \tilde{\partial}_i h^{jk} - \tilde{\partial}^j \tilde{\partial}^k h_i{}^i) - \delta^{kj}(\tilde{\partial}_i \tilde{\partial}_l h^{il} - \tilde{\partial}^l \tilde{\partial}_l h_i{}^i)}$$
$$+\underline{(\partial^k \tilde{\partial}^i B_i{}^j + \partial^j \tilde{\partial}^i B_i{}^k - \partial_i \tilde{\partial}^j B^{ik} - \partial_i \tilde{\partial}^k B^{ij}) + 2\delta^{kj} \partial_i \tilde{\partial}^j B^i{}_j} = 0$$

where $\tilde{\partial}^i = \hat{\theta}^{ij} \partial_j$

Application to Noncommutative Gauge Th.

We put our machinery in the context of noncommutative geometry / emergent geometry

It turned out that the machinery is in part well describable:

fluctuations of gauge field $\longleftrightarrow$ fluctuations of Poisson:

$S_{\rm NC}[\hat{A}] = S^{\underline{g(A)}}_{\rm curved}[A]$ [03 Rivelles] $R[\underline{g(A)}] = \bar{R}[\delta, \theta(A)]$

(*cf.* Kaneko-san's talk)

Summary

● We considered a gravity on Poisson manifolds, regarding a differential induced by Poisson bracket as a guiding object to develop differential geometry and gravity

● We constructed an Einstein-Hilbert-type action and got its field equations, and gave some solutions to the equations.

● We examined linearization of the field equations

Discussions

- Odd dimensional case
- Relaxing the Poisson condition => quasi-Poisson
- Noncommutative extension
- Applications to noncommutative gauge theory
  (*cf.* Kaneko-san's talk)
- Relation with other approaches
  (applications to another context)