Information metric for matrix geometry

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Non-locality and non-commutativity

- Non-local structures are promising candidate of the quantum structure of the space-time.

- One way to lead non-local structures is making the coordinates non-commutative:

  Madore, Connes

$$[x^\mu, x^\nu] \neq 0 \quad \longrightarrow \quad \Delta x^\mu \Delta x^\nu \gtrsim \Lambda$$

- We can realize such non-commutativity by replacing the coordinates with matrices.

Example: fuzzy 2-sphere

- 2-sphere $S^2 \subset \mathbf{R}^3$ with embedding functions,

$$(x^1, x^2, x^3) \in \mathbf{R}^3, \quad x^i x_i = 1$$

- Fuzzy 2-sphere is given by a replacing x^i with normalized N dim irrep of SU(2) generators L^i ,

Madore

$$x^i \longrightarrow X^i = \frac{2}{\sqrt{N^2 - 1}} L^i \quad (X^i X_i = 1)$$

$$[X^i, X^j] = \frac{2}{\sqrt{N^2 - 1}} i\epsilon^{ijk} X^k \xrightarrow[N \to \infty]{} 0$$

Commutative limit

Functions

2-sphere

- Functions

$$f(\Omega) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} c_{\ell m} Y_{\ell m}(\Omega)$$

- Spherical harmonics

$$Y_{\ell m}(\Omega) = \sum_{k=0}^{\ell} t^{\ell m}_{i_1 i_2 \cdots i_k} x^{i_{i_1}} \cdots x^{i_{i_k}}$$

- Algebra

$$Y_{\ell_1 m_1}(\Omega) Y_{\ell_2 m_2}(\Omega) = \sum_{\ell_3 m_3} C^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2} Y_{\ell_3 m_3}(\Omega)$$

$C^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2}$: Clebsch-Gordan coefficients

Fuzzy 2-sphere

- Matrices

UV cutoff

$$\hat{f} = \sum_{\ell=0}^{N-1} \sum_{m=-\ell}^{\ell} c_{\ell m} \hat{Y}_{\ell m}$$

- Fuzzy spherical harmonics

$$\hat{Y}_{\ell m} = \sum_{k=0}^{\ell} t^{\ell m}_{i_1 i_2 \cdots i_k} X^{i_{i_1}} \cdots X^{i_{i_k}}$$

- Algebra

$$\hat{Y}_{\ell_1 m_1} \hat{Y}_{\ell_2 m_2} = \sum_{\ell_3 m_3} \hat{C}^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2} \hat{Y}_{\ell_3 m_3}$$

$$\hat{C}^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2} \xrightarrow[N\to\infty]{} C^{\ell_3 m_3}_{\ell_1 m_1 \ell_2 m_2}$$

Ishiki-Shimasaki-Takayama-Tsuchiya

Differential and integral

2-sphere

- Poisson bracket

$$\{x^i, x^j\} = 2\epsilon^{ijk} x_k$$

- Integral

$$\int \frac{d\Omega}{2\pi} Y_{\ell m}(\Omega)$$

Fuzzy 2-sphere

- Commutator

$$-iN[X^i, X^j] \sim 2\epsilon^{ijk} X_k$$

- Trace

$$\frac{1}{N} \mathrm{Tr}\, \hat{Y}_{\ell m} \xrightarrow[N\to\infty]{} \int \frac{d\Omega}{2\pi} Y_{\ell m}(\Omega)$$

- Poisson bracket gives angular momentum operators:

$$\{x^i, \cdot\} = 2\epsilon^{ijk} x_k \partial_j$$

Dictionary of fuzzy 2-sphere

- Functions, Poisson bracket and integral on 2-sphere are replaced as

$$
\begin{aligned}
Y_{\ell m}(\Omega) &\longrightarrow \hat{Y}_{\ell m} \\
\{x^i, \cdot\} &\longrightarrow -iN[X^i, \cdot] \\
\int \frac{d\Omega}{2\pi} &\longrightarrow \frac{1}{N}\mathrm{Tr}
\end{aligned}
$$

- This replacing is known as a concrete example of matrix regularization.

Matrix regularization (MR)

- Symplectic manifold $\mathcal{M}$ with symplectic form ω and Poisson bracket $\{\cdot,\cdot\}$.

- MR is a sequence of linear maps $\{T_N\}_{N=1,2,\dots}$ as

  Goldstone-Hoppe, Arnlind-Hoppe-Huisken

$$T_N : C^\infty(\mathcal{M}) \to N \times N \text{ Hermitian matrices}$$

$$\lim_{N\to\infty} ||T_N(fg) - T_N(f)T_N(g)|| = 0$$

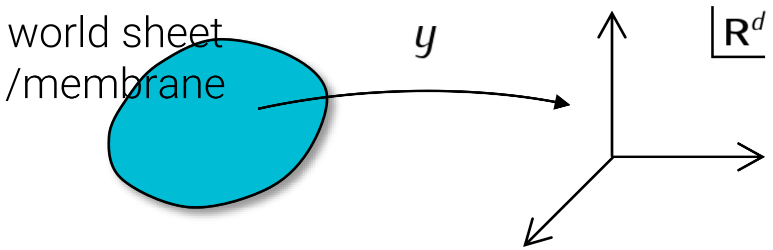
$$\lim_{N\to\infty} ||iN[T_N(f), T_N(g)] - T_N(\{f,g\})|| = 0$$

$$\lim_{N\to\infty} \frac{2\pi}{N} \mathrm{Tr}\, T_N(f) = \int_{\mathcal{M}} f\omega$$

Relation to string/M theories

- We can apply MR to the action of string/membrane after some gauge fixing and lead matrix models:

BFSS, IKKT, DVV, etc.



- They have a lot of expectations: non-perturbative formulation, 2nd quantization, emergent geometry, etc.

Hanada-Kawai-Kimura, Steinacker, Kim-Nishimura-Tsuchiya, etc.

How about the theory of gravity ??

- Can we describe gravity in terms of matrices ??

  Francesco-Ginsparge-Zinn-Justin, BFSS, IKKT, Yang, Hanada-Kawai-Kimura
  Steinacker, Fukuma-Sugishita-Umeda, etc.

$$?? \xrightarrow[N \to \infty]{} \int_M R \sqrt{g} \, d^4 x$$

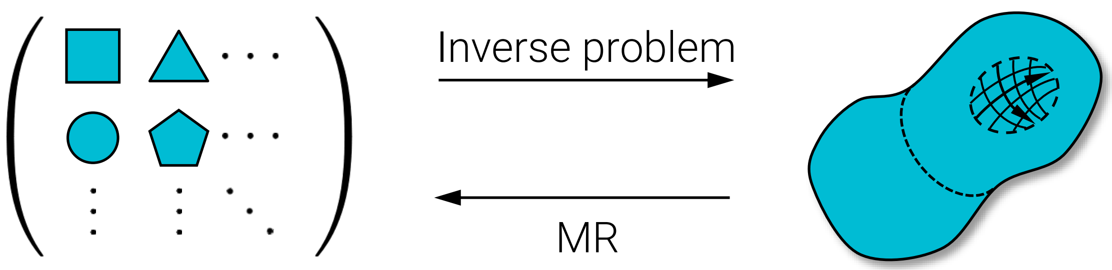
- In the theory of gravity, the fundamental structure is Riemannian structures: MR is not useful for this case.

- How can we describe Riemannian structures (metric tensor) in terms of matrices ??

Our approach

- We consider the “inverse problem” of MR: finding the corresponding space for given matrices.

Hotta-Nishimura-Tsuchiya, Asakawa-Sugimoto-Terasima, Berenstein-Dzienkowski, Ishiki, etc.



- We combine the formulation of the inverse problem with the notion of information metric.

Finding corresponding space Ishiki 2015

- Assumption: we are given a configuration of d $N \times N$ Hermitian matrices,

$$\{(X^1, X^2, \cdots, X^d) \mid N = 1, 2, \cdots\}$$

- We introduce a parameter $y \in \mathbf{R}^d$ and construct "Hamiltonian" using given matrices,

$$H(y) = \frac{1}{2} \sum_{\mu=1}^{d} (X^\mu - y^\mu)^2$$

Finding corresponding space Ishiki

- We can interpret the zeros of the energies of $H(y)$ as points on the corresponding smooth space:

$$E(y) = \langle H(y) \rangle = \frac{1}{2}\left\langle (X^\mu - y^\mu)^2 \right\rangle = \frac{1}{2}\langle (X^\mu)^2 \rangle - \langle X^\mu \rangle y_\mu + \frac{1}{2}(y^\mu)^2$$

$$= \frac{1}{2}\langle (X^\mu)^2 \rangle \underline{- \frac{1}{2}\langle X^\mu \rangle^2 + \frac{1}{2}\langle X^\mu \rangle^2} - \langle X^\mu \rangle y_\mu + \frac{1}{2}(y^\mu)^2$$

$$= \frac{1}{2}(\Delta X^\mu)^2 + \frac{1}{2}(\langle X^\mu \rangle - y^\mu)^2$$

$$E(y) \to 0 \quad \Leftrightarrow \quad \langle X^\mu \rangle \to y^\mu, \quad \Delta X^\mu \to 0$$

$\Leftrightarrow$ The wave packet shrinks to a point at $y \in \mathbf{R}^d$

Finding corresponding space Ishiki

- We define the corresponding smooth space $\mathcal{M}$ for given matrices as a set of zeros of $E(y)$:

$$\mathcal{M} = \{y \in \mathbf{R}^d | \lim_{N\to\infty} E(y) = 0\}$$

- The zero mode $|0, y\rangle$ of $H(y)$ have geometric information for given matrices X^μ.

  Cf. Berenstein-Dzienkowski 2012, Asakawa-Matsuura 2017

- This method gives the correct corresponding spaces for fuzzy sphere, fuzzy torus, etc.

Example: fuzzy 2-sphere

Given matrices

$$X^i = \frac{2}{\sqrt{N^2-1}} L^i \quad (X^i X_i = 1)$$

$L^i\,(i=1,2,3)$ are N dim irrep of SU(2) generators

Hamiltonian

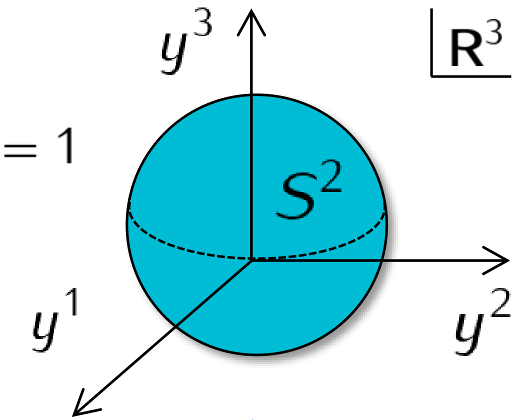
$$H(y) = \frac{1}{2}\left(\frac{2}{\sqrt{N^2-1}} L^i - y^i\right)^2$$

Zeros of the zero mode

$$E(y) = \frac{1+|y|^2}{2} - \frac{N|y|}{\sqrt{N^2-1}} \xrightarrow[N\to\infty]{} \frac{1}{2}(1-|y|)^2$$

Corresponding space

$|y| = 1$



Zero modes and density matrix

- In general, the zero mode of $H(y)$ is degenerate,

$$|0, y\rangle_\alpha \quad \alpha = 1, 2, \cdots, k$$

degree of degeneracy

- Using the zero modes, we construct a $N \times N$ density matrix that is "invariant" under the changing basis:

$$\rho(y) = \frac{1}{k} \sum_{\alpha=1}^{k} |0, y\rangle_{\alpha\,\alpha}\langle 0, y|$$

$$|0, y\rangle_\alpha \;\rightarrow\; c^\beta{}_\alpha(y)|0, y\rangle_\beta \quad (c^\dagger c = 1)$$

Information metric

- For density matrices ρ, information metric is defined by the distance between them,

$$ds^2 = \frac{1}{2}\mathrm{Tr}[G d\rho], \quad d\rho = G\rho + \rho G$$

- It gives a Riemannian metric on the space of density matrices of fixed size.

Riemannian metric via pullback

- In most cases, the density matrix using the zero modes gives an embedding,

$$\rho : y \in \mathcal{M} \;\mapsto\; \rho(y) \in \{\, \text{all } N \times N \text{ density matrices} \,\}$$

$$\rho(y) = \frac{1}{k} \sum_{\alpha=1}^{k} |0, y\rangle_{\alpha\,\alpha}\langle 0, y|$$

- We can get a Riemannian metric on $\mathcal{M}$ via the pullback of information metric induced by $\rho(y)$:

$$ds^2 = \frac{1}{2}\mathrm{Tr}[G(y)d\rho(y)] = \frac{k}{2}\mathrm{Tr}[d\rho(y)]^2$$

Example: fuzzy 4-sphere

- Fuzzy 4-sphere is defined by N-fold tensor product of five-dimensional gamma matrices Γ^μ $(\mu = 1, \ldots, 5)$,

  Castelino-Lee-Taylor

$$X^\mu = \frac{1}{N}(\Gamma^\mu \otimes 1_4 \otimes \cdots \otimes 1_4 + \cdots + 1_4 \otimes \cdots \otimes 1_4 \otimes \Gamma^\mu)_{\mathrm{sym}}$$

- It is a non-commutative version of 4-sphere $S^4 \subset \mathbf{R}^5$,

$$X^\mu X_\mu = 1 + \mathcal{O}(1/N), \quad [X^\mu, X^\nu] \to 0 \, (N \to \infty)$$

- However, fuzzy 4-sphere is not matrix regularization because 4-sphere is not symplectic manifold.

Example: fuzzy 4-sphere

- The Hamiltonian for fuzzy 4-sphere is given by

$$H(y) = \frac{1}{2}(X^\mu - y^\mu)^2, \quad y \in \mathbf{R}^5$$

- The zero modes are

$|0, y\rangle_\alpha = U\,|$ eigenstates of X^5 with +1 $\rangle_\alpha \quad \alpha = 1, 2, , \ldots, N+1$

$U = e^{-\chi\Gamma^{21}/2} e^{-\psi\Gamma^{32}/2} e^{-\phi\Gamma^{43}/2} e^{-\theta\Gamma^{54}/2}$

- Our metric is given by: $\rho = \frac{1}{N+1}\sum_\alpha |0, y\rangle_{\alpha\alpha}\langle 0, y|$

$ds^2 = (d\theta)^2 + \sin^2\theta(d\phi)^2 + \sin^2\theta\sin^2\phi(d\psi)^2 + \sin^2\theta\sin^2\phi\sin^2\psi(d\chi)^2$

Summary

- If we are given matrices, we can get a Riemannian metric on the corresponding space in terms of the matrices (in most cases).
- In the framework of MR, our metric have nice relation with the symplectic form. c.f Ishiki-TM-Muraki
- Our metric works even for fuzzy 4-sphere which is not matrix regularization.
- It is expected that our metric gives direct relation between the transformations of matrices and dffeomorphism.

Thank you very much for

nice conference !!